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EXCEPTIONAL SETS FOR LINEAR DIFFERENTIAL POLYNOMIALS

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1. Introduction

Let $f(z)$ be a non-constant entire function with Nevanlinna characteristic $T(r,f)$ (see e.g. [7]). Suppose that

$$\psi(z)=\sum_{i=0}^{k}\alpha_i(z)f^{(i)}(z) \tag{1.1}$$

is non-constant, where $\alpha_0(z), \ldots, \alpha_k(z)$ are entire functions each satisfying

$$T(r,\alpha_i)=S(r,f) \tag{1.2}$$

and, using standard notation from [7], $S(r,f)$ denotes any quantity such that

$$S(r,f)=o\big(T(r,f)\big),$$

possibly outside a set of finite linear measure. Then we have the following bound on the growth of $T(r,f)$ ([7], p. 57):

Theorem A. *If $f(z)$ is a non-constant entire function, and if $\psi(z)$ is given by* (1.1) *and* (1.2) *and is non-constant, then*

$$T(r,f)<N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\psi-1}\right)-N\left(r,\frac{1}{\psi'}\right)+S(r,f). \tag{1.3}$$

We observe that Theorem A is usually stated in a slightly different form, with common zeros of $\psi(z)-1$ and $\psi'(z)$ cancelled out, but the statement (1.3) is more convenient for our purposes here.

It follows from Theorem A (Hayman [5]) that if $g(z)$ is a transcendental entire function, and $N\geqq 2$, then $g^N(z)g'(z)$ has infinitely many 1-points. It was shown in [2] by Anderson, Baker and Clunie that infinitely many of these 1-points must lie outside certain exceptional sets. They proved:

Theorem B. *If (a_n) is a sequence converging to infinity such that, for all n,*

$$\left|\frac{a_{n+1}}{a_n}\right|>q>1 \tag{1.4}$$

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then for any $N \geqq 2$ *every transcendental entire function* $g(z)$ *must have infinitely many solutions of*

$$g^N(z)g'(z) = 1$$

outside $E = \{a_n\}$.

This result was improved in the author's Ph. D. thesis [8], written under the supervision of I. N. Baker, in which it is shown that the exceptional set E may consist of a countable union of small discs whose centres a_n satisfy (1.4). These exceptional sets are comparable to certain Picard sets for entire functions (see [3], [4], [10]) — that is, subsets of the plane outside which every transcendental entire function takes every finite value, with at most one exception, infinitely often.

In the present paper we return to the initial problem of Theorem A in the case where $\alpha_0(z), \ldots, \alpha_k(z)$ are polynomials and $\alpha_k(z) \not\equiv 0$. We prove:

Theorem 1. *Given* $\varepsilon > 0$, *there exists* $K(\varepsilon) > 0$, *depending only on* ε, *such that if* (a_n) *converges to infinity with*

$$|a_n - a_m| > \varepsilon |a_n| \tag{1.5}$$

for all $n \neq m$, *while* (ϱ_n) *satisfies*

$$\log \frac{1}{\varrho_n} > K(\varepsilon)(\log |a_n|)^2 \tag{1.6}$$

then for any polynomials $\alpha_0(z), \ldots, \alpha_k(z)$ *(with* $k \geqq 1$ *and* $\alpha_k(z) \not\equiv 0$*) and any transcendental entire function* $f(z)$ *such that*

$$\psi(z) = \sum_{i=0}^{k} \alpha_i(z) f^{(i)}(z)$$

is non-constant, either $f(z)$ *has infinitely many zeros outside the union E ot the discs*

$$B(a_n, \varrho_n) = \{z\colon |z - a_n| < \varrho_n\}$$

or $\psi(z)$ *has infinitely many* 1*-points outside E.*

The condition (1.5) is best-possible. We have, in fact:

Theorem 2. *Given any increasing, positive function* $h(r)$ *on* $0 < r < \infty$, *such that* $h(r) \to \infty$ *as* $r \to \infty$, *there exists a transcendental entire function* $f(z)$ *such that all large zeros of* $f(z)$ *and* 1*-points of* $f'(z)$ *lie in a set* $\{b_m\}$, *where, for all m,* b_m *is positive and*

$$b_{m+1} \geqq \left(1 + \frac{1}{h(b_m)}\right) b_m.$$

While the exceptional sets of Theorem 1 are comparable to certain Picard sets for entire functions (see [3]), Theorem 2 marks a departure from the situation there, insofar as the best-possible condition on the centres a_m of the exceptional discs of a Picard set for entire functions is

$$|a_n - a_m| > \varepsilon |a_n| (\log |a_n|)^{-1}$$

for $n \neq m$ (Toppila [10]).

Theorem 2 was proved in the author's Ph. D. thesis [8], while Theorem 1 is a substantial improvement of a result in [8] but makes considerable use of ideas communicated to the author by I. N. Baker.

Throughout the paper we use standard notation from [7], including

$$M(r, f) = \max\{|f(z)|: |z| = r\}$$

for an entire function $f(z)$.

2. Preliminary lemmas

Lemma 1. *Suppose that $P(z)$ is a polynomial of degree k whose zeros $\alpha_1, \ldots, \alpha_k$ lie in $|z| < R_0$. Then for $|z| = R \geqq 3R_0$,*

$$\frac{k!}{(k-i)!\,2^i R^i} \leqq \left|\frac{P^{(i)}(z)}{P(z)}\right| \leqq \frac{k!\,2^i}{(k-i)!\,R^i}$$

for $i = 1, \ldots, k$.

Proof. We proceed by induction on i. We have, for $|z| = R \geqq 3R_0$,

$$\frac{P'(z)}{P(z)} = \frac{1}{z} \sum_{j=1}^{k} \left(1 - \frac{\alpha_j}{z}\right)^{-1}.$$

But, for $|t| < 1$,

$$\mathrm{Re}\left((1-t)^{-1}\right) \geqq 1 - \frac{|t|}{1-|t|}$$

and so

$$\mathrm{Re}\left(\sum_{j=1}^{k} \left(1 - \frac{\alpha_j}{z}\right)^{-1}\right) \geqq k/2$$

and

$$\left|\frac{P'(z)}{P(z)}\right| \geqq \frac{k}{2R}.$$

For an inequality in the other direction, we just note that if $R = |z| \geqq 3R_0$, then $|z - \alpha_j| \geqq 2|z|/3$ and so

$$\left|\frac{P'(z)}{P(z)}\right| \leqq \frac{2k}{R}.$$

To prove the general case, we note that if $1 \leqq i \leqq k-1$,

$$\frac{P^{(i+1)}(z)}{P(z)} = \frac{P^{(i)}(z)}{P(z)} \frac{P^{(i+1)}(z)}{P^{(i)}(z)}$$

and since $P^{(i)}(z)$ has $k-i$ zeros, all lying in the convex hull of the set of zeros of $P(z)$ ([1], p. 29), and hence in $|z|<R_0$, we have, for z, R as above,

$$\frac{(k-i)}{2R} \leqq \left|\frac{P^{(i+1)}(z)}{P^{(i)}(z)}\right| \leqq \frac{2(k-i)}{R}$$

and Lemma 1 follows by induction.

Lemma 2. *Suppose that $R\geqq 1$ and that $h(z)$ is regular and non-zero in $|z|\leqq R$, with $\big|\log|h(z)|\big|\leqq M$ on $|z|=R$. Then in $|z|\leqq R/4$ we have for $i=1, \ldots, k$,*

$$\left|\frac{h^{(i)}(z)}{h(z)}\right| \leqq \beta_k(1+M)^i$$

where β_k depends only on k.

Proof. We have (see [7], p. 22)

$$\frac{h'(z)}{h(z)} = \frac{1}{\pi}\int_0^{2\pi} \log|h(Re^{i\varphi})| \frac{Re^{i\varphi}}{(Re^{i\varphi}-z)^2}\, d\varphi$$

in $|z|<R$, and so

$$\left|\frac{h'(z)}{h(z)}\right| \leqq \beta_1 M \tag{2.1}$$

in $|z|\leqq R/2$. But (see [7], p. 73), $h^{(i)}(z)/h(z)$ is a differential polynomial in $h'(z)/h(z)$ with constant coefficients and total degree at most i, and so Lemma 2 follows from (2.1) and Cauchy's estimate.

Lemma 3. *Let N and K be positive integers, and suppose that (a_n) is a complex sequence converging to infinity such that, for some ε with $0<\varepsilon<1$, we have*

$$|a_n-a_m| > \varepsilon|a_n|$$

for all large $n\neq m$. Suppose that the transcendental entire function $h(z)$ is given by

$$h(z) = \beta z^Q \prod_{j=1}^{\infty}\left(1-\frac{z}{\gamma_j}\right)$$

where $\beta\neq 0$, $Q\geqq 0$ and each γ_j lies in some disc $B(a_n, 1)$, while, for large n, $B(a_n, 1)$ contains at most K zeros of $h(z)$, counting multiplicities. Then there exist M_1, M_2 depending on N, K and the sequence (a_n), but not on $h(z)$, such that if $|z|=r>M_1$ and $n(r/6, 1/h)>M_2$ and z lies outside the union of the discs $B(a_n, \varepsilon|a_n|/4)$ we have, for $k=1, \ldots, N$,

$$\left(\frac{n(r)}{3r}\right)^k \leqq \left|\frac{h^{(k)}(z)}{h(z)}\right| \leqq \left(\frac{3n(r)}{r}\right)^k$$

where

$$n(r) = n(r, 1/h).$$

In addition, if $\alpha_0(z), \ldots, \alpha_N(z)$ *are polynomials, not all identically zero, then*

$$L(z) = \sum_{k=0}^{N} \alpha_k(z) h^{(k)}(z)/h(z)$$

satisfies

$$|L(z)| > \frac{1}{r^N}$$

for z *and* r *as above, provided that* $r > M_3$, *where* M_3 *depends only on* $\alpha_0(z), \ldots, \alpha_N(z)$.

Proof. For large w lying outside the union of the discs $B(a_n, \varepsilon|a_n|/4)$ we first prove by induction on k that

$$\left(\frac{n(r)}{3r}\right)^k \leqq \left|\frac{h^{(k)}(z)}{h(z)}\right| \leqq \left(\frac{3n(r)}{r}\right)^k \tag{2.2}$$

for $|z|=r$ and $|z-w| \leqq \varepsilon|w|2^{-k-4}$. Now the disc $B(w, \varepsilon|w|/16)$ does not meet any of the discs $B(a_n, 1)$. Moreover, since the discs $B(a_n, \varepsilon|a_n|/2)$ are disjoint for large n, we see that for any large R and any points a_n lying in the annulus

$$A_R = \{\zeta\colon\ R \leqq |\zeta| \leqq 2R\}$$

there exist corresponding disjoint discs of radius $\varepsilon R/4$ contained entirely in A_R, with a_n on their respective boundaries, and hence, since A_R has area $3\pi R^2$, the number of points a_n in A_R is at most $48\varepsilon^{-2}$. Thus, in $B(w, \varepsilon|w|/32)$, we have, with $|z|=r$,

$$\frac{h'(z)}{h(z)} = \Sigma_1 \frac{1}{z-\gamma_j} + \Sigma_2 \frac{1}{z-\gamma_j} + O\left(\frac{1}{r}\right), \tag{2.3}$$

where Σ_1 denotes the sum over all j with $|\gamma_j| \leqq |w|/6$ and Σ_2 denotes the sum over all j with $|\gamma_j| \geqq 3|w|$. But

$$|\Sigma_2(z-\gamma_j)^{-1}| \leqq \frac{1}{r} \Sigma_2 \left|\left(1-\frac{\gamma_j}{z}\right)^{-1}\right| < \frac{49K}{\varepsilon^2 r} \sum_{i=1}^{\infty} (2^i-1)^{-1} = O\left(\frac{1}{r}\right). \tag{2.4}$$

Also, from Lemma 1,

$$|\Sigma_1(z-\gamma_j)^{-1}| \geqq \frac{1}{2r} n(|w|/6) \tag{2.5}$$

and combining (2.3), (2.4) and (2.5), and noting that $n(|w|/6)$ and $n(r)$ differ by a bounded quantity, we obtain the left-hand inequality of (2.2) in the case $k=1$. Indeed, for $k=1$, (2.2) holds with 3 replaced by 5/2, the second inequality following from (2.3), (2.4) and the fact that

$$|\Sigma_1(z-\gamma_j)^{-1}| \leqq \frac{2}{r} n(|w|/6),$$

by Lemma 1.

Now, for $k \geqq 1$,

$$\frac{h^{(k+1)}(z)}{h(z)} = \frac{h^{(k)}(z)}{h(z)} \frac{h'(z)}{h(z)} + \frac{d}{dz}\left(\frac{h^{(k)}(z)}{h(z)}\right)$$

and since (2.2) implies that

$$\frac{d}{dz}\left(\frac{h^{(k)}(z)}{h(z)}\right) = O\left(\frac{(n(r))^k}{r^{k+1}}\right)$$

in $|z-w| \leqq \varepsilon |w| 2^{-k-5}$, using Cauchy's estimate, we obtain the first part of the lemma by induction.

To prove the second part, suppose that the polynomials $\alpha_0(z), \ldots, \alpha_N(z)$ have degrees $D_0, \ldots, D_N$, respectively, and suppose that $z = re^{i\theta}$ and $n(r/6, 1/h)$ are such that (2.2) holds for $k = 1, \ldots, N$. Set

$$d_i = D_i - i$$

and

$$d = \max \{d_i\colon \alpha_i(z) \not\equiv 0\}$$

and define s, t by

$$s = \max \{i\colon d_i = d \text{ and } \alpha_i(z) \not\equiv 0\}$$

and

$$t = \max \{s-1, 0\}.$$

Then, assuming that r is so large that

$$\log |\alpha_i(z)| = (1+o(1)) D_i \log r$$

for each i we write

$$L(z) = \Sigma_3 \alpha_j(z) \frac{h^{(j)}(z)}{h(z)} + \Sigma_4 \alpha_j(z) \frac{h^{(j)}(z)}{h(z)}$$

where Σ_3 denotes the sum over all j with $d_j = d$ and Σ_4 denotes the sum over the remaining j. But then, using (2.2) we have

$$\Sigma_4 \alpha_j(z) \frac{h^{(j)}(z)}{h(z)} = O((n(r))^N r^{d-1}) \tag{2.6}$$

while, for some $c > 0$,

$$\left|\Sigma_3 \alpha_j(z) \frac{h^{(j)}(z)}{h(z)}\right| > cr^d (n(r))^s - tO(r^d (n(r))^t). \tag{2.7}$$

Since $n(r) \to \infty$ but satisfies

$$n(r) = O(\log r)$$

we obtain the second part of Lemma 3, noting that $d \geqq -N$.

Lemma 4. *Suppose that* $q > 81$ *and that*

$$f(z) = \Pi_{n=1}^{\infty} \left(1 - \frac{z}{a_n}\right)$$

where $a_{n+1}>a_n$ *and each* a_n *lies in the set* $\{q, q^2, q^3, \ldots\}$. *Then, for large n,*

$$\min\{|f(z)|:\ |z| = q^{1/2}a_n\} > M(q^{1/4}a_n, f).$$

Proof. For $|z|=q^{1/2}|a_N|$ and $n\leqq N$,

$$\left|1-\frac{z}{a_n}\right| \geqq q^{1/2}\frac{a_N}{a_n}-1$$

while, if $|w|=q^{1/4}a_N$,

$$\left|1-\frac{w}{a_n}\right| \leqq q^{1/4}\frac{a_N}{a_n}+1.$$

Now,

$$\frac{q^{1/2}\frac{a_N}{a_n}-1}{q^{1/4}\frac{a_N}{a_n}+1} = \frac{q^{1/2}a_N-a_n}{q^{1/4}a_N+a_n} \geqq (q^{1/2}-1)(q^{1/4}+1)^{-1} > 2.$$

Also, if $|z|=q^{1/2}a_N$,

$$\left|\Pi_{n>N}\left(1-\frac{z}{a_n}\right)\right| \geqq \Pi_{n>N}\left(1-q^{1/2}\frac{a_N}{a_n}\right)$$
$$\geqq \Pi_{n=1}(1-q^{1/2-n}) = C_1,$$

say. On the other hand, for $|w|=q^{1/4}a_N$,

$$\sum_{n>N}\log\left|1-\frac{w}{a_n}\right| \leqq \sum_{n>N}\log\left(1+q^{1/4}\frac{a_N}{a_n}\right)$$
$$\leqq q^{1/4}\sum_{n>N}\frac{a_N}{a_n}$$
$$\leqq q^{1/4}\left(\frac{1}{q}+q^{-2}+q^{-3}+\ldots\right)$$
$$= C_2,$$

say. Thus

$$(\min\{|f(z)|:\ |z| = q^{1/2}a_N\})(M(q^{1/4}a_N, f))^{-1} \geqq 2^N C_1 e^{-C_2} > 1$$

for sufficiently large N.

We need also a result of Hayman [6]:

Theorem C. *Suppose that* $f(z)$ *is a non-constant entire function satisfying*

$$T(r,f) = O(\log r)^2.$$

Then

$$\log|f(re^{i\theta})| \sim \log M(r,f)$$

as $z=re^{i\theta}$ *tends to infinity outside an ε-set surrounding the large zeros of* $f(z)$.

An ε-set is defined (following Hayman [6]) to be a countable set of discs not meeting the origin which subtend angles at the origin whose sum is finite.

3. Proof of Theorem 1

Suppose that $f(z)$ is a transcendental entire function, and that

$$\psi(z)=\sum_{i=0}^{k}\alpha_i(z)f^{(i)}(z) \tag{3.1}$$

is non-constant, where $\alpha_0, \dots, \alpha_k$ are polynomials, with $\alpha_k(z)\not\equiv 0$, and $k\geqq 1$. Suppose that all but finitely many zeros of $f(z)$ and 1-points of $\psi(z)$ lie in the union of the discs $D_n=B(a_n, \varrho_n)$ where $a_n\to\infty$ such that $|a_n-a_m|>\varepsilon|a_n|$ for some $\varepsilon>0$ and for all $n\neq m$, while ϱ_n satisfies

$$\log\frac{1}{\varrho_n}>K(\varepsilon)(\log|a_n|)^2 \tag{3.2}$$

for some $K(\varepsilon)$ which we assume to be large and positive.

We use $c_1, c_2, \dots$ to denote positive constants depending only on ε. We first note that (see [7], p. 56)

$$T(r,\psi)\leqq m(r,\psi/f)+m(r,f)\leqq m(r,f)+S(r,f). \tag{3.3}$$

Also, applying Theorem A,

$$T(r,f)<N\left(r,\frac{1}{f}\right)+N\left(r,\frac{1}{\psi-1}\right)-N_1\left(r,\frac{1}{\psi'}\right)+S(r,f), \tag{3.4}$$

where $N_1(r, 1/\psi')$ counts only zeros of ψ' which lie in the discs $B(a_n, 3\varrho_n)$.

We assume, without loss of generality, that $1/\varepsilon$ and a_1 are large and that $|a_n|\leqq |a_{n+1}|$ for all n, and observe, as we saw in the proof of Lemma 3, that the annulus $\{z\colon |a_n|\leqq|z|\leqq 2|a_n|\}$ contains less than $49\varepsilon^{-2}$ of the points a_m. Thus there exists, for large n, s_n satisfying $|a_n|\leqq s_n\leqq 2|a_n|$ such that the annulus $\{z\colon s_n\leqq|z|\leqq s_n+\varepsilon^2|a_n|/50\}$ meets none of the discs $B(a_m, 3\varrho_n)$, since $\varrho_n\to 0$. We define t_n, r_n and the annulus A_n by

$$t_n=s_n(1+\varepsilon^2/100)$$

and

$$r_n=s_n(1+\varepsilon^2/200)$$

and

$$A_n=\{z\colon s_n\leqq|z|\leqq t_n\}. \tag{3.5}$$

Of course, for large n, A_n contains no zeros of $f(z)$ or $\psi(z)-1$, nor any zeros of $\psi'(z)$ which contribute to $N_1(r, 1/\psi')$.

We now estimate $T(r,f)$ for large r. Let μ_n denote the number of zeros of $f(z)$ in D_n, σ_n the number of 1-points of $\psi(z)$ in D_n, and τ_n the number of zeros of ψ' in $B(a_n, 3\varrho_n)$, in each case counting points according to multiplicity. Suppose that $\hat r$ is large, and that the annulus $\{z\colon \beta_1\hat r\leqq|z|\leqq\beta_2\hat r\}$ meets none of the discs $B(a_n, 3\varrho_n)$, where

$$\beta_1=(1+\varepsilon^2/200)^{-1}$$

and

$$\beta_2=\beta_1(1+\varepsilon^2/100).$$

Then, using (3.4), if $\hat{r} \leqq r \leqq \beta_2 \hat{r}$, we have

$$T(r,f) < \Sigma' \mu_m \left(\log \frac{r}{|a_m|} + o(1)\right) + \Sigma' \sigma_m \left(\log \frac{r}{|a_m|} + o(1)\right) \tag{3.6}$$

$$-\Sigma' \tau_m \left(\log \frac{r}{|a_m|} - o(1)\right) + O(\log r) + S(r,f)$$

where Σ' denotes the sum over all m with $|a_m| < \beta_1 \hat{r}$. But

$$\Sigma'(\mu_m + \sigma_m + \tau_m) \leqq n\left(\beta_1\ \hat{r}, \frac{1}{f}\right) + n\left(\beta_1\ \hat{r}, \frac{1}{\psi - 1}\right) + n\left(\beta_1\ \hat{r}, \frac{1}{\psi'}\right) \tag{3.7}$$

$$\leqq c_1 \left\{N\left(r, \frac{1}{f}\right) + N\left(r, \frac{1}{\psi - 1}\right) + N\left(r, \frac{1}{\psi'}\right)\right\}.$$

Combining (3.5) and (3.7), and using (3.3) and the fact that $f(z)$ is assumed to be transcendental, we obtain, with r as above, and in particular for $r = r_n$,

$$T(r,f) < \Sigma' y_m \left(\log \frac{r}{|a_m|}\right) + S(r,f) \tag{3.8}$$

where

$$y_m = \mu_m + \sigma_m - \tau_m.$$

Now, whether or not the sequence (y_m) is bounded above, we can find m_0 and an infinite set $\mathscr{H}$ of m such that

$$y_m = \max\{y_j\colon m \geqq j \geqq m_0\} \geqq 1. \tag{3.9}$$

For $m \in \mathscr{H}$ we choose $M(m)$ satisfying $M(m) \geqq m$ and $|a_{M(m)}| \leqq s_m$ and

$$y_{M(m)} = \max\{y_j\colon j \geqq m \text{ and } |a_j| \leqq s_m\} \tag{3.10}$$

and for convenience we set

$$m' = M(m). \tag{3.11}$$

We note that any absolute bound which holds on $y_{m'}$ for $m \in \mathscr{H}$ holds on y_m for all large m.

For $m \in \mathscr{H}$, we have, from (3.8), (3.9), (3.10) and (3.11),

$$T(r,f) < M' y_{m'} \left(\log \frac{r}{|a_1|}\right) + O(\log r) + S(r,f), \tag{3.12}$$

for $r_m \leqq r \leqq t_m$, where M' is the number of points a_j in $|z| < s_m$. Since for large m, $m = O(\log |a_m|)$ by virtue of the absolute bound on the number of points a_n in any annulus $r^* \leqq |z| \leqq 2r^*$, we obtain

$$T(r_m, f) + T(r_m, \psi) < c_2 y_{m'} (\log r_m)^2. \tag{3.13}$$

We now go on to establish a series of claims and finally to obtain a contradiction if $K(\varepsilon)$ is large enough in (3.2). Suppose that m is large, not necessarily in $\mathscr{H}$, and that $z_1, \ldots, z_{\mu_m}$ are the zeros of $f(z)$ in D_m, while $w_1, \ldots, w_{\sigma_m}$ are the 1-points of $\psi(z)$ there. Set

$$P(z) = \prod_{i=1}^{\mu_m} (z-z_i) \tag{3.14}$$

and

$$Q(z) = \prod_{j=1}^{\sigma_m} (z-w_j) \tag{3.15}$$

and define $h(z)$, $H(z)$ by

$$f(z) = P(z)h(z) \tag{3.16}$$

and

$$\psi(z)-1 = Q(z)H(z). \tag{3.17}$$

Claim 1. For any n, m, if $|a_m|<r_n$, with r_n as in (3.5), then

$$\big|\log|h(z)|\big| \leqq c_3 T(r_n, f)+c_4\mu_m \log r_n \tag{3.18}$$

and

$$\big|\log|H(z)|\big| \leqq c_5 T(r_n, \psi-1)+c_6\sigma_m \log r_n \tag{3.19}$$

in $|z-a_m|\leqq 16$.

Applying the Poisson—Jensen formula to $h(z)$ in $|z|<r_n$, we obtain (since $|\log x|=\log^+ x+\log^+(1/x)$)

$$\big|\log|h(z)|\big| \leqq \left(\frac{r_n+|z|}{r_n-|z|}\right)\left(m(r_n, h)+m\left(r_n, \frac{1}{h}\right)\right)+\sum_\zeta'' \log\left|\frac{r_n^2-\bar{\zeta}z}{r_n(\zeta-z)}\right| \tag{3.20}$$

where the sum Σ'' is taken over all zeros of h in $|z|<r_n$. But $h(z)$ is non-zero in the disc $B(a_m, \varepsilon|a_m|/2)$ and so, for $|z-a_m|\leqq 16$, (3.20) yields

$$\big|\log|h(z)|\big| \leqq c_7\left(m(r_n, h)+m\left(r_n, \frac{1}{h}\right)\right)+c_8 n\left(s_n, \frac{1}{f}\right). \tag{3.21}$$

Also,

$$m(r_n, h)+m\left(r_n, \frac{1}{h}\right) \leqq m(r_n, f)+m\left(r_n, \frac{1}{P}\right)+m\left(r_n, \frac{1}{f}\right)+m(r_n, P)$$

and, noting that $|P(u)|\geqq 1$ on $|u|=r_n$, and that

$$n\left(s_n, \frac{1}{f}\right) < c_9 N\left(r_n, \frac{1}{f}\right)$$

we obtain (3.18) from (3.21). The estimate (3.19) is proved identically.

Claim 2. For all large m, we have $y_m<4(k+1)$.

Suppose, on the contrary, that for some large $m\in\mathscr{H}$, m and m' satisfy (3.9), (3.10) and (3.11), and suppose that $y_{m'}\geqq 4(k+1)$. Then either

$$\mu_{m'} \geqq y_{m'}/2 \geqq 2(k+1)$$

or

$$\sigma_{m'}-\tau_{m'} \geqq y_{m'}/2 \geqq 2(k+1),$$

and we shall show that both cases are impossible. We define $P(z)$, $Q(z)$, $h(z)$ and $H(z)$ as in (3.14)—(3.17), but with m replaced by m', and note that if

$$S = \mu_{m'} \geqq y_{m'}/2$$

then from Lemma 1 we have

$$\frac{S!}{(S-i)!2^i(3\varrho_{m'})^i} \leqq \left|\frac{P^{(i)}(z)}{P(z)}\right| \leqq \frac{S!2^i}{(S-i)!(3\varrho_{m'})^i} \tag{3.22}$$

on the circle

$$C_{m'} = \{z\colon |z-a_{m'}| = 3\varrho_{m'}\}$$

for $i=1, ..., k+1$. Also, combining (3.13) and Claim 1, and noting that $r_m\leqq 3|a_{m'}|$, we have, using Lemma 2,

$$\left|\frac{h^{(i)}(z)}{h(z)}\right| \leqq c_{10}S^i(\log|a_{m'}|)^{2i} \tag{3.23}$$

on $C_{m'}$, for $i=1, ..., k+1$. But

$$\psi(z) = P(z)h(z)\left[\alpha_k(z)\left[\frac{P^{(k)}(z)}{P(z)}+...+\frac{h^{(k)}(z)}{h(z)}\right]+...+\alpha_0(z)\right] \tag{3.24}$$

and thus on $C_{m'}$, using (3.22) and (3.23) we obtain from (3.24) (with d denoting the maximum of the degrees of $\alpha_0(z), ..., \alpha_k(z)$)

$$\psi(z) = P(z)h(z)\left[\alpha_k(z)\frac{P^{(k)}(z)}{P(z)}+O\left(|z|^d S^k\frac{(\log|a_{m'}|)^{2k}}{(\varrho_{m'})^{k-1}}\right)\right].$$

But

$$\frac{S!}{(S-k)!S^k} \geqq 2^{-k}$$

and thus, using (3.22), if $K(\varepsilon)$ is large enough in (3.2) we have

$$\psi(z) = h(z)\alpha_k(z)(1+o(1))P^{(k)}(z) \tag{3.25}$$

on $C_{m'}$. It follows from (3.25) and Rouché's theorem that $\psi(z)$ has the same number of zeros inside $C_{m'}$ as $P^{(k)}(z)$; since the zeros of $P^{(k)}(z)$ lie in the convex hull of the set of zeros of $P(z)$, and hence in $D_{m'}$, we conclude that $\psi(z)$ has $S-k$ zeros in the disc $B(a_{m'}, 3\varrho_{m'})$.

Similarly

$$\begin{aligned}\psi'(z) &= P(z)h(z)\left[\alpha_k(z)\frac{P^{(k+1)}(z)}{P(z)}+...+\alpha_0'(z)\right]\\ &= (1+o(1))P^{(k+1)}(z)\alpha_k(z)h(z)\end{aligned}$$

on $C_{m'}$, and thus

$$\tau_{m'} = \mu_{m'} - (k+1).$$

But then

$$\sigma_{m'} \geqq y_{m'}/2 > 0 \tag{3.26}$$

and so $Q(z) \not\equiv 1$ in (3.17). Moreover, inside the circle $C_{m'}$ we have, using (3.3), (3.13), Claim 1 and (3.26),

$$\begin{aligned}\log|\psi(z)-1| &= \log|H(z)|+\log|Q(z)| \\ &\leqq c_5 T(r_m, \psi-1) \\ &\quad + c_6 \sigma_{m'} \log r_m \\ &\quad + \sigma_{m'} \log 6\varrho_{m'} \\ &\leqq c_{11}\sigma_{m'}(\log|a_{m'}|)^2 + \sigma_{m'} \log 6\varrho_{m'} < 0\end{aligned} \tag{3.27}$$

if $K(\varepsilon)$ is large enough in (3.2). But this implies that $\psi(z)$ has no zeros inside $C_{m'}$, contradicting the conclusion that followed (3.25).

On the other hand, suppose that $y_{m'} \geqq 4(k+1)$, and that

$$\sigma_{m'} - \tau_{m'} \geqq y_{m'}/2.$$

But then, since

$$\psi'(z) = Q'(z)H(z) + Q(z)H'(z)$$

it follows from Lemmas 1 and 2, and (3.3), (3.13), and Claim 1 that

$$\psi'(z) = Q'(z)H(z)\big(1+o(1)\big)$$

on $C_{m'}$, and hence that $\sigma_{m'} - \tau_{m'} = 1$. This establishes Claim 2.

Claim 3. We have

$$T(r,f) = O(\log r)^2$$

and in particular, for large m,

$$T(r_m, f) + T(r_m, \psi) \leqq c_{12}(\log|a_m|)^2. \tag{3.28}$$

This follows immediately from (3.8) and Claim 2.

Claim 4. For all large m, we have $\mu_m < 2(k+1)$.

For, suppose that $\mu_m \geqq 2(k+1)$ for some large m. Then on the circle

$$C_m = \{z\colon |z-a_m| = 3\varrho_m\}$$

we have, defining P, Q, h and H as in (3.14) to (3.17) and setting $S = \mu_m$,

$$\frac{S!}{(S-i)!\,2^i(3\varrho_m)^i} \leqq \left|\frac{P^{(i)}(z)}{P(z)}\right| \leqq \frac{S!\,2^i}{(S-i)!\,(3\varrho_m)^i} \tag{3.29}$$

for $i=1, \ldots, S$, which of course is just (3.22) with m' replaced by m. Also, from Claim 1, (3.28) and Lemma 2 we have, for $i=1, \ldots, k$,

$$\left|\frac{h^{(i)}(z)}{h(z)}\right| \leqq c_{13} S^i (\log |a_m|)^{2i} \tag{3.30}$$

on C_m. But then, on C_m, with d denoting the maximum of the degrees of $\alpha_0(z), \ldots, \alpha_k(z)$,

$$\begin{aligned} \psi(z) &= P(z)h(z)\left[\alpha_k(z)\frac{P^{(k)}(z)}{P(z)} + O\left(|z|^d S^k \frac{(\log|a_m|)^{2k}}{\varrho_m^{k-1}}\right)\right] \\ &= \alpha_k(z)h(z)P^{(k)}(z)(1+o(1)) \end{aligned}$$

using (3.29) and (3.30) and noting that $(S-k)>S/2$. Hence $\psi(z)$ has $S-k$ zeros in $B(a_m, 3\varrho_m)$ and so, by Rouché's theorem, $\psi(z)-1$ has at least $S-k$ zeros in D_m since by Theorem C $\psi(z)$ is large outside an ε-set. But then, using Claim 1 and (3.28), we have, in $B(a_m, 3\varrho_m)$,

$$\begin{aligned} \log|\psi(z)-1| &= \log|H(z)| + \log|Q(z)| \\ &\leqq c_{14}(\log|a_m|)^2 + c_6\sigma_m \log r_m \\ &\quad + \sigma_m \log 6\varrho_m < 0 \end{aligned}$$

if $K(\varepsilon)$ is large enough in (3.2), and we have a contradiction.

Claim 5. For large m, if $\sigma_m>0$, then $\mu_m \geqq \sigma_m+k$.

We note first that if $\mu_m=0$ then by Theorem C and Lemma 3, $\psi(z)$ is large in the disc $B(a_m, \varepsilon|a_m|/2)$ and hence $\sigma_m=0$. Thus, if $\sigma_m>0$, we set

$$c = \min\{\mu_m, k\} \geqq 1$$

and we write

$$\psi(z) = P(z)h(z)\left[\frac{P^{(c)}(z)}{P(z)}L(z) + B(z)\right] \tag{3.31}$$

where P, h, Q and H are defined as in (3.14) to (3.17) and $B(z)$, $L(z)$ are as follows. We have

$$L(z) = \left(\alpha_k(z)\frac{k!}{c!(k-c)!}\frac{h^{(k-c)}(z)}{h(z)} + \ldots + \alpha_c(z)\right)$$

if $c<k$, and

$$L(z) = \alpha_k(z)$$

if $c=k$. Thus, if we regard $\psi(z)/f(z)$ as a linear combination of terms $P^{(i)}(z)/P(z)$ with $i \leqq c$, then $L(z)$ is the coefficient of $P^{(c)}(z)/P(z)$. Accordingly, $B(z)$ is a linear combination of terms $P^{(i)}(z)/P(z)$ with $i<c$, and with coefficients which are sums of terms $\gamma_{ij}(z)h^{(j)}(z)/h(z)$ where each $\gamma_{ij}(z)$ is a constant multiple of one of $\alpha_0(z), \ldots, \alpha_k(z)$.

On the circle C_m, by Lemmas 1 and 3, and Claim 4,

(3.32) $$|B(z)| = O(|z|^d \varrho_m^{1-c})$$

where d as before is the maximum of the degrees of $\alpha_0(z), \ldots, \alpha_k(z)$. We may also apply the second part of Lemma 3 to $h(z)$ (since for all r, $n(r, 1/h)$ and $n(r, 1/f)$ differ by at most $2k+1$) to conclude that, on the circle C_m,

(3.33) $$|L(z)| \geqq c_{15} |z|^{-k}.$$

Combining (3.2), (3.29), (3.30), (3.31), (3.32) and (3.33) we see that

(3.34) $$\psi(z) = h(z) P^{(c)}(z) L(z)(1+o(1))$$

on the circle C_m and hence that $\psi(z)$ has v_m zeros in $B(a_m, 3\varrho_m)$, where

(3.35) $$v_m = \max\{0, \mu_m - k\},$$

since $h(z)$ and $L(z)$ do not vanish on or inside C_m.

Thus, if $\sigma_m > 0$, we must have $\mu_m > k$. For otherwise $P^{(c)}(z)$ is a non-zero constant and $v_m = 0$; also on the circle $\{z\colon |z-a_m| = \varepsilon |a_m|/4\}$ we have

$$\log|h(z)| \geqq \log|f(z)| - k \log|a_m| - O(1)$$

and so, by Theorem C and the minimum principle $h(z)$ is large in $|z-a_m| \leqq 3\varrho_m$. From (3.34), $\psi(z)$ must be large on C_m, and again by the minimum principle, we conclude that $\sigma_m = 0$, which is a contradiction.

Thus $\sigma_m > 0$ implies that $\mu_m \geqq k+1$. Now, we may apply the above reasoning ((3.31) to (3.35)) to $\psi'(z)$ to conclude that

(3.36) $$\tau_m = \max\{\mu_m - (k+1), 0\}.$$

Also,

(3.37) $$\tau_m = \sigma_m - 1$$

since on C_m,

$$\psi'(z) = H(z)Q(z)\left(\frac{Q'(z)}{Q(z)} + \frac{H'(z)}{H(z)}\right)$$
$$= H(z)Q'(z)(1+o(1))$$

using (3.2), Claim 1, (3.28) and Lemmas 1 and 2. Combining (3.36) and (3.37) we see that if $\sigma_m > 1$ then

$$\tau_m = \mu_m - (k+1) \quad \text{and} \quad \sigma_m = \mu_m - k.$$

Claim 5 is now proved.

We may now conclude the proof of Theorem 1 by obtaining a contradiction. From Claims 4 and 5 we have, for large m, if $\sigma_m > 0$,

$$\sigma_m \leqq \mu_m - k \leqq \frac{3}{4} \mu_m$$

and thus

$$N\left(r, \frac{1}{\psi-1}\right) < \frac{7}{8} N\left(r, \frac{1}{f}\right) \tag{3.38}$$

for large r. But for large m, using Lemma 3 and Theorem C, and recalling that the annulus A_m as defined by (3.5) meets none of the discs D_n we have

$$\begin{aligned} N\left(r_m, \frac{1}{f}\right) &< m(r_m, f) + O(1) \\ &< m(r_m, \psi) + m\left(r_m, \frac{f}{\psi}\right) + O(1) \\ &< m(r_m, \psi-1) + O(\log r_m) \\ &< N\left(r_m, \frac{1}{\psi-1}\right) + O(\log r_m) \\ &< \frac{15}{16} N\left(r_m, \frac{1}{f}\right) \end{aligned}$$

using (3.38), which is impossible.

4. Proof of Theorem 2

We set

$$f(z) = \Pi_{n=1}^{\infty}\left(1 - \frac{z}{a_n}\right) \tag{4.1}$$

where $a_{n+1} > a_n$ and the a_n lie in the set $\{q, q^2, q^3, \ldots\}$ for some $q > 81$. We will make the actual choice of the a_n later.

Consider a large zero, z_0 say, of $f'(z)$, which must be real and positive ([9], p. 266). Suppose that a_N is the nearest zero of $f(z)$ to z_0, choosing the larger if z_0 is equidistant from two zeros of $f(z)$. Now

$$\frac{f'(z_0)}{f(z_0)} = \sum_{n=1}^{\infty} \frac{1}{z_0 - a_n} = 0. \tag{4.2}$$

But

$$\sum_{n>N} \frac{1}{a_n - z_0} \leqq \frac{1}{a_{N+1} - z_0} + \sum_{k=1}^{\infty} \frac{1}{(q^k-1)z_0} \leqq \frac{c_1}{z_0}, \tag{4.3}$$

say. Also, for $n < N$, $a_n < z_0$ and so

$$\sum_{n=1}^{N-1} \frac{1}{z_0 - a_n} > \frac{N-1}{z_0}. \tag{4.4}$$

Combining (4.2), (4.3) and (4.4) we obtain

$$\frac{1}{|z_0-a_N|} > \frac{N-1}{z_0} - \frac{c_1}{z_0}$$

and so

$$|z_0-a_N| < c_2 z_0 \left(n\left(z_0, \frac{1}{f}\right)\right)^{-1}, \tag{4.5}$$

say. Thus, given a small, positive ε, all large enough zeros of $f'(z)$ lie in the union of the discs $B(a_n, \varepsilon a_n/2)$, and, observing that $T(r,f)=O(\log r)^2$, we see from Theorem C that all large 1-points of $f'(z)$ lie in the union of the discs $B(a_n, \varepsilon a_n)$. Thus, for large n, the annuli $A_n=\{z\colon q^{1/4}a_n \leqq |z| \leqq q^{1/2}a_n\}$ are free of zeros and 1-points of $f'(z)$; also, by Lemma 4 there must exist level curves J_n each closing in A_n on which $|f(z)|=M(q^{1/4}a_n, f)$. But then (see [9], p. 122) $f(z)$ and $f'(z)$ have the same number of zeros in the region between J_n and J_{n+1}, and so, for large n, $f'(z)$ has exactly one zero, x_n say, and one 1-point, z_n say, in the disc $B(a_n, \varepsilon a_n)$.

Now, $f(a_n)=f(a_{n+1})=0$, and since

$$\frac{d}{dx}\left(\frac{f'(x)}{f(x)}\right) = -\sum_{k=1}^{\infty} \frac{1}{(x-a_k)^2} < 0$$

on (a_n, a_{n+1}) we see that $f(x)$ has exactly one maximum or minimum on each interval (a_n, a_{n+1}). Suppose now that n is large and that x_n is a maximum point for $f(x)$ on (a_{n-1}, a_{n+1}). If we have $x_n > a_n$ then $f'(a_n)>0$ and $f'((1-\varepsilon)a_n)>10$ (using Theorem C) and so $(1-\varepsilon)a_n<z_n<x_n$. Similarly, if $x_n<a_n$ then $f'((1-\varepsilon)a_n)>10$ and $(1-\varepsilon)a_n<z_n<x_n$. On the other hand, if x_n is a minimum point for $f(x)$ on (a_{n-1}, a_{n+1}) then $x_n<z_n<(1+\varepsilon)a_n$. Since $|f(x_n)|>a_n^{10}$, say, we obtain, by integration,

$$|f(z_n)| > z_n^2 \tag{4.6}$$

in either case.

Thus we have shown that for all large 1-points z_n of $f'(z)$, $z_n>0$ and $|f(z_n)|$ is large. But then $f'(z_n)/f(z_n)$ is small and

$$\frac{1}{|z_n-a_n|} \leqq \frac{o(1)}{z_n} + \left|\sum_{k\neq n} \frac{1}{z_n-a_k}\right|. \tag{4.7}$$

For $k<n$,

$$|z_n-a_k| > (1-q^{-1/2})\, z_n$$

while (compare 4.3)

$$\left|\sum_{k>n} \frac{1}{z_n-a_k}\right| \leqq \frac{c_3}{z_n},$$

say. Thus (4.5) yields

$$\frac{1}{|z_n-a_n|} \leqq \frac{c_4}{z_n} n\left(z_n, \frac{1}{f}\right) \leqq \frac{c_5}{a_n} n\left(a_n, \frac{1}{f}\right)$$

for some c_4, c_5, and setting $c_6=\max\{c_4, c_5\}$ we need only choose the points a_n from the set $\{q, q^2, q^3, \ldots\}$ in such a way as to ensure that $c_6 n(r, 1/f) \leqq h(r)$ for all r. The set $\{b_m\}$ is then the set of large a_n and z_n points, arranged in order of magnitude, and we have

$$b_{m+1}-b_m \geqq b_m(h(b_m))^{-1}.$$

This proves Theorem 2.

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