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NORMALITY AND THE SHIMIZU—AHLFORS CHARATERISTIC FUNCTION

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1. Introduction

We shall propose criteria for a function meromorphic in the open unit disk D to be normal in the sense of O. Lehto and K. I. Virtanen [2] in terms of the Shimizu—Ahlfors characteristic function. The spirit of the proof is available to find criteria for a holomorphic function f in D to be Bloch in terms of mean values. Furthermore, we observe that if f is holomorphic and bounded, $|f|<1$ in D, then f is of hyperbolic Hardy class H^1 [4] in each disk of center $\neq 0$ internally tangent to ∂D.

For f meromorphic in $D=\{|z|<1\}$, the Shimizu—Ahlfors characteristic function of f is a nondecreasing function of ϱ, $0<\varrho\leqq 1$, defined by

$$T(\varrho, f) = \pi^{-1}\int_0^{\varrho} t^{-1}\left[\iint_{|z|<t} f^{\#}(z)^2\,dx\,dy\right] dt,$$

where $f^{\#}=|f'|/(1+|f|^2)$. Therefore, the Shimizu—Ahlfors characteristic function of $f(a+(1-|a|)z)$, $z\in D(a\in D)$, is

$$T(\varrho, a, f) \equiv \pi^{-1}\int_0^{\varrho} t^{-1}\left[\iint_{D(a,t)} f^{\#}(z)^2\,dx\,dy\right] dt, \quad 0<\varrho\leqq 1,$$

where $D(a,\varrho)=\{|z-a|<(1-|a|)\varrho\}$, $a\in D, 0<\varrho\leqq 1$. In particular, $T(\varrho,f)=T(\varrho,0,f)$, $0<\varrho\leqq 1$.

A necessary and sufficient condition for f meromorphic in D to be normal is that

$$\sup\nolimits_{z\in D}(1-|z|^2)f^{\#}(z)<\infty.$$

Theorem 1. *For f meromorphic in D, the following are mutually equivalent.*

(I) *f is normal in D.*

(II) *For each $c, 0<c<1$, we have*

$$\sup_{c\leqq|a|<1} T(1,a,f)<\infty. \tag{1.1}$$

(III) *There exist $c, 0<c<1$, and $\varrho, 0<\varrho<1$, such that*

$$\sup_{c\leqq|a|<1} T(\varrho,a,f)<\infty. \tag{1.2}$$

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Thus, f is normal in D if and only if f is of bounded characteristic "uniformly" in each disk $D(a, 1)$ when a is near ∂D.

What is the holomorphic analogue of Theorem 1? We shall give criteria for f holomorphic in D to be Bloch, namely,

$$\sup_{z\in D}(1-|z|^2)|f'(z)|<\infty,$$

in terms of the mean values. For u subharmonic in D we set

$$M(\varrho, u)=\frac{1}{2\pi}\int_0^{2\pi} u(\varrho e^{it})\,dt, \quad 0<\varrho<1.$$

Replacing u by $u(a+(1-|a|)z)$, $z\in D$, we set

$$M(\varrho, a, u)=\frac{1}{2\pi}\int_0^{2\pi} u\big(a+(1-|a|)\varrho e^{it}\big)\,dt.$$

Then, $M(\varrho, u)=M(\varrho, 0, u)$. We further set $M(1, a, u)=\lim_{\varrho\to 1} M(\varrho, a, u)$, so that $M(1, u)=M(1, 0, u)$.

Theorem 2. *For f holomorphic in D, the following are mutually equivalent.*

(I) *f is Bloch.*

(II) *For each c, $0<c<1$, we have*

$$\sup_{c\leqq|a|<1} M(1, a, |f-f(a)|^2)<\infty. \tag{1.3}$$

(III) *There exist c, $0<c<1$, and ϱ, $0<\varrho<1$, such that*

$$\sup_{c\leqq|a|<1} M(\varrho, a, \log|f-f(a)|)<\infty. \tag{1.4}$$

Thus, f is Bloch if and only if f is of Hardy class H^2 "uniformly" in each disk $D(a, 1)$ when a is near ∂D.

We note that (1.4) is weaker than

$$\sup_{c\leqq|a|<1} M(\varrho, a, |f-f(a)|^2)<\infty. \tag{1.5}$$

As is observed by Lehto [1] (see also [4]),

$$M(\varrho, a, |f-f(a)|^2)=2T_1(\varrho, a, f), \tag{1.6}$$

where

$$T_1(\varrho, a, f)=\pi^{-1}\int_0^{\varrho} t^{-1}\Big[\iint_{D(a,t)}|f'(z)|^2\,dx\,dy\Big]\,dt, \quad 0<\varrho\leqq 1;$$

this is a consequence of the Green formula

$$t\frac{d}{dt}M(t, a, |f-f(a)|^2)=\frac{1}{2\pi}\iint_{D(a,t)}\Delta(|f-f(a)|^2)\,dx\,dy,$$

together with $\Delta(|f-f(a)|^2)=4|f'|^2$. One can now recognize that (1.3) ((1.5), respectively) is an analogue of (1.1)((1.2)).

Finally, let $\sigma(z, w)=\tanh^{-1}|(z-w)/(1-\bar{w}z)|$ be the non-Euclidean hyperbolic distance of z and w in D. Let f be holomorphic and bounded, $|f|<1$, in D. As will be observed later, we have

$$\left|M(\varrho, a, \sigma(f, f(a)))-T_2(\varrho, a, f)\right| \leqq \log 2, \quad 0<\varrho<1, \tag{1.7}$$

where

$$T_2(\varrho, a, f)=\pi^{-1} \int_0^\varrho t^{-1}\left[\iint_{D(a, t)} f^*(z)^2 \, dx \, dy\right] dt,$$

with $f^*=|f'|/(1-|f|^2)$. Since by the Schwarz—Pick lemma

$$(1-|z|^2) f^*(z) \leqq 1, \quad z \in D,$$

we shall be able to prove

Theorem 3. *For f holomorphic and bounded, $|f|<1$, in D, and for each c, $0<c<1$, we have*

$$\sup_{c \leqq|a|<1} M(1, a, \sigma(f, f(a))) \leqq \frac{1}{\sqrt{c}(1+c)}+\log 2. \tag{1.8}$$

The condition (1.8) reads that f is of hyperbolic Hardy class H^1 [4] "uniformly" in each $D(a, 1)$ when a is near ∂D.

2. Proof of Theorem 1

Parts of the following lemmas will be of use.

Lemma 1. *The hyperbolic area*

$$S(a, \varrho)=\iint_{D(a, \varrho)}(1-|z|^2)^{-2} \, dx \, dy$$

of $D(a, \varrho)$, $a \neq 0$, $0<\varrho<1$, satisfies

$$\frac{2 \pi}{(1+|a|)(3+|a|)} \frac{\varrho^2}{(1-\varrho^2)^{1/2}} \leqq S(a, \varrho) \leqq \frac{\pi}{\sqrt{|a|}(1+|a|)} \frac{\varrho^2}{(1-\varrho^2)^{1/2}}. \tag{2.1}$$

Proof. For $w \in D$, $0<r<1$, we set

$$\Delta(w, r)=\{z \in D ;\ |z-w|/|1-\bar{w} z|<r\}.$$

With the aid of the well-known facts (see for example [3, p. 511]) we obtain $D(a, \varrho)= \Delta(p, R)$, where

$$2R=A-(A^2-4)^{1/2}, \quad A=\{1+|a|+(1-|a|) \varrho^2\}/\varrho$$

(we do not need the expression of p). Since

$$S \equiv S(a, \varrho) = \pi R^2/(1-R^2)$$

[3, p. 509], it follows that

$$\frac{2\pi\varrho^2}{(1-\varrho^2)^{1/2}} S^{-1} = \sqrt{Q}\,[1+|a|+(1-|a|)\varrho^2+(Q(1-\varrho^2))^{1/2}],$$

where $Q=(1+|a|)^2-(1-|a|)^2\varrho^2$. We thus obtain (2.1).

Lemma 2. *For f meromorphic in D the following hold.*
(a) *f is normal in D if and only if there exist c, $0<c<1$, and r, $0<r<1$, such that*

$$\sup_{c\leqq|w|<1} \iint_{\Delta(w,r)} f^{\#}(z)^2\, dx\, dy < \pi.$$

(b) $\lim_{|z|\to1}(1-|z|^2)f^{\#}(z)=0$ *if and only if there exists r, $0<r<1$, such that*

$$\lim_{|w|\to1} \iint_{\Delta(w,r)} f^{\#}(z)^2\, dx\, dy = 0.$$

This is [5, Lemma 3.2]; our Lemma 2 is worded somewhat differently in (a), but the proof is the same as in [5, p. 354] because $(1-|z|^2)f^{\#}(z)$ is continuous in D.

We begin with the proof of (I)⇒(II) in Theorem 1.

There exists $K>0$ such that

$$f^{\#}(z)^2 \leqq K(1-|z|^2)^{-2}, \quad z\in D,$$

so that, by (2.1) with $c\leqq|a|<1$, we have

$$\iint_{D(a,\varrho)} f^{\#}(z)^2 dx\, dy \leqq \frac{\pi K}{\sqrt{c}(1+c)}\,\frac{\varrho^2}{(1-\varrho^2)^{1/2}}.$$

Consequently, $T(1, a, f)\leqq K/\{\sqrt{c}(1+c)\}$ for $c\leqq|a|<1$.

Since (II)⇒(III) is trivial, what remains for us to prove is (III)⇒(I). The function $X(z; a, t)$ is defined for $z\in D$ to be one if $z\in D(a, t)$ and zero otherwise. Then

$$T(\varrho, a, f) = \pi^{-1}\iint_D \left[\int_0^{\varrho} t^{-1}X(z;\ a, t)\, dt\right] f^{\#}(z)^2\, dx\, dy$$
$$= \pi^{-1}\iint_{D(a,\varrho)} f^{\#}(z)^2 \log\frac{\varrho(1-|a|)}{|z-a|}\, dx\, dy.$$

Letting μ be the supremum in (1.2) we choose δ such that $0<\delta<e^{-2\mu}$. Since

$$\Delta(a, \varrho\delta/3) \subset D(a, \varrho\delta) \subset D(a, \varrho),$$

it then follows that

$$\mu \geqq \pi^{-1} \iint_{D(a,\varrho\delta)} f^{\#}(z)^2 \log \frac{\varrho(1-|a|)}{|z-a|}\, dx\, dy$$

$$\geqq \pi^{-1}(-\log \delta) \iint_{D(a,\varrho\delta)} f^{\#}(z)^2\, dx\, dy$$

$$\geqq \pi^{-1}(-\log \delta) \iint_{\Delta(a,\varrho\delta/3)} f^{\#}(z)^2\, dx\, dy,$$

or,

$$\iint_{\Delta(a,\varrho\delta/3)} f^{\#}(z)^2\, dx\, dy < \pi/2 < \pi.$$

It follows from the "if" part in (a) of Lemma 2 that f is normal in D. This completes the proof of Theorem 1.

It is now an easy exercise to prove

Theorem 4. *For f meromorphic in D, the following are mutually equivalent.*

(I) $\lim_{|z|\to 1} (1-|z|^2) f^{\#}(z) = 0.$

(II) $\lim_{|a|\to 1} T(1, a, f) = 0.$

(III) *There exists* ϱ, $0<\varrho<1$, *such that*

$$\lim_{|a|\to 1} T(\varrho, a, f) = 0.$$

As is observed in Section 1, there is a simple relation (1.6) for holomorphic functions. This is also the case for meromorphic functions.

A meromorphic function f in D can be expressed as $f=f_1/f_2$, where f_1 and f_2 are holomorphic with no common zero in D. Then $F=\log(|f_1|^2+|f_2|^2)$ is subharmonic in D with $\Delta F=4f^{\#2}$. With the aid of the Green formula

$$r\frac{d}{dr} M(r, a, F-F(a)) = 2\pi^{-1} \iint_{D(a,r)} f^{\#}(z)^2\, dx\, dy,$$

one can obtain

$$M(\varrho, a, F-F(a)) = 2T(\varrho, a, f), \quad 0<\varrho\leqq 1.$$

3. Proof of Theorem 2

The proof of (I)$\Rightarrow$(II) is similar to that of Theorem 1 in view of (1.6).

Since (II)$\Rightarrow$(III) is trivial, we shall show that (III)$\Rightarrow$(I). There exists a holomorphic function g on $\bar{D}$ such that

$$zg(z) = f(a+(1-|a|)\varrho z) - f(a), \quad z\in \bar{D},$$

so that $g(0)=(1-|a|)\varrho f'(a)$. Since $\log|g|$ is subharmonic on $\bar{D}$, it follows that

$$\log|g(0)| \leqq M(1, \log|g|) = M(\varrho, a, \log|f-f(a)|) \leqq K,$$

where K is the supremum in (1.4) and $c \leqq |a| < 1$. Consequently,

$$\sup_{c \leqq |a| < 1} (1-|a|^2)|f'(a)| \leqq 2e^K/\varrho,$$

which, together with the continuity of $(1-|z|^2)|f'(z)|$ in D, shows that f is Bloch. This completes the proof of Theorem 2.

In a similar manner we can prove

Theorem 5. *For f holomorphic in D, the following are mutually equivalent.*

(I) $\lim_{|z|\to 1} (1-|z|^2)|f'(z)| = 0.$

(II) $\lim_{|a|\to 1} M(1, a, |f-f(a)|^2) = 0.$

(III) *There exists* ϱ, $0<\varrho<1$, *such that*

$$\lim_{|a|\to 1} M(\varrho, a, \log|f-f(a)|) = -\infty.$$

For each p, $0<p<\infty$, we have

$$\exp[pM(\varrho, a, \log|f-f(a)|)] \leqq M(\varrho, a, |f-f(a)|^p)$$
$$\leqq M(1, a, |f-f(a)|^p),$$

so that, setting $p=2$, one can conclude that (II) implies (III).

4. Proof of Theorem 3

First of all, if f is holomorphic and bounded, $|f|<1$, in D, then both

$$\log \sigma(f, 0) \quad \text{and} \quad \log\left(\log \frac{1}{1-|f|^2}\right)$$

are subharmonic in D; see [4]. Setting

$$\varphi_w(z) = (z-w)/(1-\bar{w}z), \quad z,\ w \in D,$$

we have

$$-\log(1-|\varphi_w \circ f|^2) \leqq 2\sigma(f, w) \leqq -\log(1-|\varphi_w \circ f|^2) + \log 4,$$

because $-\log(1-x^2) \leqq 2\sigma(x, 0) \leqq -\log(1-x^2) + \log 4$ for $0 \leqq x < 1$. Since $-\Delta \log(1-|\varphi_w \circ f|^2) = 4f^{*2}$, $w \in D$, it follows that

$$r\frac{d}{dr}(Mr, a, -(1/2)\log(1-|\varphi_{f(a)} \circ f|^2)) = \pi^{-1} \iint_{D(a, r)} f^*(z)^2\, dx\, dy.$$

We thus obtain

$$M(\varrho, a, -(1/2)\log(1-|\varphi_{f(a)} \circ f|^2)) = T_2(\varrho, a, f),$$

whence (1.7). Theorem 3 is now easily proved.

We finally propose

Theorem 6. *For f holomorphic and bounded,* $|f|<1$, *in D, the following are mutually equivalent.*

(I) $\lim_{|z|\to 1}(1-|z|^2)f^*(z)=0.$

(II) $\lim_{|a|\to 1} M(1, a, -\log(1-|\varphi_{f(a)}\circ f|^2))=0.$

(III) *There exists* ϱ, $0<\varrho<1$, *such that*

$$\lim_{|a|\to 1} M(\varrho, a, \log\{-\log(1-|\varphi_{f(a)}\circ f|^2)\})=-\infty.$$

It is easy to see that (II) implies (III). The detailed proof of (III)⇒(I) must be given.

There exists a holomorphic function g on $\bar{D}$ such that

$$zg(z)=(\varphi_{f(a)}\circ f)(a+(1-|a|)\varrho z), \quad z\in\bar{D},$$

so that $|g|<1$ in D, and $|g(0)|=(1-|a|)\varrho f^*(a)$. Since

$$|g(0)|^2 \leqq -\log(1-|g(0)|^2),$$

it follows that

$$2\log|g(0)| \leqq M(1, \log\{-\log(1-|g|^2)\})$$
$$= M(\varrho, a, \log\{-\log(1-|\varphi_{f(a)}\circ f|^2)\}),$$

whence $\lim_{|a|\to 1}(1-|a|)f^*(a)=0$.

References

[1] Lehto, O.: A majorant principle in the theory of functions. - Math. Scand. 1, 1953, 5—17.

[2] Lehto, O., and K. I. Virtanen: Boundary behaviour and normal meromorphic functions. - Acta Math. 97, 1957, 47—65.

[3] Tsuji, M.: Potential theory in modern function theory. - Maruzen, Tokyo, 1959.

[4] Yamashita, S.: Hyperbolic Hardy class H^1. - Math. Scand. 45, 1979, 261—266.

[5] Yamashita, S.: Functions of uniformly bounded characteristic. - Ann. Acad. Sci. Fenn. Ser. A I Math. 7, 1982, 349—367.

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