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LINEARLY LOCALLY CONNECTED SETS AND QUASICONFORMAL MAPPINGS

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1. Introduction

The purpose of this paper is to study the linear local connectivity of sets in Euclidean n-space in relation to their boundaries, and to prove that linear local connectivity is a quasiconformal invariant which characterizes quasiconformal mappings.

The notion of linearly locally connected sets arose in the work of F. W. Gehring and J. Väisälä [7] in their investigation of quasiconformal mappings in Euclidean three-space, and appeared, under the name of strongly locally connected sets, in a paper of Gehring [3] as a means of proving that a quasiconformal mapping of a Jordan domain D in three-space onto the unit ball can be extended to a quasiconformal mapping of the whole space if and only if the exterior of D is quasiconformally equivalent to the unit ball. Since then, the concept has been used [4], [5], [6] in studying the univalence of analytic functions and in characterizing quasidisks.

2. Notation

For each integer $n\geqq 1$, let R^n denote Euclidean n-dimensional space, and $\bar{R}^n$ the one-point compactification $R^n\cup\{\infty\}$. Points of R^n will be denoted by letters such as P, Q, x, y. The coordinates of x will be $(x_1, \ldots, x_n)$, $|P|$ denotes the norm of P, and $B^n(P, r)$ is the ball $\{x\colon |x-P|<r\}$ in R^n. For a set $E\subset\bar{R}^n$, $\bar{E}$, $C(E)$, ∂E will denote the closure, complement and boundary of E. If $P\in R^n$ and E, F are sets in $\bar{R}^n$, then $d(P, E)$ and $d(E, F)$ denote the Euclidean distances from E to P and F respectively. The Euclidean diameter of a bounded set E is denoted by diam (E).

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3. Definition and examples

3.1. Definition. A set $E\subset\bar{R}^n$ is *c-locally connected*, where $1\leqq c<\infty$, if, for each $P\in R^n$ and each $r>0$, each pair of points in $E\cap\bar{B}^n(P,r)$ is joined by a continuum in $E\cap\bar{B}^n(P,cr)$, and each pair of points in $E\setminus B^n(P,r)$ is joined by a continuum in $E\setminus B^n(P,r/c)$.

E is *linearly locally connected* if there is a number c, $1\leqq c<\infty$ for which E is c-locally connected.

Quasidisks, uniform domains, the complement in $\bar{R}^n$ $(n\geqq 3)$ of the quasiconformal image of a ball are all linearly locally connected sets.

It is easy to check that any linearly locally connected set is connected, locally connected at each point of $\bar{R}^n$ and free of cut-points. If E is c-locally connected and $c_1>c$, E is c_1-locally connected.

4. Boundary and closure relations

4.1. Proposition. *Let E be a set in $\bar{R}^n$. Then*

(a) *if ∂E is c-locally connected, so is $\bar{E}$,*

(b) *if E is c-locally connected, so is $\bar{E}$.*

Proof. (a) Let $P\neq\infty$ and $r>0$ be given, and $x,y\in\bar{E}\cap\bar{B}^n(P,r)$. If $\bar{B}^n(P,r)\subset\bar{E}$, then the segment $[x,y]$ joins x, y in $\bar{E}\cap\bar{B}^n(P,cr)$. If not, then $\partial E\cap\bar{B}^n(P,r)$ is non-empty, so there are (possibly trivial) segments in $\bar{E}\cap\bar{B}^n(P,r)$ joining x, y to points $\bar{x},\bar{y}\in\partial E\cap\bar{B}^n(P,r)$. The c-local connectivity of ∂E yields a continuum γ in $\partial E\cap\bar{B}^n(P,cr)$ joining $\bar{x}$, $\bar{y}$, so $[x,\bar{x}]\cup\gamma\cup[\bar{y},y]$ is a continuum joining x, y in $\bar{E}\cap\bar{B}^n(P,cr)$.

A similar argument applies to $x,y\in\bar{E}\setminus B^n(P,r)$.

(b) If $x,y\in\bar{E}\cap\bar{B}^n(P,r)$, there are sequences (x_m), (y_m) in E with $\lim_{m\to\infty}x_m=x$, $\lim_{m\to\infty}y_m=y$, and we may assume $x_m,y_m\in E\cap\bar{B}^n(P,r+1/m)$. By the c-local connectivity of E, x_m and y_m can be joined by a continuum γ_m in $E\cap\bar{B}^n(P,c(r+1/m))$. Let

$$F_k=\overline{\bigcup_{m=k}^{\infty}\gamma_m}\quad\text{and}\quad F=\bigcap_{k=1}^{\infty}F_k.$$

Then $F_k\subset\bar{E}\cap\bar{B}^n(P,c(r+1/k))$, and (F_k) is a decreasing sequence of compact sets containing x and y. Thus F is a compact set in $\bar{E}\cap\bar{B}^n(P,cr)$ containing x, y.

To show F is connected, we first show that, for each $\varepsilon>0$, F_k is ε-connected for each $k\geqq k_0$, where k_0 is such that $|x-x_m|<\varepsilon$ whenever $m\geqq k_0$. For, if $Q_1,Q_2\in F_k$ and $k\geqq k_0$, then, for $i=1,2$ we have integers $k_i\geqq k$ and $Q_i'\in\gamma_{k_i}$ with $|Q_i-Q_i'|<\varepsilon$. As γ_{k_1}, γ_{k_2} are continua, they are ε-connected, so there is an ε-chain from Q_1 to Q_2 via Q_1', x, Q_2' in F_k. F_k is thus ε-connected.

It now follows that F is 2ε-connected for each $\varepsilon>0$, and being compact, is a continuum joining x, y in $\bar{E}\cap\bar{B}^n(P,cr)$.

A similar argument applies to $x, y \in \bar{E} \setminus B^n(P, r)$.

Simple examples show that neither converse in Proposition 4.1 is true.

4.2. Proposition. *If E is a uniformly locally connected set, and ∂E is c-locally connected, then E is c_1-locally connected for any $c_1 > c$.*

Proof. Let $P \in R^n$, $r > 0$ and $x, y \in E \cap \bar{B}^n(P, r)$. If the segment $[x, y]$ is in $E \cap \bar{B}^n(P, r)$, this is a continuum of the required type.

If not, let x_0, y_0 be the first and last points of $[x, y] \cap \partial E$ in passing from x to y. As ∂E is c-locally connected, there is a continuum γ joining x_0 to y_0 in $\partial E \cap \bar{B}^n(P, cr)$.

As E is uniformly locally connected, for any fixed $c_1 > c$, there is a $\delta > 0$ such that, for any point Q of γ, any two points of $E \cap B^n(Q, \delta)$ can be joined by a continuum in $E \cap B^n(Q, (c_1 - c)r)$. Since γ is a continuum, there is a $\delta/2$-chain $x_0 = Q_0, Q_1, \dots, Q_m = y_0$ joining x_0 to y_0 in γ.

We use this chain to construct a continuum joining x, y in $E \cap B^n(P, c_1 r)$. For $i = 0, 1, \dots, m$, let $\Delta_i = B^n(Q_i, \delta)$. Choose $P_0 \in [x, x_0] \cap \Delta_0$. If P_k has been chosen, let $P_{k+1} \in \Delta_k \cap \Delta_{k+1} \cap E$ if $k \leqq m-1$, and let $P_{m+1} \in \Delta_m \cap [y_0, y]$. By the local connectivity of E, there is a continuum $\gamma_k \subset B(Q_k, (c_1 - c)r) \cap E$ joining P_k to P_{k+1}. Hence $\gamma_0 \cup \dots \cup \gamma_m$ is a continuum joining P_0 to P_m in E, with each point less than $(c_1 - c)r$ from ∂E and so $[x, x_0] \cup \gamma_0 \cup \dots \cup \gamma_m \cup [y_0, y]$ is a continuum joining x, y in $E \cap B^n(P, c_1 r)$.

A similar argument applies to $x, y \in E \setminus B^n(P, r)$.

Remarks. 1. If, in Proposition 4.2, E is a domain, the conclusion holds with $c_1 = c$.

2. For $n \geqq 3$, there are linearly locally connected domains whose boundaries fail to be linearly locally connected.

4.3 Proposition. *If D is a plane domain and ∂D is c-locally connected, and consists of more than one point, then D is a c-locally connected Jordan domain.*

*Proof**). Since ∂D is connected, locally connected and free of cut-points, Theorem 9.9 of [8, p. 281] implies that D is a Jordan domain. Thus D is locally connected at each boundary point, indeed uniformly locally connected, so from Proposition 4.2 and the subsequent Remark we see that D is c-locally connected.

4.4. Corollary. *A domain $D \subset \bar{R}^2$ is a quasidisk if and only if ∂D is linearly locally connected and consists of more than one point.*

Proof. This follows from a theorem of Gehring [6, p. 31] if we merely apply Proposition 4.3 when ∂D is linearly locally connected.

Remark. In fact [F. W. Gehring], $D \subset \bar{R}^2$ is a K-quasicircle domain if and only if D is c-locally connected, where c depends only on K.

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*) Dr. Kari Hag has kindly pointed out this method of proof.

5. Invariance under quasiconformal mappings

We begin with [3].

5.1. Theorem. *If E is a c-locally connected set in $\bar{R}^n$ and E' is the image of E under a Möbius transformation, then E' is $g(c)$-locally connected, where g is the inverse of the increasing function h given by*

$$h(t) = t^{1/2}+t^{-1/2}-1, \quad 1 \leqq t < \infty.$$

For a detailed proof, see [12].

Thus linearly local connectivity is invariant under Möbius transformations.

In dealing with the effect of quasiconformal mappings, other characterizations of linear local connectivity are sometimes more convenient.

If Γ denotes a family of locally σ-rectifiable arcs in $\bar{R}^n$, and $F(\Gamma)$ the family of non-negative Borel-measurable functions ϱ for which $\int_\gamma \varrho\, ds \geqq 1$ for each $\gamma \in \Gamma$, we define the *modulus* of Γ, $M(\Gamma)$ by

$$M(\Gamma) = \inf_{\gamma\in\Gamma} \int_{R^n} \varrho^n\, dm$$

where m denotes n-dimensional Lebesgue measure.

5.2. Definition. Given two sets F_0, F, in $\bar{R}^n$, the *extremal distance*, $\lambda(F_0, F_1)$, between F_0, F_1 is defined by

$$\lambda(F_0, F_1) = \left(\frac{\sigma_{n-1}}{M(\Gamma)}\right)^{1/(n-1)}$$

where Γ is the family of arcs joining F_0, F_1 in $\bar{R}^n$ and σ_{n-1} is the $(n-1)$-dimensional measure of the unit $(n-1)$-sphere $\partial B^n(0, 1)$.

We note that $\lambda(F_0, F_1)$ is invariant under Möbius transformations.

5.3. Definition. A set $E\subset\bar{R}^n$ is *linearly locally connected (in the extremal distance sense)* if there is a number $s>0$ such that, if F_0, F_1 are continua in $\bar{R}^n$ with

$$\lambda(F_0, F_1) \geqq s,$$

then $E\cap F_i$ can be joined by a continuum in $E\setminus F_{1-i}$, for $i=0, 1$.

We relate this definition to the original (Euclidean) definition of linear local connectivity by means of rings.

An open connected set $R\subset\bar{R}^n$ is a *ring* if $C(R)$ consists of two components, which we usually denote by C_0, C_1. We define the *modulus* of a ring R as

$$\operatorname{mod} R = \lambda(C_0, C_1).$$

If $R=\{x\colon a<|x-P|<b\}$ where $0<a<b<\infty$, then

$$\operatorname{mod} R = \log\frac{b}{a}.$$

The n-dimensional Teichmüller ring $R_T(u)$ is the ring with complementary components the segment

$$\{x:\ -1 \leqq x_1 \leqq 0,\ x_2 = \ldots = x_n = 0\}$$

and the ray

$$\{x:\ u \leqq x_1 \leqq \infty,\ x_2 = \ldots = x_n = 0\},$$

where $u>0$. For such u, we define the function Ψ_n by

$$\log \Psi_n(u) = \operatorname{mod} R_T(u).$$

Note that Ψ_n is increasing and $\Psi_n(1)>1$. See [1] and references therein.

The link between the two definitions of linear local connectivity is provided by

5.4. Lemma. *Let F_0, F_1 be continua in $\bar{R}^n$ with $\infty \in F_1$ and*

$$\lambda(F_0, F_1) \geqq \log \Psi_n(c),$$

where $c>1$. Then for any $P \in F_0$, there is a number $r>0$ such that the ring $\{x:\ r<|x-P|<rc\}$ separates F_0, F_1 in $C(F_0 \cup F_1)$.

Proof. Since $\lambda(F_0, F_1)>0$, F_0, F_1 are disjoint, so Lemma 3.5 of [7] implies there is a ring R with complementary components C_0, C_1 such that, for $i=0, 1$, $\partial C_i \subset F_i \subset C_i$. As every arc joining F_0, F_1 joins C_0, C_1 it follows easily that

$$\lambda(F_0, F_1) = \lambda(C_0, C_1) = \operatorname{mod} R.$$

Let r be the radius of the smallest ball with centre P containing C_0. The extremal property of the Teichmüller ring ([10] for R^2, [2] for R^3 and [9] for R^n) then implies the result.

5.5. Theorem. *A set E is linearly locally connected in the Euclidean sense if and only if E is linearly locally connected in the extremal distance sense.*

Proof. Suppose E is linearly locally connected in the extremal distance sense, with constant s. Given $P \in R^n$ and $r>0$, let $F_0 = \bar{B}^n(P, r)$ and $F_1 = C(B^n(P, re^s))$. Then $\lambda(F_0, F_1) = s$, so any two points of $E \cap F_i$ can be joined by a continuum in $E \setminus F_{1-i}$. Therefore E is e^s-locally connected.

For the converse, let E be c-locally connected, where we may assume $c>1$, and let F_0, F_1 be continua in $\bar{R}^n$ with $\lambda(F_0, F_1) \geqq s$, where $s = \log \Psi_n(c)$.

Suppose first that $\infty \in F_1$. Choose $P \in F_0$ and r as in Lemma 5.4, so that $\{x:\ r<|x-P|<rc\}$ separates F_0, F_1 in $C(F_0 \cup F_1)$. Then $E \cap F_0 \subset E \cap \bar{B}^n(P, r)$, and so is in a continuum in $E \cap \bar{B}^n(P, rc) \subset E \setminus F_1$; and $E \cap F_1 \subset E \setminus B^n(P, rc)$ is in a continuum in $E \setminus B^n(P, r) \subset E \setminus F_0$.

If $\infty \notin F_1$, let M be a Möbius transformation such that $\infty \in M(F_1)$. By Theorem 5.1, $M(E)$ is linearly locally connected, so the above argument applies to $M(E)$. The conclusion follows because M^{-1} is a homeomorphism.

Remark. It is evident from the proofs of Lemma 5.4 and Theorem 5.5 that the definition of linear local connectivity in terms of extremal distance is equivalent to the analogous definition in terms of rings: a set $E\subset\bar{R}^n$ is linearly locally connected if and only if there is a number $s>0$ such that, for each ring R with complementary components C_0, C_1, and $\operatorname{mod} R\geqq s$, $E\cap C_i$ is in a component of $E\setminus C_{1-i}$ for $i=0, 1$.

If D, D' are domains in $\bar{R}^n$, we say the homeomorphism $f\colon D\to D'$ is a *K-quasiconformal mapping* of D onto D', where $K\geqq 1$, if, for each disjoint pair of continua F_0, F_1 with the ring R of the proof for Lemma 5.4 satisfying $\bar{R}\subset D$,

$$\frac{1}{K}\lambda(F_0, F_1)\leqq\lambda\big(f(F_0\cap D), f(F_1\cap D)\big)\leqq K\lambda(F_0, F_1).$$

The mapping f is *quasiconformal* if f is K-quasiconformal for some $K\geqq 1$.

For $\bar{B}^n(P, r)\subset D$, let

$$L(P,r) = \max\{|f(x)-f(P)|\colon\ |x-P| = r\},$$
$$l(P,r) = \min\{|f(x)-f(P)|\colon\ |x-P| = r\}$$

and

$$H(P) = \limsup_{r\to 0}\frac{L(P,r)}{l(P,r)}.$$

Then it is known [11] that f is K-quasiconformal if

$$\sup_{P\in D} H(P) < \infty$$

and

$$H(P)\leqq K$$

almost everywhere.

5.6. Theorem. *Let D be R^n or $\bar{R}^n$ and $f\colon D\to D$ K-quasiconformal mapping. If $E\subset D$ is c-locally connected, then $E'=f(E)$ is c'-locally connected, where c' depends only on c, K and n.*

Proof. If $D=R^n$, then we can extend f to a K-quasiconformal mapping of $\bar{R}^n$ by removing the singularity at ∞. So we assume $D=\bar{R}^n$.

Let F_0', F_1' be continua with

$$\lambda(F_0', F_1')\geqq Ks$$

where $s=\log\Psi_n(c)$. If $f^{-1}(F_i')=F_i$, $i=0, 1$, then

$$\lambda(F_0, F_1)\geqq s$$

and so $E\cap F_i$ lies in a continuum in $E\setminus F_{1-i}$, $i=0, 1$. Since f is a homeomorphism, $E'\cap F_i'$ lies in a continuum in $E'\setminus F_{1-i}'$, for $i=0, 1$. It follows that E is c'-locally connected, where

$$c' = e^{sK} = \big(\Psi_n(c)\big)^K.$$

If the domain D is a proper subset of R^n, the distance to the boundary plays a role. We first prove

5.7. Lemma. *Let E be a bounded connected set in R^n which satisfies the definition of c-local connectivity for all $P\in R^n$ and all r, $0<r\leqq\delta$ where $\delta>0$ is fixed. Then E is $2c(1+d/\delta)$-locally connected, where $d=\operatorname{diam}(E)$.*

Proof. Suppose $P\in R^n$, $r>\delta$ are given.

(i) If $E\cap\bar{B}^n(P,r)\neq\emptyset$, then the connected set

$$E\subseteqq E\cap\bar{B}^n(P,r+d)\subseteqq E\cap\bar{B}^n(P,(1+d/\delta)r),$$

so $E\cap\bar{B}^n(P,r)$ is in the connected set $E\cap\bar{B}^n(P,(1+d/\delta)r)$.

(ii) If $E\setminus B^n(P,r)\neq\emptyset$, then either $r\geqq 2d$, in which case $E\subset C(B^n(P,r-d))\subset C(B^n(P,r/2))$, so that $E\setminus B^n(P,r)$ lies in the connected set $E\setminus B^n(P,r/2)$; or else $\delta<r<2d$, when $E\setminus B^n(P,r)\subset E\setminus B^n(P,\delta)$, and by hypothesis any two points of this set are joined by a continuum in $E\setminus B^n(P,\delta/c)\subset E\setminus B^n(P,r\delta/(2cd))$.

The result now follows readily.

5.8. Theorem. *Let D, D' be proper subdomains of R^n, and $f\colon D\to D'$ a K-quasiconformal mapping. If E is a c-locally connected set with $\bar{E}\subset D$, then $E'=f(E)$ is c'-locally connected, where $c'=c'(c,K,\operatorname{diam}(E)/d(E,\partial D),n)$.*

Proof. We first exhibit numbers $c''>1$ and $\delta'>0$ for which Lemma 5.7 applies to E'.

Since $E'\subset D'$, E' is bounded and $0<d(E',\partial D')<\infty$. Let $c''=(\Psi_n(c))^K$, and choose $\delta'<d(E',\partial D')/(c''+1)$.

Let $P'\in R^n$, and $0<r'\leqq\delta'$. If $E'\cap\bar{B}^n(P',r')\neq\emptyset$ then $d(P',E')\leqq r'$, and so

$$d(\partial D',\bar{B}^n(P',r'c''))\geqq d(\partial D',E')-(c''+1)r'>0.$$

Hence $\bar{B}^n(P',r'c'')\subset D'$. Denoting by R' the ring $\{x\colon r'<|x-P'|<c''r'\}$, we see that $R=f^{-1}(R')$ has modulus at least

$$\frac{1}{K}\operatorname{mod}R'=\frac{1}{K}\log c''=\log\Psi_n(c).$$

But if R has complementary components C_0, C_1, where $C_0=f^{-1}(\bar{B}^n(P',r'))$, we see $E\cap C_i$ is in a continuum in $E\setminus C_{1-i}$ for $i=0,1$. Since f is a homeomorphism, $E'\cap\bar{B}^n(P',r')$ is in a component of $E'\cap\bar{B}^n(P',r'c'')$, and $E'\setminus B^n(P',r'c'')$ is in a component of $E'\setminus B^n(P',r')$.

Lemma 5.7 now shows E is c'-locally connected, where c' depends on c, K and $\operatorname{diam}E'/d(E',\partial D')$.

It now suffices to show that $\operatorname{diam}E'/d(E',\partial D')$ depends only on K, $\operatorname{diam}E/d(E,\partial D)$ and n. For this, let α be the number such that

$$\Theta_K^n(\alpha)=\frac{1}{3}$$

where Θ_K^n is the distortion function of [11, p. 63], and let $\delta = d(E, \partial D)$. Consider a grid of closed n-cubes Q meeting E, with disjoint interiors, of side-length $\alpha\delta/((\alpha+1)\sqrt{n})$. The number of such cubes clearly cannot exceed a bound depending only on n, K and $\operatorname{diam}(E)/d(E, \partial D)$. If $x, y \in Q$, then

$$d(Q, \partial D) \geqq d(x, \partial D) - \operatorname{diam} Q \geqq \delta - \frac{\alpha\delta}{\alpha+1} = \frac{\delta}{\alpha+1}$$

so that

$$\frac{|x-y|}{d(x, \partial D)} \leqq \frac{\operatorname{diam} Q}{d(Q, \partial D)} \leqq \alpha.$$

Denoting $f(x), f(y), f(Q)$ by x', y', Q', we have

$$\frac{|x'-y'|}{d(x', \partial D')}, \frac{|x'-y'|}{d(y', \partial D')} \leqq \Theta_K^n\left(\frac{|x-y|}{d(x, \partial D)}\right) \leqq \Theta_K^n(\alpha) = \frac{1}{3},$$

by Theorem 18.1 of [11]. This implies

$$\frac{\operatorname{diam} Q'}{d(Q', \partial D')} \leqq \frac{1}{2}. \tag{5.9}$$

Since E is connected, there is a simple arc γ joining points x, y in E with $|f(x)-f(y)| = \operatorname{diam}(E')$. Clearly, γ can be covered by a sequence $Q_1, \ldots, Q_N$ of cubes of the grid such that $x \in Q_1$, $y \in Q_N$ and $F_k \cap Q_{k+1} \neq \emptyset$ where $F_k = \bigcup_{j=1}^k Q_j$. Let $F_k' = f(F_k)$. Simple geometric considerations show that

$$\frac{\operatorname{diam}(F_k')}{d(F_k', \partial D')} \leqq m_k, \quad k = 1, \ldots, N,$$

where $m_1 = 1/2$ and $m_{k+1} = 3m_k/2 + 1/2$, $k = 1, \ldots, N-1$. But then

$$\frac{\operatorname{diam}(E')}{d(E', \partial D')} \leqq \frac{\operatorname{diam}(F_N')}{d(F_N', \partial D')} \leqq m_N,$$

a number with an upper bound depending only on n, K and $\operatorname{diam}(E)/d(E, \partial D)$. The proof of Theorem 5.8 is complete.

5.10. Corollary. *Let f be a quasiconformal mapping of the domain D onto the domain D' in $\bar{R}^n$. If E is a linearly locally connected set with $\bar{E} \subset D$, then $f(E)$ is linearly locally connected.*

Proof. This follows from Theorem 5.6, Theorem 5.8 and the invariance of linear local connectivity under Möbius transformations.

We have the following characterization of quasiconformal mappings:

5.11. Theorem. *Let f be a homeomorphism of a domain D onto a domain D' in $\bar{R}^n$. If f maps each c-locally connected set E with $\bar{E} \subset D \setminus \{\infty\}$ onto a c'-locally connected set where c' depends only on c, f and* $\operatorname{diam}(E)/d(E, \partial D)$ *(in case $R^n \subset D$, we take the last to be 0), then f is a quasiconformal mapping of D onto D'.*

Proof. Write $D_1 = D \setminus \{\infty, f^{-1}(\infty)\}$ and let $P \in D_1$. Choose a so that $0 < a < d(P, \partial D_1)/2$, where the last may be ∞. For $0 < r < a/2$, let $P_1, P_2 \in \partial B^n(P, r)$ so that the acute angle between the segments PP_1, P_1P_2 is at least $\pi/4$, and let P_3, P_4 be the points of intersection of the rays P_1P, P_1P_2 with $\partial B^n(P, a)$. The set E consisting of the segments P_1P_3, P_4P_1 and the minor arc of the great circle of $\partial B^n(P, a)$ through P_3, P_4 is then $\operatorname{cosec} \pi/8$-locally connected, and, by choice of a, $\operatorname{diam}(E)/d(E, \partial D) < 1/2$. Then $f(E)$ is c'-locally connected, where c' is independent of P and the particular choice of E.

Now, if r is so small that

$$\tfrac{1}{2}(c'+1)L(P, r) \leqq l(P, a),$$

it is not difficult to show that

$$|f(P) - f(P_1)| \leqq \tfrac{1}{2}(c'+1)\,|f(P) - f(P_2)|$$

from the c'-local connectivity of $f(E)$. But then it follows that

$$L(P, r) \leqq \tfrac{1}{4}(c'+1)^2\, l(P, r)$$

so that

$$H(P) = \limsup_{r \to 0} \frac{L(P, r)}{l(P, r)} \leqq \frac{1}{4}(c'+1)^2$$

for all points $P \in D_1$.

It follows that $f|_{D_1}$ is quasiconformal. But then f is quasiconformal on D, since if $\infty \in D$ or D', $\infty, f^{-1}(\infty)$ are removable singularities.

Remark. We have only used the hypothesis of Theorem 5.11 when $c = \operatorname{cosec}(\pi/8)$. When $n = 2$, this can be sharpened to $c = 1$.

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