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ON THE EXTREMALITY AND UNIQUE EXTREMALITY OF AFFINE MAPPINGS IN SPACE

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1. Introduction and historical remarks. In the beginning of the theory of quasiconformal mappings in the plane, Grötzsch considered the problem of finding a mapping which is "as conformal as possible" among all mappings from one rectangle onto another one with fixed given side-correspondence [G]. The solution is the affine mapping, and we call it an *extremal* mapping. Furthermore, it is the only mapping with this property, wherefore it is called *uniquely* extremal. We scarcely need point out how important it is in Teichmüller theory that this mapping is not only extremal but also uniquely extremal.

In space one can certainly pose the analogous problem. But now the question of how to measure the "distance" from a quasiconformal mapping to the class of conformal mappings becomes more of an issue. Whereas in the plane, the ratio of major to minor axes of the infinitesimal ellipses which are mapped on circles provides a widely accepted standard, in higher dimensions several "dilatations" have been used, which reflect in various ways the values of the intermediate axes of the infinitesimal ellipsoids. Initially the above mentioned "linear" dilatation was still in use, and E. Zimmermann attacked the rectangular box problem with respect to this dilatation in his papers in 1955 and 1959 for the case of three dimensions, $[Z_1, Z_2]$. He succeeded in proving the unique extremality of the affine mapping in the unit cube only in the very special case where two sides of the image box have the same length. Then in 1962 R. Kühnau showed that the affine mapping is extremal in general, but that there exist infinitely many extremals when the sides of the image box are all different [K].

Kühnau also considered the problem with respect to the "inner" and "outer" dilatations, which were just creeping into the literature [N, V, Ša]. Again he found that the affine mapping is only one extremal among infinitely many, except for very special cases. For example, in three space the affine mapping is the only mapping which is simultaneously extremal for both inner and outer dilatations, but even this property is not true for dimensions greater than three.

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In 1975, L. Ahlfors introduced a new dilatation which is due to C. Earle [Ah]. As Ahlfors noted, a crucial test for this "logarithmic" dilatation is whether the affine mapping is extremal for the rectangular box problem. In 1980 the first author showed that in three-space the answer is affirmative provided that the dilatation is small [Ag_2].

In the present paper we make two advances. First, extremality is retained by the affine mapping (with small dilatation) in any dimension. Second, the affine mapping is in addition *uniquely* extremal (under the same proviso). The questions raised are substantial: many of the methods used are precise, yet the nonextremality is not exhibited for any affine mapping. We begin with a review of the definitions of the various dilatations, a precise statement of the problems to be considered, and an application. The proofs of the main results occupy the central Sections 3, 4, 5, and we conclude with some remarks on the limitations of the method.

2. Dilatations and extremal problems. A *scalar dilatation* is best viewed as a numerical quantity K associated to each matrix $A \in \mathrm{GL}(n, R)$. Thus we write $K[A]$, and by its magnitude infer how close A is to a conformal matrix. More specifically, as is well known, there exist orthogonal U, V such that UAV is diagonal, i.e.,

$$\frac{UAV}{\det^{1/n} A} = \begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix} = \mathrm{diag}(\lambda) \quad \left(\Pi_{i=1}^n \lambda_i = 1\right).$$

These numbers λ are not unique, but their squares are unique, as the eigenvalues of the positive definite symmetric normalized *dilatation matrix*

$$X[A] = \frac{A^T A}{\det^{2/n} A}$$

in which the columns of V are the eigenvectors of $A^T A$.

By adjustments in U, V we can further assume the λ's are all positive, which we do in practice and throughout this paper. Most scalar dilatations, then, are functions of the λ's, and we mention five for reference:

$K_0 = \max_i \{\lambda_i^n\}$, the "outer" dilatation,

$K_I = \max_i \{\lambda_i^{-n}\}$, the "inner" dilatation,

$K_L = \max_{i,j} \{\lambda_i/\lambda_j\}$, the "linear" dilatation,

$K_T = \frac{1}{n} \sum_{i=1}^n \lambda_i^2$, the Kreines, or "trace" dilatation,

$K_E = \exp \sqrt{c_n \sum_{i=1}^n \log^2 \lambda_i}$, the Earle/Ahlfors or "logarithmic" dilatation.

Each of these has $K \geqq 1$, with equality if and only if all λ's equal one, a fact which is obvious for all except K_T. For K_T however, we observe: $X[UAV] = X[AV]$, and trace $X[AV]=\text{trace}\, X[A]$. Thus

$$K_T[A] = \frac{1}{n}\text{trace diag}\,(\lambda_1^2, \lambda_2^2, \ldots, \lambda_n^2) = \frac{1}{n}\text{trace}\, X[UAV]$$

$$= \frac{1}{n}\text{trace}\, X[A] = \frac{1}{n}\,(\textstyle\sum_{i,j} a_{ij}^2)/\det^{2/n} A.$$

This makes K_T very attractive in the sense that one need not process A in any way (other than to calculate its determinant).

It is well to note that for $A=\text{diag}\,(\lambda, 1, \ldots, 1)$ $(\lambda>1)$, we have $K_0=\lambda^{n-1}$, $K_I=\lambda$, $K_L=\lambda$, $K_T=\dfrac{1}{n}(\lambda^{2(n-1)/n}+(n-1)\lambda^{-2/n})$, and the normalizing constant c_n in K_E is chosen $(n/(n-1))$ so that K_E is also λ. Thus when $n=2$, all are equal to λ, except for K_T which is $(\lambda+1/\lambda)/2$.

We further recall some of the most basic inequalities:

$$M(\Gamma)/K_0 \leqq M(\Gamma^*) \leqq K_I M(\Gamma), \tag{2.1}$$

$$\alpha/K_L \leqq \alpha^* \leqq K_L \alpha,$$

in which Γ is a path family, M the modulus, α the radian measure of an angle between two smooth curves, and $M(\Gamma^*)$, α^* the corresponding quantities after transformation by A([G—V], [Ag$_1$]).

For a quasiconformal mapping f and given dilatation K, the differential matrix $f'(u)$ belongs to GL (n, R) for a.e. u, and the eigenvalues λ_i^2 for $X[f'(u)]$ are measurable functions, so we obtain the measurable *dilatation function*

$$K_f(u) = K[f'(u)].$$

The quantity $K[f]=\|K_f\|_\infty=\text{ess sup}\, K_f(u)$ is then a measure of how "close to conformal" is the mapping f. The inequalities (2.1) remain valid for f, although the latter only makes sense at vertices of differentiability. We note that $K_f(u)$ is independent of conformal change in the variable u, and therefore these quantities make sense on a Riemann surface, or more generally an n-manifold with conformal structure class.

If a class Q of quasiconformal mappings is specified, one may pose the problem: calculate

$$K^*(Q) = \inf\,\{K[f]\colon f\in Q\},$$

and try to find all the *extremal* f, i.e., those $f\in Q$ with $K[f]=K^*(Q)$. If such an f is unique, it is then called a *unique extremal.*

Two problems which motivated much of the subsequent theory were the problems of *Teichmüller* and of *Grötzsch.* The problem of Teichmüller originates with tori: given two tori S, T let Q be a class of quasiconformal mappings $f\colon S\to T$, and try to find the extremals. We shall comment more on the subtleties of this problem in another article, but for now we take two normalized n-tuples

$$a = (a_1, a_2, \ldots, a_n), \quad b = (b_1, b_2, \ldots, b_n) \quad (\Pi_{i=1}^{n} a_i = 1 = \Pi_{i=1}^{n} b_i)$$

of positive numbers and consider the class Q of mappings of R^n with the reproducing property

$$f(u+a_i e_i) = f(u)+b_i e_i \quad (i = 1, 2, \ldots, n; \text{ all } u).$$

Here, $\{e_1, e_2, \ldots, e_n\}$ are orthogonal unit basis vectors in R^n.

If we set $\lambda_i = b_i/a_i$, we see that the linear map f_λ which takes $\sum u_i e_i$ to $\sum u_i \lambda_i e_i$ is a member of the class Q. The *Teichmüller* problem to which we refer is then simply stated: is f_λ extremal or uniquely extremal in Q? (For uniqueness one must normalize, for example by requiring $f(0)=0$, which we do implicitly throughout.)

The related *Grötzsch* problem assumes, rather than the reproducing property, the simpler requirement that f map the rectangular box

$$C_a = \{u\colon\ 0 \leqq u_i \leqq a_i, \text{ all } i\}$$

onto the rectangular box $C_b = f_\lambda(C_a)$ with face-correspondence

$$\{u_i = 0\} \leftrightarrow \{w_i = 0\}, \quad \{u_i = a_i\} \leftrightarrow \{w_i = b_i\}.$$

As is well known, and as it is shown in [Ag$_2$], if f competes in a Grötzsch problem, it induces by reflection an extension with the same dilatation which competes in a related Teichmüller problem (e_i replaced by $2e_i$). We therefore make very few additional comments about the Grötzsch problem in this article, except to observe that extremality or unique extremality for the Grötzsch problem follows from the corresponding property for the Teichmüller problem.

A third problem is the *boundary value* problem. Given a domain D and a mapping $g\colon \bar{D} \to R^n$, one considers the collection Q of mappings $f\colon \bar{D} \to R^n$ which are homeomorphic, quasiconformal in D, and with

$$f|_{\partial D} = g|_{\partial D}.$$

An extremal is said to be "extremal for the boundary values $g|_{\partial D}$" and the question is often whether or not a given qc mapping g is extremal for its own boundary values.

This question in the plane has been studied in great detail and with great success in a long series of papers by E. Reich and K. Strebel. (See [R] for a good bibliography of their work.) One usually assumes that g is a so-called Teichmüller mapping to begin with, since this is often a necessary condition for extremality. Little is known about this problem for $n>2$, although Ahlfors [Ah] has proved one theorem in this direction for mappings of the unit ball and the dilatation K_E.

To highlight the importance of *unique* extremality, we prove for any dilatation K whose essential boundedness implies the essential boundedness of the outer dilatation K_0:

Theorem 2. *If the affine mapping f_λ above is uniquely K-extremal for the Grötzsch problem, then f_λ is uniquely K-extremal for its own boundary values on any bounded domain D.*

Proof. By the scale invariance of the problem, we may assume that the bounded domain D lies inside the rectangular box C_a. Given $f\in Q$, we define a new mapping $\hat{f}\colon C_a\to C_b$ by

$$\hat{f}(u)=\begin{cases} f(u) & (u\in D),\\ f_\lambda(u) & (u\in C_a\setminus D).\end{cases}$$

Now it may be perfectly obvious, thanks to the geometry of D, that $\hat{f}$ is quasiconformal. Obvious or not, the methods of proof in a theorem of J. Väisälä [V_2, Theorem 2] make it so that $\hat{f}$ is indeed quasiconformal in C_a, and furthermore

$$(2.2)\qquad K[\hat{f}]\leqq K[f_\lambda]\quad\text{if and only if}\quad K[f]\leqq K[f_\lambda].$$

Since $\hat{f}$ competes in the Grötzsch problem, it follows by the K-extremality of f_λ that $K[f_\lambda]\leqq K[\hat{f}]$. But should nonequality occur, then it follows by (2.2) that $K[f]>K[f_\lambda]$. On the other hand, should equality occur, then by the unique K-extremality, we have $\hat{f}=f_\lambda$, hence $f=f_\lambda$ in D, and $K[f]=K[f_\lambda]$. □

We observe that without the *unique* extremality, it is not even possible to deduce *extremality*, for a priori

$$K[f]=\operatorname*{ess\,sup}_{u\in D}K_f(u)<K[f_\lambda]$$

is compatible with $K[\hat{f}]=K[f_\lambda]$.

In summary, the unique extremality is a highly desirable property for a given dilatation. Coupled with subsequent results regarding K_E, Theorem 2 gives a space-formulation of one of the most basic Reich—Strebel foundations [St].

We note that this particular property for f_λ (unique K-extremality for its own boundary values) has been proved by O. Taari [T] for the outer dilatation, even only requiring that D have finite n-volume. The results by Taari and Kühnau taken together show that our condition (unique K-extremality for the Grötzsch problem) is not necessary for unique K-extremality for the boundary value problem when $K=K_0$. On the other hand, we certainly know of no other proof for $K=K_E$.

3. Axiomatic approach to the Teichmüller problem. We now become more specific. Having used u as a prototypical variable in R^n, we also work in

$$R^n_+=\{x=(x_1,x_2,\ldots,x_n)\colon x_i>0\}.$$

We denote by $\mathscr{H}_+$ the set $\{x\in R^n_+\colon \prod_{i=1}^n x_i\geqq 1\}$, with its boundary $\mathscr{H}(\Pi x_i=1)$. For any $p\in R$, $x\in R_n^+$, we denote the point $(x_1^p, x_2^p, \ldots, x_n^p)$ by x^p. In particular, $x^0=(1,1,\ldots,1)$ which we call **1**, and also $x^{-1}=(1/x_1, 1/x_2, \ldots, 1/x_n)$. We say

$x \leqq y$ if $x_i \leqq y_i$ for each i. Finally, we shall denote by $\hat{x}_i$ the point

$$(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n) \in R_+^{n-1},$$

in which the coordinate x_i is omitted.

We define a function δ: $\mathrm{GL}(n, R) \to \mathscr{H}_+$ as follows: $\delta[A]$ is the point $x \in \mathscr{H}_+$ whose coordinates are the diagonal entries of $X[A]$. Suppose now that $\lambda^2 = (\lambda_1^2, \ldots, \lambda_n^2)$ are the eigenvalues of $X[A]$. Then we say that λ^2 is a *representative* of A. We also consider the point set

$$\mathscr{S}(\lambda^2) = \{\delta[AV] \colon V \in O(n)\}.$$

If we temporarily assume $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$, then it is clear that for any $x \in \mathscr{S}(\lambda^2)$, we have the relation $\sum_{i=1}^n x_i = \sum_{i=1}^n \lambda_i^2$. Less obvious but also true are the inequalities

$$\sum_{j=1}^k x_{i_j} \leqq \sum_{i=1}^k \lambda_i^2,$$

for each set of indices $\{i_1, i_2, \ldots, i_k\}$. From these we infer:

Proposition 3.1. *The set $\mathscr{S}(\lambda^2)$ is a subset of the convex hull of the permutations of λ^2.*

Proof. We had devised a simple proof of this fact, but apparently these matters arise in other areas of mathematics. The interested reader may consult [M—O, Corollary B3, p. 23].

We now formulate a set of axioms relating a dilatation K and a real valued function Φ defined on $\mathscr{H}_+$. We assume

A1. Φ is related to K in the sense that there exists a strictly increasing function α: $[1, \infty) \to R_+$ such that

$$\Phi(\lambda^{n/n-1}) = \alpha(K[A])$$

whenever $\lambda^2 \in \mathscr{H}$ is a representative of $A \in \mathrm{GL}(n, R)$.

A2. Φ is convex on a convex subset $\mathscr{E} \subseteqq \mathscr{H}_+$.

A3. The pair $(\Phi, \mathscr{E})$ has the property that the conditions

$$\left.\begin{matrix} \lambda^2 \in \mathscr{E} \cap \mathscr{H} \\ x \in \mathscr{S}(\lambda^2) \end{matrix}\right\} \quad \text{imply} \quad \left\{\begin{matrix} \lambda^{n/n-1} \in \mathscr{E} \\ x^{n/2(n-1)} \in \mathscr{E}, \end{matrix}\right. \quad \text{and}$$

$$\Phi(x^{n/2(n-1)}) \leqq \Phi(\lambda^{n/n-1}).$$

A4. $\Phi(x) \leqq \Phi(y)$ whenever $x \in \mathscr{E} \cap \mathscr{H}$ and $x \leqq y$.

Lemma 3.2. *Assume that a dilatation K has an associated function Φ satisfying axioms* A1—A4. *Assume that f_λ as above is the linear map with matrix* $\mathrm{diag}(\lambda)$. *Assume that $\lambda^2 \in \mathscr{E} \cap \mathscr{H}$, that f competes with f_λ in the Teichmüller problem, and that $q^2 \in \mathscr{E}$ whenever q^2 is a representative of $f'(u)$ (a.e. $u \in C_a$). Then $K[f] \geqq K[f_\lambda]$.*

Proof. Due to the reproductive nature of f, one can see that

$$\operatorname{vol} f(C_a) = \operatorname{vol} f_\lambda(C_a) = \operatorname{vol} C_b = 1.$$

Hence, setting $J_f = \det f'$, we have

$$1 = \int_{C_a} J_f.$$

We also have

$$b_i e_i = \int_0^{a_i} \partial_i f(u_i, \hat{u}_i)\, du_i \quad (\text{all } \hat{u}_i, \text{ all } i).$$

Integrating over $\{\hat{u}_i: u \in C_a\}$ (a set of $(n-1)$-volume $1/a_i$), we find

$$\lambda_i e_i = \frac{b_i}{a_i} e_i = \int_{C_a} \partial_i f,$$

hence by the modulus inequality and Hölder's inequality,

$$\lambda_i^n \leqq \Big(\int_{C_a} \|\partial_i f\|\Big)^n \leqq \left(\int_{C_a} \frac{\|\partial_i f\|^{n/n-1}}{J_f^{1/n-1}}\right)^{n-1}$$

or, as points in $\mathscr{H}_+$,

(3.0) $$\lambda^{n/n-1} \leqq \int_{C_a} \delta^{n/2(n-1)}(f').$$

By A3 we see $\lambda^{n/n-1} \in \mathscr{E} \cap \mathscr{H}$, and by A4 and (3.0):

(3.1) $$\Phi(\lambda^{n/n-1}) \leqq \Phi\Big(\int_{C_a} \delta^{n/2(n-1)}(f')\Big).$$

Now let $q^2(u)$ be a representative of $f'(u)$. We have assumed $q^2 \in \mathscr{E}$. By A3 we know

$$\delta^{n/2(n-1)}(f'(u)) \in \mathscr{E}, \quad q^{n/n-1}(u) \in \mathscr{E},$$

and

(3.2) $$\Phi(\delta^{n/2(n-1)}(f'(u))) \leqq \Phi(q^{n/n-1}(u)) \quad (\text{a.e. } u \in C_a).$$

From (3.1), Jensen's inequality, and (3.2), we infer

$$\Phi(\lambda^{n/n-1}) \leqq \Phi\Big(\int_{C_a} \delta^{n/2(n-1)}(f')\Big) \leqq \int_{C_a} \Phi(\delta^{n/2(n-1)}(f')) \leqq \int_{C_a} \Phi(q^{n/n-1}),$$

hence in summary

$$\Phi(\lambda^{n/n-1}) \leqq \operatorname*{ess\,sup}_{u \in C_a} \Phi(q^{n/n-1}(u)).$$

Finally by A1, it follows that

$$\alpha(K[f_\lambda]) \leqq \operatorname{ess\,sup} \alpha(K_f(u)),$$

and since α is strictly increasing, we finally conclude $K[f_\lambda] \leqq K[f]$. □

Example. For the outer dilatation K_0, one may take

$$\Phi_0(x) = \max\{x_i\}, \quad \alpha(u) = u^{1/n-1}, \quad \mathscr{E} = \mathscr{H}_+.$$

The details are straightforward, and as we commented in the introduction, the extremality of the affine mapping for the outer dilatation has been studied.

However, our main thrust is toward K_E. Here, for each $c<2/3$ and t sufficiently large, we develop a system Φ_t, $\mathscr{E}_c$, α satisfying axioms A1—A4. We first present the formulae and a summary of the argument, leaving the technical details (Lemmas 4.1 and 5.1) to the next sections. Thus for $x\in R_+^{n-1}$ $(n\geqq 3)$, set

$$\psi(x_1, x_2, \dots, x_{n-1}) = \sum_{i=1}^{n-1} \log^2 x_i + \log^2 \big(\prod_{i=1}^{n-1} x_i\big).$$

Back in R_+^n, set

$$\Psi(x) = \max\{\psi(\hat{x}_1), \psi(\hat{x}_2), \dots, \psi(\hat{x}_n)\}.$$

Evidently for $x\in\mathscr{H}$, all competitors are the same, and thus

$$\Psi(x) = \sum_{i=1}^{n} \log^2 x_i \quad (x\in\mathscr{H}). \tag{3.3}$$

Lemma 5.1. *ψ has convex sublevel sets*

$$\mathscr{D}_c = \{x\in R_+^{n-1}\colon \psi(x) \leqq c\}$$

for $c\leqq 2/3$. For each $c<2/3$, there is a number $p(c)$ such that for $t>p(c)$, the function $\varphi_t(x)=\exp t\sqrt{\psi(x)}$ is convex in $\mathscr{D}_c$.

Taking this result for granted, we now define for $c<2/3$, $t>p(c)$:

$$\mathscr{E}_c = \bigcap_{i=1}^{n} \{x\colon \hat{x}_i\in\mathscr{D}_c\};$$

$$\Phi_t(x) = \max\{\varphi_t(\hat{x}_1), \varphi_t(\hat{x}_2), \dots, \varphi_t(\hat{x}_n)\} = \exp t\sqrt{\Psi(x)};$$

$$\alpha(u) = u^{t\sqrt{c_n}}.$$

Proposition 3.3. *The trio $\{\Phi_t, \mathscr{E}_c, \alpha\}$ satisfy axioms* A1—A4.

Proof. Axiom A1 is clear from the construction and (3.3). Next, Lemma 5.1 together with the relation

$$\Phi_t(x) = \max\{\varphi_t(\hat{x}_i)\},$$

where each constituent is convex in $\{x\colon \hat{x}_i\in\mathscr{D}_c\}$, shows that Φ_t is convex in $\mathscr{E}_c$.

For Axiom A3, we make use of the self-evident identities

$$\Psi(x^p) = p^2\Psi(x), \quad \Phi_t(x^p) = [\Phi_t(x)]^p. \tag{3.4}$$

If indeed $\lambda^2\in\mathscr{E}_c$, then so is every permutation of λ^2, and hence the entire convex hull $\mathscr{K}$ of these permutations lies in $\mathscr{E}_c$. Being convex, Φ_t takes its max on $\mathscr{K}$ at the extreme points, where it is obviously constant. Thus for $x\in\mathscr{S}(\lambda^2)$, we have $x\in\mathscr{K}$ (Proposition 3.1) and

$$\Phi_t(x) \leqq \Phi_t(\lambda^2). \tag{3.5}$$

Using (3.4), we gain the important conclusion

$$\Phi_t(x^{n/2(n-1)}) = [\Phi_t(x)]^{n/2(n-1)} \leqq [\Phi_t(\lambda^2)]^{n/2(n-1)} = \Phi_t(\lambda^{n/n-1}),$$

and moreover

$$\Psi(\lambda^{n/n-1}) = \frac{n^2}{4(n-1)^2}\,\Psi(\lambda^2) \leqq \frac{n^2 c}{4(n-1)^2} \leqq c.$$

The last inequality is simply because $n/2(n-1) \leqq 1$, but suffices to give the inclusions of Axiom A3. □

We remark that the inclusions also follow from the weaker assumption $\Psi(\lambda^2) \leqq 4(n-1)^2 c/n^2$, but we have not been able to give an independent proof of (3.5) under this weaker assumption, which is not adequate to place $\lambda^2 \in \mathscr{E}_c$ if $n \geqq 3$. More seriously, for c sufficiently close to 2/3, the number $4(n-1)^2 c/n^2$ may exceed 2/3, at which point all hope is lost. We would consider this a minor improvement in any case. See Section 6 for additional comments on the limitations of our method.

Finally, Axiom A4 is a consequence of Lemma 4.1, which it is not necessary to state here. □

Theorem 3. *If $K_E[f_\lambda] < \exp\sqrt{n/6(n-1)}$, then f_λ is K_E-extremal for the Teichmüller and Grötzsch problems.*

Proof. To apply the arguments above, it is only necessary to show that our condition will imply that $\lambda^2 \in \mathscr{E}_c$ for some $c < 2/3$. Hence we require

$$K_E[f_\lambda] = \exp\sqrt{c_n \Psi(\lambda)} = \exp\sqrt{\frac{1}{4}\,c_n \Psi(\lambda^2)}$$

$$< \exp\sqrt{\frac{1}{4}\,c_n \frac{2}{3}} = \exp\sqrt{n/6(n-1)}.$$

We arrive at the point where f_λ is extremal among competitors f with $K[f] < \exp\sqrt{n/6(n-1)}$. But the others are no competition because $K[f_\lambda] < \exp\sqrt{n/6(n-1)}$ to begin with. □

4. The uniqueness. The uniqueness proof now proceeds in three stages. First, two technical lemmas regarding Ψ and K_E. We then formulate two more natural axioms A4′, A5 (satisfied by Φ_t) which, in conjunction with the first three axioms, will make any extremal for the Teichmüller problem into a "diagonal" map. Finally, unique extremality in the class of diagonal mappings is proved. These steps, taken individually, do not require that K_E be small, and we anticipate further applications.

4A. Preliminaries. We begin this section with the crucial lemma which assures that both A4 and A4′ are satisfied by $(\Phi_t, \mathscr{E}_c, \alpha)$. It really only concerns our function Ψ, and we remark that the hypothesis $\hat{x}_n \leqq \hat{y}_n$ is weaker than $x \leqq y$.

Lemma 4.1. *Let $x \in \mathscr{H}$, normalized by $x_1 \geqq x_2 \geqq \ldots \geqq x_n$, and suppose $y \in \mathscr{H}_+$ with $\hat{y}_n \geqq \hat{x}_n$. Then $\Psi(y) \geqq \Psi(x)$, with equality only if $\hat{x}_n = \hat{y}_n$.*

Proof. We write, for $u=(u_1, u_2, \dots, u_{n-1})\in R^{n-1}$:

$$g(u) = \sum_{j=1}^{n-1} u_j^2 + \left(\sum_{j=1}^{n-1} u_j\right)^2$$

and observe that in $E=\{u\colon u_j \geqq \log x_j\}$, the inequality

$$\frac{\partial g}{\partial u_i}(u) = 2u_i + 2\sum_{j=1}^{n-1} u_j \geqq 2\log x_i + 2\sum_{j=1}^{n-1}\log x_j$$

$$\geqq 2\log x_n + 2\sum_{j=1}^{n-1}\log x_j = 0$$

holds for every $i\ (1\leqq i\leqq n-1)$, and if for even *one* such index i we have $u_i > \log x_i$, we get strict inequality. Hence we conclude that the function $u_i \mapsto g(u)$ is strictly monotonic increasing for $u_i\in[\log x_i, \infty)$ provided that $u\in E$. We find in consequence:

$$\Psi(y) \geqq \psi(\hat{y}_n) = g(\log y_1, \log y_2, \dots, \log y_{n-1})$$

$$\geqq g(\log x_1, \log x_2, \dots, \log x_{n-1}) = \Psi(x),$$

where equality implies (in the second inequality symbol) $\hat{x}_n = \hat{y}_n$. □

We add the remark that under the given assumptions, it can happen that $x_n < y_n$ and $\Psi(x) = \Psi(y)$ occur simultaneously.

We next require a formula for K_E in the diagonal but not normalized case. Accordingly, we assume $\xi\in R_+^n$.

Lemma 4.2. $\log^2 K_E[\operatorname{diag}(\xi)] = \dfrac{1}{n-1}\sum_{i<j}\log^2 \xi_i/\xi_j$.

Proof. Since both sides are independent of scale, it suffices at once to assume $\xi\in\mathscr{H}$. Then set $u_i = \log \xi_i$, hence

$$\sum_{i=1}^n u_i = 0, \quad \sum_{i=1}^n u_i^2 = -2\sum_{i<j} u_i u_j,$$

and

$$\sum_{i<j}(u_i-u_j)^2 = (n-1)\sum_{i=1}^n u_i^2 - 2\sum_{i<j} u_i u_j = n\sum_{i=1}^n u_i^2.$$

Division by $n-1$ makes the last expression into $\log^2 K_E[\operatorname{diag}(\xi)]$. □

4B. Axiomatics for partial uniqueness. We propose the following axiom A4′ (which follows for Φ_t by Lemma 4.1):

A4′ The conditions $x\in\mathscr{H}$, $x\leqq y$ imply $\Phi(x)\leqq\Phi(y)$, with equality only if $x_i = y_i$ with the possible exception of the smallest x_i.

In addition, we assume

A5 If f is K-extremal for the Teichmüller problem, then f^{-1} is K-extremal in the reversed Teichmüller problem (b's and a's interchanged).

We note that A5 is true for K_E or any dilatation K such that

$$K[A] = K[A^{-1}].$$

Indeed, f competes in the Teichmüller problem if and only if f^{-1} competes in the reversed Teichmüller problem. Thus the families for the two problems have collectively the same spectrum of dilatations.

Lemma 4.3. *Assume that* $(K, \Phi, \mathscr{E})$ *satisfy axioms* A1, 2, 3, 4′, 5, *that* f *and* f_λ *satisfy the same conditions as in Lemma* 3.2, *and that* $K[f]=K[f_\lambda]$. *Then* $f'(u)$ *has a diagonal structure for a.e. u.*

Proof. We study the case of equality in the proof of Lemma 3.2. We assume $\lambda_1 \geqq \lambda_2 \geqq \ldots \geqq \lambda_n$, and we must have equality in the expression (3.1). Therefore by A4′ and (3.0), we conclude that

$$\lambda_i^{n/n-1} = \int_{C_a} \frac{\|\partial_i f\|^{n/n-1}}{J_f^{1/n-1}} \quad (i = 1, 2, \ldots, n-1).$$

But this implies (reviewing the circumstances):

$$\|\lambda_i e_i\| = \left\|\int_{C_a} \partial_i f\right\| = \int_{C_a} \|\partial_i f\|, \quad (i = 1, 2, \ldots, n-1),$$

and hence the vector $\partial_i f$ fulfills

$$\partial_i f(u) = \xi_i(u) e_i \quad (\text{a.e. } u)$$

where $\xi_i(u)$ is a real valued function with $\xi_i(u) \geqq 0$ a.e.

This already makes $f'(u)$ a.e. diagonal with the possible exception of the last column, which column is of course associated to λ_n, the smallest of the λ's. Now the inverse of such a matrix has exactly the same form. Yet in the reversed Teichmüller problem, λ is replaced by λ^{-1}, and the "smallest" will now be $1/\lambda_1$ associated to the first column. In other words, $[f'(u)]^{-1} = (f^{-1})'(w)$ is diagonal except for the first column (a.e. w). These mappings preserve sets of measure zero, and so in sum, f' is a.e. diagonal as soon as $n \geqq 2$. □

4C. The uniqueness for K_E. As a result of the previous sections, each K_E-extremal f for the Teichmüller problem when

$$K_E[f_\lambda] < \exp \sqrt{n/6(n-1)} \tag{4C.1}$$

has diagonal structure. This alone will be sufficient to complete the uniqueness proof, i.e., we no longer make use of (4C.1).

The ACL property for f makes each coordinate function f_i for $f(u)$ an absolutely continuous function of u_i *only*, and still:

$$\int_0^{a_i} f_i'(u_i)\, du_i = b_i \quad (i = 1, 2, \ldots, n). \tag{4C.2}$$

Let us set $\xi_i = f_i'$. We now apply Lemma 4.2, with extremality, to conclude

$$\sum_{i<j} \log^2 \xi_i/\xi_j = (n-1) \log^2 K_f(u) \leqq (n-1) \log^2 K_E[f_\lambda] = \sum_{i<j} \log^2 \lambda_i/\lambda_j \tag{4C.3}$$

for a.e. u. The left side is

$$\sum_{i<j}\left(\log\frac{\xi_i\lambda_j}{\lambda_i\xi_j}+\log\frac{\lambda_i}{\lambda_j}\right)^2$$

$$=\sum_{i<j}\left[\log^2\frac{\xi_i\lambda_j}{\lambda_i\xi_j}+2\log\frac{\xi_i\lambda_j}{\lambda_i\xi_j}\log\frac{\lambda_i}{\lambda_j}+\log^2\frac{\lambda_i}{\lambda_j}\right]$$

$$=\sum_{i<j}\log^2\frac{\xi_i\lambda_j}{\lambda_i\xi_j}+2\sum_{i<j}\log\frac{\xi_i}{\lambda_i}\log\frac{\lambda_i}{\lambda_j}+2\sum_{i<j}\log\frac{\xi_j}{\lambda_j}\log\frac{\lambda_j}{\lambda_i}+\sum_{i<j}\log^2\frac{\lambda_i}{\lambda_j}.$$

Hence by (4C.3), we have a.e.:

$$\sum_{i<j}\log^2\frac{\xi_i\lambda_j}{\lambda_i\xi_j}+2\sum_{i<j}\log\frac{\xi_i}{\lambda_i}\log\frac{\lambda_i}{\lambda_j}+2\sum_{i>j}\log\frac{\xi_i}{\lambda_i}\log\frac{\lambda_i}{\lambda_j}\leqq 0.$$

The left side can be written

$$\sum_{i<j}\log^2\frac{\xi_i\lambda_j}{\lambda_i\xi_j}+2\sum_{\substack{i=1\\ j=1}}^{n}\log\frac{\xi_i}{\lambda_i}\log\frac{\lambda_i}{\lambda_j},$$

where the second term can be written as

$$2\sum_{i=1}^{n}\left(\log\frac{\xi_i}{\lambda_i}\sum_{j=1}^{n}\log\frac{\lambda_i}{\lambda_j}\right)=2\sum_{i=1}^{n}\log\frac{\xi_i}{\lambda_i}\log\lambda_i^n,$$

since $\prod_{j=1}^{n}\lambda_j=1$. We therefore have a.e.:

$$\sum_{i<j}\log^2\frac{\xi_i\lambda_j}{\lambda_i\xi_j}+2\sum_{i=1}^{n}\log\frac{\xi_i}{\lambda_i}\log\lambda_i^n\leqq 0. \tag{4C.4}$$

Now fix an integer k, $1\leqq k<n$, such that

$$\lambda_k\geqq 1\geqq\lambda_{k+1}$$

and we define for $1\leqq i\leqq n$:

$$U_i=\begin{cases}\{u_i\in[0,a_i]:\ \xi_i(u_i)\geqq\lambda_i\} & (1\leqq i\leqq k),\\ \{u_i\in[0,a_i]:\ \xi_i(u_i)\leqq\lambda_i\} & (k<i\leqq n).\end{cases}$$

We point out explicitly that $\xi_1(u_1)\geqq\lambda_1$ in U_1 and $\xi_n(u_n)\leqq\lambda_n$ in U_n. Because of (4C.2), each set U_i has positive one-dimensional measure, and letting $U=\times_{i=1}^{n}U_i=\{u:\ u_i\in U_i \text{ for all } i\}$, we have

$$\sum_{i=1}^{n}\log\frac{\xi_i}{\lambda_i}\log\lambda_i^n\geqq 0\quad(\text{a.e. } u\in U). \tag{4C.5}$$

By (4C.4) we therefore conclude

$$\sum_{i<j}\log^2\frac{\xi_i\lambda_j}{\lambda_i\xi_j}=0\quad(\text{a.e. } u\in U),$$

or, for every pair i, j with $i<j$, we have

$$\frac{\xi_i(u_i)}{\lambda_i} = \frac{\xi_j(u_j)}{\lambda_j} \quad (\text{a.e. } (u_i, u_j)\in U_i\times U_j).$$

Since the variables u_i and u_j appear on different sides of this equation, we conclude that there is a constant c with

$$\frac{\xi_i(u_i)}{\lambda_i} = c \quad (\text{a.e. } u_i\in U_i, \text{ all } i).$$

For $i=1$ we get $c\geqq 1$, and for $i=n$ we get $c\leqq 1$, so we infer

$$\xi_i(u_i) = \lambda_i \quad (\text{a.e. } u_i\in U_i, \text{ all } i).$$

But then by the definitions of U_i, ξ_i and by (4C.2), we find (up to a set of one-dimensional measure zero):

$$U_i = [0, a_i] \quad (\text{all } i),$$

and hence $\xi_i(u_i)=\lambda_i$ a.e. in $[0, a_i]$, so by the absolute continuity of f_i we get $f_i(u_i)=\lambda_i u_i$ in $[0, a_i]$, which finishes the proof. We summarize:

Theorem 4. *If the affine mapping f_λ satisfies*

$$K_E[f_\lambda] < \exp\sqrt{n/6(n-1)},$$

then it is uniquely K_E-extremal for the Teichmüller problem, the Grötzsch problem, and for its own boundary values on any bounded domain.

5. The convexity. Although the application of this section is to dimension $n-1$, it is cumbersome and pointless to retain this notation. Accordingly we treat R^n, this time assuming $n\geqq 2$. Since linear algebra is the main tool, we agree that points are to be thought of as column vectors. We use $\cdot$ for the traditional "dot" product $x\cdot y=\sum x_i y_i=x^T y$. The differential of a real valued function is represented as a row matrix, and the second differential as a square matrix M which operates on pairs of vectors u, v by $D^2\psi(x)\ u, v=v^T Mu$. The unit sphere is S.

Recall that we are working with the function ψ, whose formula is

$$\psi(x) = \sum_{i=1}^n \log^2 x_i+\log^2 y \quad (x\in R^n_+,\ y = \prod_{i=1}^n x_i\in R_+).$$

Lemma 5.1. *For $0\leqq c\leqq 2/3$, ψ has convex sublevel sets*

$$\mathcal{D}_c = \{x\colon \psi(x) \leqq c\}.$$

For $0<c<2/3$, there is a number $p=p(c)$ such that for $t>p(c)$, the function

$$\varphi_t = \exp\left(t\sqrt{\psi}\right)$$

is convex in $\mathcal{D}_c$.

Proof. The convexity of the sublevel sets is a by-product of the convexity, and is really only used to establish the convexity of the domain $\mathscr{E}_c$ in Section 3. Therefore, as observed in the earlier paper [Ag$_2$], (pp. 98 and 100) it suffices to show

$$\frac{1-t\sqrt{c}}{2c} \leqq \inf\left\{\frac{D^2\psi(x)u, u}{(D\psi(x)u)^2}: D\psi(x)u \neq 0\right\} \tag{5.1}$$

whenever x lies on the level surface $\psi(x)=c$.

Thus our first task is to analyze and evaluate the right hand side of (5.1), which we denote by $\theta(x)$. We easily find matrix representations

$$D\psi(x) = 2\left[\frac{\log x_1 y}{x_1} \frac{\log x_2 y}{x_2} \dots \frac{\log x_n y}{x_n}\right]$$

$$D^2\psi(x) = 2\begin{bmatrix} \frac{2-\log x_1 y}{x_1^2} & \frac{1}{x_1x_2} & \cdots \\ \frac{1}{x_1x_2} & \frac{2-\log x_2 y}{x_2^2} & \\ \vdots & & \ddots \end{bmatrix}.$$

We introduce the variables $y_i=x_i y$, and consider the apparently simpler expressions $a=a(x)$ and $B=B(x)$ with

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix} = a = \begin{bmatrix} \log y_1 \\ \log y_2 \\ \vdots \\ \log y_n \end{bmatrix}, \quad B = \begin{bmatrix} 2-a_1 & 1 & \dots & 1 \\ 1 & 2-a_2 & & \\ \vdots & & \ddots & \vdots \\ 1 & 1 & \dots & 2-a_n \end{bmatrix}.$$

Now fix $x\in R^n_+$, and define for any vector $u\in R^n$, the vector $\tilde{u}\in R^n$, in which $\tilde{u}_i=u_i/x_i$. It is very clear that

$$2a(x)\cdot\tilde{u} = D\psi(x)u, \quad 2\tilde{u}^T B(x)\tilde{u} = D^2\psi(x)u, u,$$

and so we may in fact consider the simpler problem R: calculate

$$\eta(a, B) = \inf\left\{\frac{\tilde{u}^T B\tilde{u}}{(a\cdot\tilde{u})^2}: a\cdot\tilde{u} \neq 0\right\} = \inf\left\{\frac{u^T Bu}{(a\cdot u)^2}: u\in S,\ a\cdot u \neq 0\right\}, \tag{R}$$

which has the value $2\theta(x)$ when $a=a(x)$ and $B=B(x)$.

Assuming that an absolute minimum is known to exist, the location and value are easily found by Lagrange multipliers. The extremal configuration is among the critical points (u, μ) for

$$w(u)-\mu(\|u\|^2-1), \left(w = \frac{u^T Bu}{(a\cdot u)^2}\right).$$

We find easily

$$\frac{\partial w}{\partial u_i} = 2\frac{(a\cdot u)(e_i^T Bu)-(u^T Bu)a_i}{(a\cdot u)^3}.$$

Hence, in addition to $\|u\|^2=1$ we have the conditions

(5.2) $$(a \cdot u)(e_i^T Bu)-(u^T Bu)a_i = \mu u_i(a \cdot u)^3.$$

We multiply by u_i and sum in i:

$$0 = (a \cdot u)(u^T Bu)-(u^T Bu)(a \cdot u) = \mu(a \cdot u)^3.$$

But $a \cdot u=0$ is disallowed, and therefore $\mu=0$. Now $e_i^T B$ is the ith row of the matrix B. With $\mu=0$, the equalities (5.2) read

(5.3) $$Bu = (a \cdot u)wa,$$

hence operating on (5.3) with the so-called "adjugate", or transposed cofactor matrix B^*, we find

$$(\det B)u = (a \cdot u)wB^* a.$$

It follows for the extremal configuration, that either $\det B=0$, or u is uniquely determined (up to sign) and a multiple of B^*a. The latter case $(\det B\neq 0)$ gives (recall B is symmetric)

$$w = \frac{a^T B^{*T} BB^* a}{(a^T B^* a)^2} = \frac{\det B}{a^T B^* a}.$$

On the other hand, if $\det B$ *is* zero, then either $w=0$, or $B^*a=0$. In the former case the (nonunique) extremal vectors comprise $\operatorname{Ker}(B)$. The latter will eventually prove unacceptable.

Now we digress and discuss the general question of the existence of a finite minimum. To this end we consider another problem in linear algebra. Here, $a \in R^n$ and B a symmetric n by n matrix are fixed. We call this problem P: calculate

(P) $$\xi(a, B) = \min \{u^T Bu\colon\ u \in S,\ au = 0\}.$$

It is clear that problem P always has a solution, and that the value ξ depends continuously on the data (a, B) as long as $a\neq 0$. But we also claim that a sufficient condition for problem R to have a *finite* solution when $a\neq 0$, is that ξ be *positive*. For then $u^T Bu$ will be *positive* everywhere on some compact set $\{u \in S\colon\ a \cdot u \leqq \varepsilon\}$, and if $u^T Bu$ is indeed negative outside that set, we still have the uniform lower bound

$$\frac{u^T Bu}{(a \cdot u)^2} \geqq \frac{1}{\varepsilon^2} \min \{u^T Bu\colon\ u \in S\} > -\infty.$$

Finally, because ξ is continuous, the collection

$\mathscr{G} = \{(a, B)\colon\ \xi(a, B) > 0,\ a \neq 0\}$ is open, with $\partial\mathscr{G} \subseteqq \{(a, B)\colon\ a = 0 \text{ or } \xi = 0\}$,

and the problem R has a finite value on $\mathscr{G}$. Some routine calculations show that $\operatorname{sgn}(a^T B^* a)$ is a partial indicator for $\mathscr{G}$ to the extent that

(5.4′) $$a^T B^* a > 0 \text{ on } \mathscr{G} \quad\text{and}\quad a^T B^* a = 0 \text{ on } \partial\mathscr{G}.$$

In our application a, B are as we have seen, continuous maps from R^n_+. We set $\mathscr{E}=\{x\in R^n_+ : \psi(x)<2/3\}$, and we shall eventually prove (5B):

$$a^T B^* a > 0 \quad \text{for} \quad x\in\mathscr{E}\setminus\{1\}. \tag{5.4}$$

(This is why we rejected $B^*a=0$ earlier.) However, our real objective is to prove that $(a, B)\in\mathscr{G}$ whenever $x\in\mathscr{E}\setminus\{1\}$, and we accomplish this by proving the inclusion $\mathscr{E}\subseteqq\mathscr{J}=\{x: \xi(a, B)>0\}$.

Indeed, for any fixed $x\in\mathscr{E}\setminus\{1\}$ and thanks to the identity (3.4) the path $p\mapsto x^p$ $(0\leqq p<\infty)$ lies entirely in $\mathscr{E}$ for $p\leqq 1$ and departs $\mathscr{E}$ at some unique $p=p_0(x)>1$. If this path departs $\mathscr{J}$ for the first time at $p=p_1(x)$, then $p_1>0$ and $(a, B)(x^{p_1})$ belongs to $\partial\mathscr{G}$ with $a^T B^* a(x^{p_1})=0$ by (5.4'). But in this case $x^{p_1}\notin\mathscr{E}$ by (5.4), and therefore $p_1\geqq p_0$.

We have thus proved that $\mathscr{E}\subseteqq\mathscr{J}$, and we summarize the tentative conclusions, it being understood that (5.4) is not yet proved.

Proposition 5.2. *Problem* R *has a finite value* $\theta(x)$ *whenever* $x\in\mathscr{E}\setminus\{1\}$, *and indeed*

$$\theta(x)=\frac{1}{2}\eta\big(a(x), B(x)\big)=\frac{\det B}{2a^T B^* a}.$$

Next, we pose one last problem S: calculate

$$\beta(c)=\min\{\theta(x):\ \psi(x)=c\}\quad (0<c<2/3). \tag{S}$$

It is clear by our formulae and by basic compactness that β is a continuous function of c on $(0, 2/3)$. With all the calculations and labels, the objective inequality (5.1) now amounts to:

$$\frac{1-t\sqrt{c}}{2c}\leqq\beta(c). \tag{5.5}$$

Evidently this inequality is established on any closed subinterval $[c_1, c_2]\subseteqq(0, 2/3)$ by the use of sufficiently large t, hence the condition $t>p(c)$ which appears in the statement. Our task is now twofold:

(A) Low end analysis: Establish (5.1) for *some* t on *some* interval $[0, c_1]$.

(B) High end analysis: Prove inequality (5.4).

5A. Low end analysis. We find an effective choice is

$$c_1=\inf\{\psi(x):\ \det B(x)=0\}.$$

Evidently c_1 is positive, and for $\psi(x)<c_1$, the eigenvalues of B are positive, B is positive definite, and in particular invertible. Now the linear transformation $M: R^n\to R^n$ which carries $u=\sum u_i e_i$ to $a=\sum a_i e_i$ $(u_i=\log x_i,\ a_i=\log y_i)$ has

matrix

$$M = \begin{bmatrix} 2 & 1 & 1 \dots 1 \\ 1 & 2 & & 1 \\ \vdots & & \ddots & \\ 1 & 1 \dots & & 2 \end{bmatrix}$$

with eigenvalues 1 (multiplicity $n-1$) and $n+1$ (eigenvector **1**). The situation is quite simple: we have

$$\psi(x) = \sum u_i^2 + \left(\sum u_i\right)^2 = u^T M u = u \cdot a.$$

Thus if $\psi(x)=c$, we have

$$c = u \cdot a \leqq \|u\| \, \|a\| \leqq \|a\|^2.$$

We easily see

$$M - B = \operatorname{diag}(a),$$

and therefore as regards the sup-norm, $\|M-B\| = \max |a_i| = O(\|a\|)$, and by the usual methods we also find

$$\|M^{-1} - B^{-1}\| = O(\|a\|).$$

Now, if $\psi(x)=c$, then

$$\begin{aligned} \frac{1}{2\theta(x)} = \frac{a^T B^* a}{\det B} &= a^T B^{-1} a = a \cdot B^{-1} a \\ &= a \cdot M^{-1} a + a \cdot (B^{-1} - M^{-1}) a \\ &= a \cdot u + O(\|a\|^3) \\ &= c + O(c^{3/2}) \\ &= c(1 + O(\sqrt{c})), \end{aligned}$$

and

$$\theta(x) = \frac{1}{2c(1+O(\sqrt{c}))} = \frac{1}{2c}(1+O(\sqrt{c})) \geqq \frac{1 - t\sqrt{c}}{2c}$$

for some $t > 0$. This completes the low end analysis.

5B. High end analysis. Let us denote the quantity $a^T B^* a$ by the symbol $\varphi = \varphi(x)$. Our objective is to prove that $\varphi(x) > 0$ when $x \in \mathscr{E} \setminus \{\mathbf{1}\}$, and it clearly suffices to prove that $\psi(x) \geqq 2/3$ when $\varphi(x) = 0$. (Recall that $B^* \to \det(M) M^{-1}$ which is positive definite as $x \to \mathbf{1}$.) Accordingly, we develop various new independent variables with which to describe the "constraint" $\varphi(x)=0$, and use the method of Lagrange to minimize ψ (as expressed in formulae involving the new independent variables).

The first change is a simple translation: we set $w = \mathbf{1} - a$. By this scheme $B = M - \operatorname{diag}(a)$ goes over into $\mathbf{J} + \operatorname{diag}(w)$, where $\mathbf{J}$ is the n by n matrix with all ones. Adopting the symbols $s_1, \dots, s_n$ for the usual symmetric functions of the

w's*), and $s_k^{(i)}$ (or $s_k^{(i,j)}$) for the corresponding expression if w_i is zero (or if w_i and w_j are both zero), one finds without difficulty that

$$[\mathbf{J}+\operatorname{diag}(w)]^* = S(w) = \begin{bmatrix} s_{n-1}^{(1)}+s_{n-2}^{(1)} & -s_{n-2}^{(1,2)} & \cdots \\ -s_{n-2}^{(1,2)} & s_{n-1}^{(2)}+s_{n-2}^{(2)} & \\ \vdots & & \ddots \end{bmatrix}.$$

(The easiest way to see this is to prove by induction that $\det(\mathbf{J}+\operatorname{diag}(w))=s_n+s_{n-1}$, and to verify by direct calculation that the matrix product $[\mathbf{J}+\operatorname{diag}(w)][S(w)]$ is $(s_n+s_{n-1})I_n$. By symmetry, only positions $(1, 1)$ and $(1, 2)$ need be checked.)

Meanwhile, what of ψ? One recalls that $\psi=a\cdot u=a\cdot M^{-1}a$, and easily verifies that

$$M^{-1} = \frac{1}{n+1}\begin{bmatrix} n & -1 & -1 & \cdots \\ -1 & n & -1 & \\ -1 & -1 & n & \\ \vdots & & & \ddots \end{bmatrix}.$$

Thus we have the formulae $(w=\mathbf{1}-a, a=Mu)$

$$\varphi = (\mathbf{1}-w)^T S(w)(\mathbf{1}-w) = s_1(s_n+s_{n-1})-n(n+2)s_n+s_{n-1},$$

$$\psi = (\mathbf{1}-w)^T M^{-1}(\mathbf{1}-w) = \frac{1}{n+1}[ns_1^2-2(n+1)s_2-2s_1+n].$$

The configuration which minimizes ψ subject to $\varphi=0$ may be found among the critical points (w, μ) for the function $\psi-\mu\varphi$. In view of the obvious formulae $\partial s_k/\partial w_i=s_{k-1}^{(i)}$, we find we must have

$$0 = \frac{\partial}{\partial w_i}(\psi-\mu\varphi) = \frac{1}{n+1}[2ns_1-2(n+1)s_1^{(i)}-2]$$
$$-\mu[s_1(s_{n-1}^{(i)}+s_{n-2}^{(i)})+(s_n+s_{n-1})-n(n+2)s_{n-1}^{(i)}+s_{n-2}^{(i)}],$$

or in view of $s_1^{(i)}=s_1-w_i$, we find

(5B.0)
$$\frac{2}{n+1}[(n+1)w_i-(s_1+1)] = \mu[s_1(s_{n-1}^{(i)}+s_{n-2}^{(i)})+(s_n+s_{n-1})-n(n+2)s_{n-1}^{(i)}+s_{n-2}^{(i)}]$$

for $i=1, 2, \ldots, n$, as well, of course, as the constraint

(5B.1) $$0 = s_1(s_n+s_{n-1})-n(n+2)s_n+s_{n-1}.$$

We consider first the possibility that two w's are zero, without loss of generality w_n and w_{n-1}. The constraint is no help or problem $(s_n=s_{n-1}=0=s_{n-1}^{(i)})$ and the equations become

$$\frac{2}{n+1}[(n+1)w_i-(s_1+1)] = \mu(s_1+1)s_{n-2}^{(i)}.$$

* $s_1 = \sum_i w_i$, $s_2 = \sum_{i<j} w_i w_j$, ..., $s_n = \Pi_i w_i$, $s_0 = 1 = s_0^{(i)}$.

But if $i \leqq n-2$, then $s^{(i)}_{n-2}$ is also zero, so all such w_i are equal to one another and to the number α where

$$(n+1)\alpha = s_1+1 = (n-2)\alpha+1,$$

hence $\alpha=1/3$, $s_1=(n-2)/3$, $s_2=(n-2)(n-3)/18$, and

$$\psi = \frac{1}{n+1}\left[\frac{n(n-2)^2}{9} - \frac{(n+1)(n-2)(n-3)}{9} - \frac{2(n-2)}{3} + n\right] = 2/3.$$

This configuration is uniquely minimal for ψ in all dimensions. We note that two w's are zero (the rest are 1/3); hence two a's are one (the rest are 2/3); hence two u's are 1/3, (the rest are zero).

As we pursue other possibilities we next assume that exactly one w is zero, without loss of generality w_n. The constraint (5B.1) reads

$$0 = s_{n-1}(s_1+1),$$

hence $s_1=-1$. The equations (5B.0) read $(i<n,\ s_n=s^{(i)}_{n-1}=0)$:

$$2w_i = \mu[(-1)s^{(i)}_{n-2}+s_{n-1}+s^{(i)}_{n-2}] = \mu s_{n-1}.$$

In particular, these w's are equal to one another, and equal to $-1/(n-1)$. We find $s_2=(n-2)/2(n-1)$, and

$$\psi = \frac{1}{n+1}\left[n - \frac{(n+1)(n-2)}{n-1} + 2 + n\right] = \frac{n}{n-1} > 2/3.$$

With none of the w's equal to zero, one may write

$$s^{(i)}_{n-1} = \frac{s_n}{w_i}, \quad s^{(i)}_{n-2} = \frac{s_{n-1}}{w_i} - \frac{s_n}{w_i^2}$$

and the equations (5B.0) become

$$\frac{2}{n+1}[(n+1)w_i-(s_1+1)] = \mu\left[s_1\left(\frac{s_n}{w_i}+\frac{s_{n-1}}{w_i}\right) - n(n+2)\frac{s_n}{w_i} + \frac{s_{n-1}}{w_i}\right]$$
$$+\mu\left[-\frac{s_1 s_n}{w_i^2} + (s_n+s_{n-1}) - \frac{s_n}{w_i^2}\right]$$
$$= \mu\left[-(s_1+1)\frac{s_n}{w_i^2} + (s_n+s_{n-1})\right],$$

because the topmost expression on the right is zero by the constraint.

We now see that each w_i satisfies the same cubic equation

$$2x^3 - \left[\frac{2}{n+1}(s_1+1) + \mu(s_n+s_{n-1})\right]x^2 + \mu(s_1+1)s_n = 0,$$

from which it follows that either

(I) all are the same (call them α); or

(II) there are exactly two distinct w's (call them α, β); or

(III) there are exactly three distinct w's, (call them α, β, γ).

Before proceeding, we remind the reader that none of the trio α, β, γ are zero.

Case (I): The constraint reads

$$\begin{aligned} 0 &= n\alpha(\alpha^n + n\alpha^{n-1}) - n(n+2)\alpha^n + n\alpha^{n-1} \\ &= n\alpha^{n-1}(\alpha^2 - 2\alpha + 1). \end{aligned}$$

Hence $\alpha = 1$ and $x = \mathbf{1}$, a point not under consideration.

Case (II): Assuming a pair (α, β) with relative frequencies k, m $(k+m=n)$, we have

$$\begin{aligned} s_1 &= k\alpha + m\beta, \\ s_{n-1} &= k\alpha^{k-1}\beta^m + m\alpha^k\beta^{m-1}, \\ s_n &= \alpha^k\beta^m, \\ n(n+2) &= k^2 + m^2 + 2(km + k + m) \end{aligned}$$

and

$$\begin{aligned} \varphi &= \alpha^{k-1}\beta^{m-1}\begin{Bmatrix} k\alpha^2\beta + m\alpha\beta^2 + m\alpha + k\beta + \\ km(\alpha^2+\beta^2) + \left(k^2+m^2-n(n+2)\right)\alpha\beta \end{Bmatrix} \\ &= \alpha^{k-1}\beta^{m-1}\{k\beta(\alpha-1)^2 + m\alpha(\beta-1)^2 + km(\alpha-\beta)^2\}. \end{aligned}$$

We see that the bracketed part of φ is never zero because $\alpha \neq \beta$. Hence $\varphi \neq 0$, and this case does not occur.

Case (III): The situation is similar, only the formulae are longer. With α, β, γ distinct, with relative frequencies k, m, p $(k+m+p=n)$, we find

$$\begin{aligned} s_1 &= k\alpha + m\beta + p\gamma, \\ s_{n-1} &= k\alpha^{k-1}\beta^m\gamma^p + m\alpha^k\beta^{m-1}\gamma^p + p\alpha^k\beta^m\gamma^{p-1}, \\ s_n &= \alpha^k\beta^m\gamma^p, \\ n(n+2) &= k^2 + m^2 + p^2 + 2(km + kp + mp + k + m + p) \end{aligned}$$

and

$$\varphi = \alpha^{k-1}\beta^{m-1}\gamma^{p-1}\begin{Bmatrix} k\beta\gamma(\alpha-1)^2 + m\alpha\gamma(\beta-1)^2 + p\alpha\beta(\gamma-1)^2 + \\ km\gamma(\alpha-\beta)^2 + kp\beta(\alpha-\gamma)^2 + mp\alpha(\beta-\gamma)^2 \end{Bmatrix}.$$

As before, the bracketed part of φ is not zero, nor is φ.

We conclude this section by pointing out the fact that in using Lagrange multipliers as we have, we are potentially overlooking singular places in the constraint locus, i.e., points x where φ and $\operatorname{grad}\varphi$ are both zero. We have taken this liberty because for any such points $x \neq \mathbf{1}$, one can prove directly that $\psi(x) \geqq 2/3$.

6. Precision; an example. We fix a parameter $a>1$, and take $0<\varepsilon<1$ as another parameter. We consider the four points in R^n_+:

$$z = \left(a(1+\varepsilon), a(1-\varepsilon), 1, \ldots, 1, \frac{1}{a^2(1-\varepsilon^2)}\right) \in \mathcal{H},$$

$$y = \left(a(1-\varepsilon), a(1+\varepsilon), 1, \ldots, 1, \frac{1}{a^2(1-\varepsilon^2)}\right) \in \mathcal{H},$$

$$x = \left(a, a, 1, \ldots, 1, \frac{1}{a^2(1-\varepsilon^2)}\right) \in \mathcal{H}_+,$$

$$w = \left(a, a, 1, \ldots, 1, \frac{1}{a^2}\right) \in \mathcal{H}.$$

We calculate easily (with $\alpha=\log a$, and neglecting terms $O(\varepsilon^3)$):

$$\Psi(z) = \psi(\hat{z}_n) = 6\alpha^2\left[1+\frac{2\varepsilon^2(1-3\alpha)}{6\alpha^2}\right],$$

$$\sqrt{\Psi(z)} = \sqrt{6}\,\alpha+\frac{\sqrt{6}(1-3\alpha)\varepsilon^2}{6\alpha},$$

$$\Phi_t(z) = \exp\left\{t\sqrt{\Psi(z)}\right\} = e^{t\sqrt{6}\alpha}\left[1+\frac{t\sqrt{6}(1-3\alpha)\varepsilon^2}{6\alpha}\right].$$

Our first observation is that if $3\alpha>1$, then for all sufficiently small $\varepsilon>0$, this number is strictly less than when $\varepsilon=0$ $(z=w)$. Under these circumstances, then:

$$\Phi_t(z) < \Phi_t(w) = \varphi_t(\hat{w}_n) = \varphi_t(\hat{x}_n) \leqq \Phi_t(x).$$

On the other hand, it is fairly easy to see that $x\in\mathcal{S}(z)$, and the preceding inequality then shows that the final inequality in Axiom A3 for Φ_t may fail as soon as $3\alpha>1$, which amounts (letting $\varepsilon\to 0$) to

$$\Psi(x) \to 6\alpha^2 > \frac{6}{9} = \frac{2}{3}.$$

This same reasoning shows that the constant 2/3 which appears in Lemma 5.2 cannot be replaced by a larger number, no matter what the dimension. Indeed, holding $3\alpha>1$, ε sufficiently small, we see that $\hat{x}_n$ is the midpoint of the line segment in R^{n-1} joining $\hat{z}_n$ and $\hat{y}_n$. Yet

$$\psi(\hat{y}_n) = \psi(\hat{z}_n) < \psi(\hat{x}_n).$$

It follows that the sublevel set $\{x\in R^{n-1}\colon \psi(x)\leqq\psi(\hat{z}_n)\}$ is not convex for ε sufficiently small. By appropriately varying a and ε these sublevel sets include all $\mathcal{D}_c$ with $c>2/3$. □

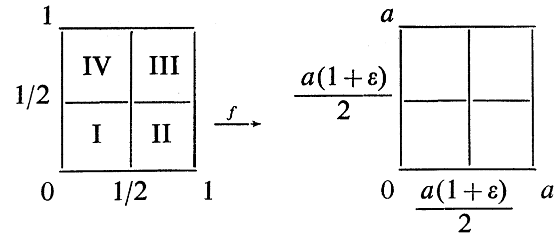
Next, we observe that Lemma 3.2 and in particular the first displayed line below (3.2) with the substitution $K=K_E$, $\Phi=\Phi_t$, is basically the mean-power-inequality for a competitor f in the Teichmüller problem based on f_λ:

$$\left\{\int_C K_f^m(u)\,du\right\}^{1/m} \geqq K_E[f_\lambda] \quad \left(m = t\sqrt{n/(n-1)}\right). \tag{6.1}$$

Its importance for us was that under the circumstances $K_E[f]<\exp\sqrt{n/6(n-1)}$ we could let $m\to\infty$, which as usual makes the left side approach $\|K_f\|_\infty=K_E[f]$.

While we are not able to give an example (some f) where the inequality (6.1) fails for any large m, we can at least give an example (in contrast to the outer dilatation) where the inequality (6.1) fails for all small m. We confine the discussion to the case $n=3$.

We fix $\varepsilon>0$ and $a>1$, and construct a piecewise affine mapping f which competes in the Grötzsch problem based on $f_0=\mathrm{diag}\,(w)$.



The unit cube in R^3 is divided into four vertical quarters by the planes $x_1=1/2$, $x_2=1/2$, and in each we specify f' as follows:

$$\begin{aligned}
&\text{In} && \text{I}, && f' = \mathrm{diag}\,\{a(1+\varepsilon), a(1+\varepsilon), 1/a^2\},\\
&\text{in} && \text{II}, && f' = \mathrm{diag}\,\{a(1-\varepsilon), a(1+\varepsilon), 1/a^2\},\\
&\text{in} && \text{III}, && f' = \mathrm{diag}\,\{a(1-\varepsilon), a(1-\varepsilon), 1/a^2\},\\
&\text{in} && \text{IV}, && f' = \mathrm{diag}\,\{a(1+\varepsilon), a(1-\varepsilon), 1/a^2\}.
\end{aligned}$$

It is clear that these can be sewed together to give a piecewise affine f, whose dilatation we now calculate with the aid of Lemma 4.2.

Quarter	$\log^2 K_f$	K_f^m (up to $O(\varepsilon^3)$)
I	$\frac{1}{2}[\log^2 a^3(1+\varepsilon)+\log^2 a^3(1+\varepsilon)]$	$a^{3m}\left[1+m\varepsilon+\frac{m}{2}(m-1)\varepsilon^2\right]$
II, IV	$\frac{1}{2}\left[\log^2\frac{1+\varepsilon}{1-\varepsilon}+\log^2 a^3(1+\varepsilon)+\log^2 a^3(1-\varepsilon)\right]$	$a^{3m}\left[1+\frac{m}{2}\left(\frac{1}{\log a}-1\right)\varepsilon^2\right]$
III	$\frac{1}{2}[\log^2 a^3(1-\varepsilon)+\log^2 a^3(1-\varepsilon)]$	$a^{3m}\left[1-m\varepsilon+\frac{m}{2}(m-1)\varepsilon^2\right]$.

Integration of K_f^m over C amounts to averaging these four cases:

$$\int_C K_f^m(u)\,du = a^{3m}\left[1+\frac{m}{4}\left[m-2+\frac{1}{\log a}\right]\varepsilon^2\right]+O(\varepsilon^3)$$

which is, for suitable small $\varepsilon>0$, less than $a^{3m}=K_E[f_0]^m$ as soon as $m<2-1/\log a$. This includes some $m>1$ as soon as $\log a>1\,(K_E>e^3)$.

In the earlier paper [Ag$_2$], we were able to prove (6.1) for sufficiently large m even when $K_E[f_0]=\sqrt{e}$. This was possible in that case $(n=3)$ because of a slightly more precise estimate on the growth of $\varphi(x)$ as $\psi(x)\to 2/3$ than we chose to attempt in this already overlong paper. Such a thing may still be true. We have our sights set higher however: we conjecture a mean inequality such as (6.1) may be true for sufficiently large m without any further restriction, as it is for our example.

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