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ANGLES AND THE INNER RADIUS OF UNIVALENCY

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1. Introduction

Let A be a quasidisc. Denote by ϱ_A the density of the Poincaré metric in A, so normalized that $\varrho_H(x+iy)=1/(2y)$ for the upper half-plane H. The inner radius of univalency $\sigma_I(A)$ of A, introduced by Lehto [6], is the supremum of numbers a such that every locally injective meromorphic function f defined in A, which satisfies $\|S_f\|_A \leqq a$, is univalent. Here S_f stands for the Schwarzian derivative of f and the norm $\|S_f\|_A$ is the supremum of the numbers $|S_f(z)|/\varrho_A(z)^2$ for z in A. It follows from the definition and the transformation properties of the Schwarzian that the inner radius of univalency is invariant under Möbius transformations. It can be shown that $\sigma_I(A)$ equals the infimum of $\|S_f\|_A$ for those conformal maps f of A for which $f(A)$ is not a quasidisc.

It is known that $0<\sigma_I(A)\leqq 2$ for every quasidisc A and $\sigma_I(A)=2$ if and only if A is a disc or a half-plane. The only other cases in which the inner radii of univalency are known exactly are those of an angular domain $A=A_k=\{z \mid |\arg z|<k\pi/2\}$, $0<k<2$; for it

$$\sigma_I(A_k) = 2k(1-|1-k|), \tag{1}$$

[5, 2], and the one in which A is a domain bounded by a branch of a hyperbola [4]. Upper and lower estimates, based on (1), have been obtained for $\sigma_I(A)$ when A is one of the domains bounded by an ellipse [6, 3, 4]. In this paper, (1) together with elementary geometrical considerations is further utilized in order to determine $\sigma_I(A)$ when A is a domain bounded by a triangle or a regular n-gon, $n\geqq 3$. An upper estimate of the inner radius of univalency for general domains bounded by a curve possessing an angle is given in Section 4.

For any plane domain A we denote by A^* the complementary domain $\bar{C}\setminus A$.

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2. A majorization principle

We estimate $\sigma_I(A)$ by comparing A to angular domains in the interior or exterior of A. The general procedure is described by the following two lemmas.

Lemma 1. *Let A be a quasidisc and let $A\subset B_k$ where B_k is a domain Möbius equivalent to A_k for some k. If $0<k<1$ and a vertex v of B_k is on ∂A, then $\sigma_I(A)\leqq 2k^2$. If $1<k<2$ and there exist points z_1, z_2 in A and a Möbius transformation g satisfying $g(B_k)=A_k$ and $g(z_1)=e^{ik\pi/2}$, $g(z_2)=e^{-ik\pi/2}$, then $\sigma_I(A)\leqq 4k-2k^2$.*

Proof. In the case $0<k<1$ we let g, $g(v)=0$, be a Möbius transformation carrying B_k onto A_k. Set $f(z)=\log g(z)$. Then, by the quasicircle criterion of Ahlfors, $f(A)$ is no quasidisc, and $\sigma_I(A)\leqq\|S_f\|_A$. But, by the Schwarz lemma, $\sigma_A(z)\geqq\varrho_{B_k}(z)$ for $z\in A$, whence $\|S_f\|_A\leqq\|S_f\|_{B_k}=2k^2$. In the second case, there exists a conformal $f\colon A_k\to\bar{C}$ such that $\|S_f\|_{A_k}=4k-2k^2$ and $f(e^{ik\pi/2})=f(e^{-ik\pi/2})=\infty$ [2]. Then $f(g(A))$ is not a Jordan domain, and a reasoning similar to the one above shows that $\sigma_I(A)\leqq\|S_f\|_A$.

Lemma 2. *Let A be a quasidisc. If each two-element subset of A is contained in the closure of a quasidisc $B\subset A$ such that $\sigma_I(B)\geqq m$, then $\sigma_I(A)\geqq m$.*

Proof. Let an $\varepsilon>0$ be given. By definition, a locally injective $f\colon A\to C$ exists such that $\|S_f\|_A<\sigma_I(A)+\varepsilon$ and f is not univalent. Then $f(z)=f(w)$ for some $z\neq w$. There is a quasidisc B in A, such that $\{z, w\}\subset B$ and $\sigma_I(B)\geqq m$. Since either f is not univalent in B or $f(B)$ is not a quasidisc, $\|S_f\|_B\geqq\sigma_I(B)$. The monotony of the Poincaré metric again implies that $\|S_f\|_A\geqq\|S_f\|_B$. Hence $\sigma_I(A)>m-\varepsilon$, and the assertion follows.

3. Domains bounded by a triangle or a regular n-gon

Lemmas 1 and 2 combined with elementary geometrical considerations yield almost immediately

Theorem 1. *If T is the finite domain bounded by a triangle with smallest angle $k\pi$, then $\sigma_I(T)=2k^2$ and $\sigma_I(T^*)=4k-2k^2$.*

Proof. That $\sigma_I(T)\leqq 2k^2$ is an immediate consequence of Lemma 1. In the opposite direction, one observes that any two points z_1, z_2 in T are on the boundary of a triangle T' similar to T and lying inside T in such a way that one of the points z_1, z_2 is a vertex of T'. It is easy to convince oneself that a vertex and a point on the perimeter of a triangle can be joined in the triangle by two circular arcs (or an arc and a part of a side of the triangle) meeting at an angle at least as large as the smallest angle in the triangle. The desired conclusion follows from Lemma 2.

To prove the equality for $\sigma_I(T^*)$, we first observe that each pair of points in T^* is in one of the three angular domains determined by the exteriors of the angles of ∂T. Hence $\sigma_I(T^*)\geqq 4(2-k)-2(2-k)^2=4k-2k^2$. On the other hand, for each $k'>k$ one can draw two symmetric circular arcs in T meeting at the angle $k'\pi$ and tangent to ∂T at two points equidistant from the common points of the arcs. By Lemma 1, then, $\sigma_I(T^*)\leqq 4k'-2k'^2$, and the assertion follows.

By a similar argument one proves also

Theorem 2. *Let P_n be the finite domain bounded by a regular n-gon. Then $\sigma_I(P_n)=2((n-2)/n)^2$ and $\sigma_I(P_n^*)=2-8/n^2$.*

Proof. Upper estimates for both $\sigma_I(P_n)$ and $\sigma_I(P_n^*)$ are obtained exactly as in the proof of Theorem 1.

For a lower estimate of $\sigma_I(P_n)$, consider two points z_1 and z_2 in P_n. Let the line through these points intersect ∂P_n at w_1 and w_2. Observe that an appropriate homothety or translation applied to the circle inscribed in P_n shows that for any two vertices of ∂P_n there exist two circular arcs in P_n (or an arc and a side of P_n) joining the vertices and meeting each other in the angle $(1-2/n)\pi$. Assume first that w_1 is a vertex and that w_2 lies on a side whose endpoints are v_1 and v_2. Homotheties with center w_1 transform arcs joining w_1 to v_1 and v_2 into arcs joining w_1 to w_2 in P_n preserving the angles at w_1. Finally, let also w_1 be an interior point of a side, with endpoints v_3 and v_4. We can now join w_2 to v_3 and v_4 by pairs of arcs meeting at the angle $(1-2/n)\pi$. The arcs can be shown to be pairwise tangent to each other at w_2, and thus an angle-preserving homothety with center w_2 can be performed in order to obtain arcs joining w_2 to w_1 in P_n. By Lemma 2, $\sigma_I(P_n)\geqq 2(1-2/n)^2$.

For P_n^*, one observes that any two points in the domain are either in a half-plane contained in the domain or (as can be seen by applying two consequtive homotheties to the figure) on the boundary of a regular n-gon which contains P_n. We may thus consider only pairs of points w_1, w_2 on ∂P_n. Assume that w_1 is a vertex and w_2 is on a side with endpoints v_1 and v_2. Circular arcs through w_2, v_1, w_1 and w_2, v_2, w_1, respectively, meet the circle circumscribed around P_n at two points each and are contained in P_n^*. The larger angle between them is at most $(1+2/n)\pi$, as can be seen by comparing it to the angle between the circumscribed circle and the sides of ∂P_n. If w_1 is on a side with endpoints v_3 and v_4, an arc joining w_2, v_2, w_1 in P_n^* is found between the arcs joining w_2, v_2, v_3 and w_2, v_2, v_4. Similarly, an arc joining w_2, v_1 and w_1 is found; these arcs meet again with larger angle at most $(1+2/n)\pi$. The estimate follows from Lemma 2.

Remark. The results in Theorems 1 and 2, in the case of a finite domain, have been obtained also by D. Calvis [1].

4. An upper estimate for domains with a boundary angle

The majorization principle incorporated in Lemma 1 immediately yields an upper estimate for $\sigma_I(A)$ in the case of an angle less than π in ∂A. For our purposes the following definition of a boundary angle is appropriate: we say that ∂A possesses an angle $k\pi$ at z_0 if for every $\varepsilon \in (0, \min\{k, 2-k\})$ there exists a Möbius transformation g_ε such that $g_\varepsilon(z_0)=\infty$ and

$$A_{k-\varepsilon} \cap D^* \subset g_\varepsilon(A) \cap D^* \subset A_{k+\varepsilon} \cap D^*,$$

where D is the unit disc.

Theorem 3. [3] *Assume A has a boundary angle $k\pi$, $0<k<1$, at a boundary point z_0. Then $\sigma_I(A) \leqq 2k^2$.*

Proof. Assume $0<\varepsilon<1-k$. Without loss of generality, assume $z_0=0$, $A \cap D \subset A_{k+\varepsilon} \cap D$. There is a Möbius transformation g such that $g(A) \subset A_{k+\varepsilon}$, $g(0)=0$. By Lemma 1, $\sigma_I(g(A))=\sigma_I(A) \leqq 2(k+\varepsilon)^2$.

For a corresponding result concerning boundary angles exceeding π, it is useful to recall a few facts concerning the universal Teichmüller space T (see e.g. [5]). We let T be the space whose points are equivalence classes of complex dilatations μ defined in $\boldsymbol{C}$, zero in H^*, the equivalence of μ and ν being defined by $f_\mu|\boldsymbol{R}=f_\nu|\boldsymbol{R}$, where f_μ is a quasiconformal mapping of the plane with complex dilatation μ, normalized at three points on the real axis. The distance of μ and ν is the minimum of numbers $(1/2)\log K_{f_\mu \circ f_\nu}-1$. The space $T(1)$ of S_f, where f is conformal in H^* and maps H^* onto a quasidisc, endowed with the norm $\|S_f\|_{H^*}$, is homeomorphic to T; the homeomorphism Φ is the one that attaches S_{f_μ} to the equivalence class of μ. If f is a conformal map of the lower half-plane onto A, the inner radius of univalency of A is the distance of S_f from the boundary of $T(1)$.

Lemma 3. *Let A be a quasidisc. If $c<\sigma_I(A)$ and f is a conformal map of A such that $\|S_f\|_A \leqq c$, then $f(A)$ is a K-quasidisc, where K is bounded by a constant depending on A and c only.*

Proof. Let $g\colon H^* \to A$ be conformal. By the transformation rules of the Schwarzian, $\|S_f\|_A=\|S_{f\circ g}-S_g\|_{H^*}$. The preimage under Φ of the compact c-neighborhood of S_g is compact in T. It follows that $f\circ g$ has a K-quasiconformal extension to H with K bounded by a constant depending on A and c. Hence $f(A)$ is a K-quasidisc.

To prove the upper estimate for $\sigma_I(A)$ for domains having an angle larger than π in the boundary, we still need the following observation. For each $k \in (1, 2)$, let E_k be the domain $A_k \cap \{z | 1-z \in A_k\}$ and f_k the conformal map of A_k onto E_k taking 0 to 1, infinity to 0 and $e^{\pm ik\pi/2}$ to infinity [2]. Then f_k maps the arc of radius 1 and center 0 onto the imaginary axis.

Theorem 4. *Let A be a quasidisc possessing at $z_0 \in \partial A$ the angle $k\pi$, $1<k<2$. Then $\sigma_I(A) \leqq 4k-2k^2$.*

Proof. Assume that $\sigma_I(A)>4k-2k^2$. Let (k_n) be a decreasing sequence tending to k. By a suitable Möbius transformation we may arrange A to lie in A_{k_n} in such a way that there are points in A closer to the points $e^{\pm ik_n\pi/2}$ than, say, $2(k_n-k)$. Then the closure of $f_n(A)$ contains points $0, iy_n, -iy_n$ and x, $0<x\leqq 1$, and y_n tends to infinity together with n. Clearly, $f_n(A)$ cannot be a K-quasidisc with a fixed K for all n. On the other hand, $\|S_f\|_A\leqq\|S_{f_n}\|_{A_n}=4k_n-2k_n^2\leqq c<\sigma_I(A)$ for n large enough. This is a contradiction with Lemma 3.

Of course, estimates of $\sigma_I(A)$ based on local properties of ∂A are in general not very sharp. Let, for example, A be the exterior of a rectangle with sides 1 and $a>1$. By Theorem 4, $\sigma_I(A)\leqq 3/2$. On the other hand, one can inscribe in ∂A two symmetric arcs which meet at the angle $k\pi=2\pi-2\arctan(1/a)$. By Lemma 1, $\sigma_I(A)\leqq 4k-2k^2$, and $\sigma_I(A)$ tends to 0 as a grows to infinity.

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