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# A REMARK ON 1-QUASICONFORMAL MAPS

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1. *Introduction.* It is well known that if $n \geqq 3$, every 1-quasiconformal map of a domain $D \subset R^n$ is the restriction of a Möbius transformation. For $C^3$-maps this was already proved by Liouville in 1850. The general result is due to Gehring [$\mathrm{Ge}_1$] and Rešetnjak [Re]. Their proofs are very deep; a more elementary proof has recently been given by Bojarski and Iwaniec [BI]. Mostow [Mo, (12.2)] pointed out that the case $D = R^n$ is much easier; another proof for this case has been given by Gehring [$\mathrm{Ge}_2$]. The purpose of this note is to give a new and simple proof for this special case. It is based on the compactness properties of quasiconformal maps and on the fact that the 1-quasiconformal maps of $R^n$ form a group. It is also valid for $n=2$.

2. *Notation.* For $x \in R^n$ and $r > 0$ we let $S(x, r)$ denote the sphere $\{y \in R^n : |y-x| = r\}$.

3. Lemma. *Let $f\colon R^n \to R^n$ be a homeomorphism such that the image of each sphere $S(x, r)$ is a sphere $S(f(x), r_x)$. Then $f$ is a similarity.*

*Proof.* Let $x, y \in R^n$ with $|x-y| = 2r > 0$, and let $z = (x+y)/2$. Consider the sphere $S_0$ of radius $r/2$ which touches the spheres $S_1 = S(x, r)$ and $S_2 = S(x, 2r)$ at $z$ and $y$. Since $fS_0$ touches $fS_1$ and $fS_2$, $f(z)$ lies on the line segment $f(x)f(y)$. Since $fS(z, r)$ is a sphere centered at $f(z)$, $f(z) = (f(x)+f(y))/2$. Hence $f$ preserves the midpoint of every line segment. By iteration and continuity, this implies that $f$ is affine on every line. For each line $L$, there is thus a number $\lambda_L > 0$ such that $|f(a)-f(b)| = \lambda_L |a-b|$ for all $a, b \in L$. Moreover, if the lines $L$ and $M$ intersect, $\lambda_L = \lambda_M$. It follows that $\lambda_L = \lambda$ is independent of $L$. □

4. Theorem. *Let $n \geqq 2$ and let $f\colon R^n \to R^n$ be 1-quasiconformal. Then $f$ is a similarity.*

*Proof.* By the preceding lemma, it suffices to show that $f$ maps every sphere $S(x, r)$ onto a sphere centered at $f(x)$. With the aid of auxiliary similarity maps, we may assume that $x = 0 = f(x)$, that $r = 1$, that $f(e_1) = e_1$, and that the open unit ball $B^n$ is contained in $fB^n$. Let $W$ be the family of all 1-quasiconformal maps $g\colon R^n \to R^n$ such that $g(0) = 0$, $g(e_1) = e_1$, and $B^n \subset gB^n$. Since $W$ is a closed nonempty normal family [Vä, 19.4, 21.3, 37.4], there is $h \in W$ for which

$$m(h\bar{B}^n) = \max\{m(g\bar{B}^n)\colon g \in W\} = M < \infty.$$

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It suffices to show that $M=m(\bar{B}^n)$. If $M>m(\bar{B}^n)$, $\bar{B}^n$ is a proper subset of $h\bar{B}^n$, and hence $h\bar{B}^n$ is a proper subset of $hh\bar{B}^n$, which implies $m(hh\bar{B}^n)>M$. Since $hh\in W$, this is a contradiction. □

5. Remark. The preceding theorem is also trivially true for $n=1$, if we, as usual, interpret the $K$-quasisymmetric functions $f$: $R^1\to R^1$ as one-dimensional $K$-quasiconformal maps. On the other hand, the same proof gives the following more general result, which is nontrivial also for $n=1$. We allow the possibility that a quasiconformal map is sense-reversing.

6. Theorem. *Let $n\geqq 1$, let $K\geqq 1$, and let $G$ be a group of $K$-quasiconformal maps of $R^n$ such that $G$ contains all similarity maps. Then $G$ is precisely the group of all similarity maps of $R^n$.*

In the proof, we may assume that $G$ is closed, replacing it by $\bar{G}$. □

Actually, it is sufficient to assume that $G$ contains a group $S$ of similarities such that for each pair of distinct points $x, y\in R^n$ there is $g\in S$ such that $g(0)=x$, $g(e_1)=y$. The proof shows that every element of $G$ is then a similarity.

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