Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 11, 1986, 275—294

# ON THE REAL ZEROS OF SOLUTIONS OF $f''+A(z)f=0$ WHERE $A(z)$ IS ENTIRE

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## 1. Introduction

If $A(z)$ is entire then it is well-known that all the solutions of the second-order linear differential equation

$$f''+A(z)f=0 \tag{1.1}$$

are entire. In a recent paper, Hellerstein, Shen, and Williamson proved the following result [9, Theorem 3]: If $A(z)$ is a nonconstant polynomial, then the differential equation (1.1) cannot posses wo linearly independent solutions each having only real zeros. This result raises a natural question, namely to determine the frequency of nonreal zeros of solutions of equation (1.1). Our first result addresses this question. We prove:

Theorem 1. *Let $A(z)$ be a polynomial of degree $n\geqq 1$, and let $f_1, f_2$ be any two linearly independent solutions of equation* (1.1). *Then at least one of $f_1, f_2$ has the property that its sequence of nonreal zeros has exponent of convergence equal to* $(n+2)/2$.

Throughout this paper we will assume that the reader is familiar with the fundamental results and standard notations $m(r,f)$, $N(r,f)$, $T(r,f)$, $N(r,f,c)$, $\delta(c,f)$, etc. of R. Nevanlinna's theory of meromorphic functions (see [8] and [12]).

We make two remarks concerning Theorem 1. First, it is well-known that the order of any nontrivial solution of (1.1) is $(n+2)/2$ (see Theorem 6 in Section 4 below). Second, the basic idea behind the proof of Theorem 1 involves considering the product $E=f_1 f_2$ of the solutions. Bank and Laine [1, Theorem 1] showed that the exponent of convergence of the zeros of $E$ is $(n+2)/2$, and that [1, p. 354] $\delta(0, E)=0$. Then it will follow from results of Edrei, Fuchs, and Hellerstein in [4] that $E$ cannot have most of its zeros being real.

The result presented in Theorem 1 suggests an investigation into those equations (1.1) where $A(z)$ is a polynomial of degree $n\geqq 1$, which possess solutions that are

*) The research of the author was supported in part by the National Science Foundation (MCS-8200497).

doi:10.5186/aasfm.1986.1105

exceptional in the sense that their nonreal zero-sequence has exponent of convergence less than $(n+2)/2$. Of course, there are trivial examples where solutions have only finitely many zeros, and such examples can occur for every even degree $n$ by considering $f=qe^p$ where $p$ and $q$ are polynomials with degree $(p)=(n+2)/2$. Leaving such examples aside, we seek examples where the exceptional solutions have infinitely many zeros. To the author's knowledge, the only known examples occur when $n=1$ (from Airy's differential equation) and when $n=4$ (from an equation studied by Titchmarsh), which we discuss in Examples 1 and 2 in Section 8 below. Our second theorem shows that such examples cannot occur in any of the degrees 2, 6, 10, .... We prove:

Theorem 2. *Let $A(z)$ be a polynomial of degree $n$ where $n=2+4k$ for some nonnegative integer $k$. Let $f\not\equiv 0$ be a solution of equation* (1.1). *Then either $f$ has only finitely many zeros, or the exponent of convergence of the nonreal zero-sequence of $f$ is $(n+2)/2$.*

For the proof of Theorem 2, we assume that the conclusion does not hold, i.e. that such an exceptional solution $f$ exists, and we apply the results in Section 4 to obtain that $N(r,f,0)=(1+o(1))4(n+4)^{-1}T(r,f)$ as $r\to\infty$. We then show that this equation cannot hold by using [1, Theorem 1] and a precise estimate of Shea [15, Corollary 2.1] for the Valiron deficiency (see Section 2) of the value zero for entire functions of certain finite orders having only real zeros.

In the cases when the degree of a polynomial $A(z)$ is either odd or a nonzero multiple of four, the results in Section 4 will easily give the exact asymptotic growths of both $N(r,f,0)$ and $T(r,f)$ as $r\to\infty$ for any exceptional solution $f$ of (1.1) (see Theorems 7 and 8 in Section 7), and it turns out that

$$\lim_{r\to\infty}\frac{N(r,f,0)}{T(r,f)}=\begin{cases}2/(n+3) & \text{if } n \text{ is odd,}\\ 4/(n+4) & \text{if } n \text{ is a nonzero multiple of 4.}\end{cases}$$

We observe that $N(r,f,0)=(1/2+o(1))T(r,f)$ as $r\to\infty$ for all the known exceptional solutions $f$ in Examples 1 and 2 in Section 8.

The results in Section 4 have independent interest. Theorem 5 in Section 4 (which was proved jointly by Steven B. Bank, Simon Hellerstein, John Rossi, and the author) gives useful expressions for the growths of both $T(r,f)$ and $N(r,f,0)$ as $r\to\infty$ when $f\not\equiv 0$ is any solution of equation (1.1) where $A(z)\not\equiv 0$ is a polynomial. It turns out that both $T(r,f)$ and $N(r,f,0)$ always have perfectly regular growth. We will obtain Theorem 5 and two corollaries by combining results of Hille, F. Nevanlinna, and Fuchs.

We now turn our attention to the problem of estimating the frequency of the real zeros of solutions of the general equation (1.1), where $A(z)$ is an entire function. If $A(z)$ is real (i.e. real on the real axis) then it is easy to see (Lemma 5 in Section 4) that any solution $f$ of (1.1) that possesses a real zero must be a constant multiple of a real solution $f_1$ of (1.1). In this case, very powerful techniques (e.g. the Sturm comparison theorem [10], and a formula due to Wiman [18] — Lemma 4 in Section 4) already exist to estimate the frequency of real zeros of $f_1$, and hence of $f$ (see Corollary 3 in Section 4). In the case when $A(z)$ is not real, we will use the classical method of the Green's transform (see [11, pp. 508—509]) to prove the following two theorems which address both the polynomial case and the transcendental case:

Theorem 3. *Let $A(z)$ be a nonreal polynomial of degree $n$, and set*

$$F(z) = \frac{A(z)-\overline{A(\bar{z})}}{2i}.$$

*Let $p$ denote the number of distinct real zeros of the polynomial $F(z)$. Then for any solution $f\not\equiv 0$ of* (1.1), *the number of real zeros $k$ of $f$ is finite, and we have $k \leqq p+1$. In particular, we have $k \leqq n+1$.*

Theorem 4. *Let $A(z)$ be an entire transcendental function of order $\sigma(A)$ where $0 \leqq \sigma(A) \leqq \infty$, and assume that $A(z)$ is not real. Let $f\not\equiv 0$ be a solution of* (1.1), *and let $\lambda_R(f)$ denote the exponent of convergence of the sequence of real zeros of $f$. Then we have*

$$\lambda_R(f) \leqq \sigma(A). \tag{1.2}$$

*More precisely, we have as $r\to\infty$,*

$$N_R(r, 1/f) \leqq 2T(r, A)+O(\log r) \tag{1.3}$$

*where $N_R(r, 1/f)$ refers to only the real zeros in $N(r, 1/f)$.*

Theorem 3 is sharp in the sense that in the situation of Theorem 3 we can have $k=p+1=1$ because $f(z)=z\exp(-iz^2)$ satisfies the equation $f''+(6i+4z^2)f=0$. The estimate (1.2) in Theorem 4 is sharp in the sense that in the situation of Theorem 4 we can have $\lambda_R(f)=\sigma(A)$ by Example 5 in Section 8.

This paper is organized as follows. In Section 2 we give notation that is used in the paper. In Section 3 we prove Theorems 3 and 4. In Section 4 we prove Theorem 5 mentioned above plus some related results. In Section 5 we prove Theorem 2, in Section 6 we prove Theorem 1, and in Section 7 we prove Theorems 7 and 8 mentioned above. In Section 8 we give several examples concerning this theory. Finally, in Section 9 we consider the case of equation (1.1) when $A(z)$ is a transcendental entire periodic function.

The author takes pleasure in thanking Steven B. Bank for numerous important contributions to this paper. The author would also like to thank Robert P. Kaufman for some valuable conversations.

## 2. Notation

As stated in Section 1 we will assume that the reader is familiar with the usual notations of the Nevanlinna theory. We will also use

$$\Delta(c, g) = 1 - \varliminf_{r\to\infty} \frac{N(r, g, c)}{T(r, g)}$$

which denotes the Valiron deficiency of the value $c$ for a nonconstant meromorphic function g.

For a nonconstant meromorphic function $f$, we will use the following notations:

1. $\sigma(f)$ will denote the order of $f$.
2. $\lambda(f)$ will denote the exponent of convergence of the sequence of zeros of $f$.
3. $\lambda_R(f)$ will denote the exponent of convergence of the sequence of real zeros of $f$.
4. $\lambda_{NR}(f)$ will denote the exponent of convergence of the sequence of nonreal zeros of $f$.
5. $n_R(r,f,0)$ and $N_R(r,f,0)$ will refer only to the real zeros of $f$ in $n(r,f,0)$ and $N(r,f,0)$ respectively. (Note: $n(r,f,0)$ will denote the number of zeros of $f$ in $|z|\leqq r$.)

## 3. Proofs of Theorems 3 and 4

We will use the Green's transform to prove these results.

Lemma 1. *Let $A(z)$ be an entire function which is not real, and set*

$$F(z) = \frac{A(z)-\overline{A(\bar{z})}}{2i}. \tag{3.1}$$

*If $f\not\equiv 0$ is any solution of equation* (1.1), *then between any two consecutive real zeros of $f$, there must be a real zero of $F$.*

*Proof.* Suppose that $x_1$ and $x_2$ $(x_1<x_2)$ are consecutive real zeros of $f$. Then by taking the imaginary part of the Green's transform [11, p. 509] of equation (1.1) we obtain

$$\int_{x_1}^{x_2} \mathrm{Im}(A(x))|f(x)|^2\,dx = 0. \tag{3.2}$$

Now if $\mathrm{Im}\,(A(x))\equiv 0$ on the segment $x_1\leqq x\leqq x_2$ then it follows that $\mathrm{Im}\,(A(x))\equiv 0$ for all real $x$, which contradicts the hypothesis that $A(z)$ is nonreal. Thus $\mathrm{Im}\,(A(x))\not\equiv 0$ on $x_1\leqq x\leqq x_2$. Then from (3.2), $\mathrm{Im}\,(A(x))$ must change sign on $x_1<x<x_2$. Since $\mathrm{Im}\,(A(x))\equiv F(x)$ for all real $x$ where $F$ is given by (3.1), the assertion follows.

Theorem 3 follows immediately from Lemma 1.

*Proof of Theorem* 4. If $f$ has $k$ real zeros in some interval $-r\leqq x\leqq r$, then by Lemma 1, $F(z)$ in (3.1) must have at least $k-1$ real zeros in $-r<x<r$. Thus

$$n_R(r, f, 0) \leqq n_R(r, F, 0)+1 \leqq n(r, F, 0)+1.$$

Hence as $r\to\infty$,

$$N_R(r, f, 0) \leqq N(r, F, 0)+O(\log r). \tag{3.3}$$

By using Jensen's formula, (3.1), and the observation that $T(r, A(z))=T(r, \overline{A(\bar{z})})$, we obtain

$$N(r, F, 0) \leqq T(r, F)+O(1) \leqq 2T(r, A)+O(1) \tag{3.4}$$

as $r\to\infty$. By combining (3.3) and (3.4) we obtain (1.2) and (1.3), and Theorem 4 is thus proven.

## 4. The growth properties of solutions of (1.1)

Let $f\not\equiv 0$ be a solution of equation (1.1) where $A(z)\not\equiv 0$ is a polynomial. We will now derive expressions for $T(r,f)$ and $N(r,f,0)$ as $r\to\infty$, which show, among other things, that the growths of both $T(r,f)$ and $N(r,f,0)$ are perfectly regular (see Theorem 5 below). These asymptotic expressions for $T(r,f)$ and $N(r,f,0)$ will depend on both $A(z)$ and on $f$. In contrast, the asymptotic growth of $\log M(r,f)$, where $M(r,f)$ is the maximum modulus function, depends only on $A(z)$ and not on $f$ (see Theorem 6 below). In this section we will prove these results and some related results.

Hille applied his theory of asymptotic integration together with the Liouville transformation to obtain many basic asymptotic properties of the nontrivial solutions $f$ of the equation $f''+Q(z)f=0$ where $Q(z)\not\equiv 0$ is a polynomial (see Chapter 7.4 of [10]). One of these properties is the following result.

Lemma 2. ([10, pp. 340—342]) *Consider the equation*

$$f''+Q(z)f=0 \tag{4.1}$$

*where $Q(z)=q_nz^n+\ldots+q_0$ is a polynomial of degree $n$ and $q_n>0$ is a positive number. Set $\alpha=(n+2)/2$. For $0<\delta<\pi/2\alpha$ and $j=0, 1, \ldots, n+1$, let $S_j(\delta)$ denote the sector*

$$j\pi/\alpha+\delta \leqq \arg z \leqq (j+2)\pi/\alpha-\delta. \tag{4.2}$$

*If $f\not\equiv 0$ is a solution of equation* (4.1) *that has infinitely many zeros in some sector $S_j(\delta)$, then for any $\varepsilon>0$, all but finitely many of these zeros must lie in the sector $W_j(\varepsilon)$ given by*

$$(j+1)\pi/\alpha-\varepsilon < \arg z < (j+1)\pi/\alpha+\varepsilon, \tag{4.3}$$

*and furthermore, as* $r\to\infty$,

$$n_j(r, f, 0) = (1+o(1))\frac{\sqrt{q_n}}{\pi\alpha}r^{\alpha}, \tag{4.4}$$

*where* $n_j(r,f,0)$ *refers only to those zeros of* $f$ *in* $W_j(\varepsilon)$.

Now suppose that $f\not\equiv 0$ is a solution of the more general equation (1.1) where $A(z)=a_nz^n+\ldots+a_0$ is any polynomial of degree $n$ with $a_n\neq 0$. If $c$ is any constant that satisfies

$$c^{n+2} = a_n, \tag{4.5}$$

and $F(z)=f(z/c)$, then $F$ satisfies an equation of the form (4.1) with $q_n=1$. Hence we can easily transform the result in Lemma 2 to the more general equation (1.1). It will be convenient to make the following definition.

Definition. Let $f\not\equiv 0$ be a solution of equation (1.1) where $A(z)=a_nz^n+\ldots+a_0$ is a polynomial of degree $n$ with $a_n\neq 0$, and for $\varepsilon>0$ and $j=0, 1, \ldots, n+1$, let $V_j(\varepsilon)$ denote the sector

$$\left|\arg z-\frac{2\pi(j+1)-\arg(a_n)}{n+2}\right| < \varepsilon. \tag{4.6}$$

Let $J(f)$ denote the set of all $j\in\{0, 1,\ldots, n+1\}$ with the property that for some $\varepsilon>0$, $f$ has only finitely many zeros in $V_j(\varepsilon)$. We will call the cardinal number of $J(f)$ the *shortage* of $f$, and denote it by $p(f)$. Hence $0\leqq p(f)\leqq n+2$.

Remark. The collection of sectors $W_0(\varepsilon)$, $W_1(\varepsilon), \ldots, W_{n+1}(\varepsilon)$ given by (4.3) are in a one-to-one correspondence with the collection of sectors $V_0(\varepsilon)$, $V_1(\varepsilon), \ldots,$ $V_{n+1}(\varepsilon)$ given by (4.6) under the transformation $z\to z/c$ where $c$ is a constant that satisfies (4.5).

We will now prove the following theorem which (as mentioned in Section 1) was proved jointly by Bank, Hellerstein, Rossi, and the author.

Theorem 5. *Let* $f\not\equiv 0$ *be any solution of equation* (1.1) *where* $A(z)=a_nz^n+\ldots+a_0$ *is a polynomial of degree n with* $a_n\neq 0$, *and set* $\alpha=(n+2)/2$. *Then* $p(f)$ *is an even number and as* $r\to\infty$, *the following three formulas hold:*

$$n(r, f, 0) = (1+o(1))\frac{2\alpha-p(f)}{\pi\alpha}\sqrt{|a_n|}\,r^{\alpha}, \tag{4.7}$$

$$N(r, f, 0) = (1+o(1))\frac{2\alpha-p(f)}{\pi\alpha^2}\sqrt{|a_n|}\,r^{\alpha}, \tag{4.8}$$

$$T(r, f) = (1+o(1))\frac{4\alpha-p(f)}{2\pi\alpha^2}\sqrt{|a_n|}\,r^{\alpha}. \tag{4.9}$$

*Hence*

$$\delta(0, f) = \Delta(0, f) = \frac{p(f)}{4\alpha - p(f)}.$$

For the proof of Theorem 5, we shall use Lemma 2 and the following lemma which comes from work of Hille, F. Nevanlinna, and Fuchs.

Lemma 3. *Let $f$ and $g$ be linearly independent solutions of equation* (1.1) *where $A(z)=a_n z^n + \ldots + a_0$ is a polynomial of degree $n$ with $a_n \neq 0$. Set $h=g/f$ and $\alpha=(n+2)/2$. Then the following hold:*

1. $$T(r, h) = (1+o(1)) \frac{2\sqrt{|a_n|}}{\pi\alpha} r^{\alpha} \quad as \quad r \to \infty.$$

2. *There exists at most $n+2$ distinct values $b_1, b_2, \ldots, b_m$ in the extended complex plane with the following properties:*

(i) *For each $k$, $\delta(b_k, h) = \Delta(b_k, h)$, and $\delta(b_k, h)$ is a positive integral multiple of $1/\alpha$.*

(ii) $\sum_{k=1}^{m} \delta(b_k, h) = 2.$

(iii) *If $b \neq b_k$ for $k=1, \ldots, m$, then as $r \to \infty$, $N(r, h, b) = (1+o(1)) T(r, h)$.*

*Proof.* From formula (2.8) on page 9 of [5], we have that as $r \to \infty$,

$$T(r, h) = (1+o(1)) \frac{2B}{\pi} r^{\alpha},$$

where the constant $B$ can be determined to be equal to $\sqrt{|a_n|}/\alpha$ from pages 6 and 7 of [5]. Thus part 1 holds. Part 2 can be found on pages 8—10 of [5].

*Proof of Theorem* 5. Suppose first that $a_n > 0$. From Lemma 2, if some sector $S_j(\delta)$ in (4.2) contains infinitely many zeros of $f$, then for any $\varepsilon > 0$, all but finitely many of these zeros will lie in the sector $W_j(\varepsilon)$ given by (4.3), and the contribution to $n(r, f, 0)$ from these zeros will be

$$(1+o(1)) \frac{\sqrt{a_n}}{\pi\alpha} r^{\alpha}$$

as $r \to \infty$, from (4.4). Since there are exactly $n+2-p(f)$ such contributions, and the sectors $S_j(\delta)$ $(j=0, \ldots, n+1)$ in (4.2) cover the punctured plane, we see that (4.7) holds.

Now suppose that $a_n \neq 0$ is arbitrary. If we set $F(z) = f(z/c)$ where $c$ satisfies (4.5), then $F$ satisfies an equation of the form (4.1) with $q_n = 1$, and $p(f) = p(F)$. Hence by using the above result on $F$, we will obtain that (4.7) holds for $f$. Then (4.8) follows from (4.7).

Now let $g$ be a solution of the same equation (1.1) as $f$, such that $f$ and $g$ are linearly independent, and set $h = g/f$. Then from Lemma 3, the sum of the Nevanlinna deficiencies of $h$ is 2, and so by formula (1.4) on page 4 of [5],

$$T(r, h') = (1+o(1))(2-\delta(\infty, h))T(r, h) \tag{4.10}$$

as $r\to\infty$. From Abel's identity, $h'=d/f^2$ where $d\neq 0$ is a constant. Thus

$$T(r, h') = 2T(r,f)+O(1) \tag{4.11}$$

as $r\to\infty$. From (4.8) we obtain

$$N(r, h) = (1+o(1))\frac{2\alpha-p(f)}{\pi\alpha^2}\sqrt{|a_n|}\,r^\alpha \tag{4.12}$$

as $r\to\infty$, since $N(r, h)=N(r,f, 0)$. Combining (4.12) and Lemma 3 gives

$$\delta(\infty, h) = \frac{p(f)}{2\alpha}. \tag{4.13}$$

From (4.13) and Lemma 3(2.) we obtain that $p(f)$ must be an even number. By combining (4.13), (4.11), (4.10), and Lemma 3(1.) we will obtain (4.9). The proof of Theorem 5 is now complete.

We will now prove two corollaries of Theorem 5 and Lemma 3.

Corollary 1. *With the hypothesis and notation of Lemma* 3, *if* $b\neq b_k$ *for* $k=1,\ldots,m$, *then the following hold:*

(i) $T(r, g-bf) = (1+o(1))\dfrac{2\sqrt{|a_n|}}{\pi\alpha}r^\alpha$ *as* $r\to\infty$.

(ii) $N(r, g-bf, 0) = (1+o(1))\dfrac{2\sqrt{|a_n|}}{\pi\alpha}r^\alpha$ *as* $r\to\infty$.

(iii) $p(g-bf) = 0$.

*Proof.* From Lemma 3 we obtain (ii). From (4.8) and (ii) we obtain $p(g-bf)=0$. Then (i) follows from (iii) and (4.9).

Corollary 2. *Let $f$ and $g$ be linearly independent solutions of equation* (1.1) *where $A(z)\not\equiv 0$ is a polynomial of degree n, such that $p(f)=0$ and $p(g)=0$ (there exist such $f$ and $g$ from Corollary* 1). *Let $b_1, \ldots, b_m$ be the deficient values of $h=g/f$, as in Lemma* 3. *Then the following hold:*

(i) *$m=2$ if $A(z)$ is a constant and $m\geqq 3$ if $A(z)$ is nonconstant.*

(ii) *$b_1, \ldots, b_m$ are all finite.*

(iii) *$p(g-b_kf)=(n+2)\delta(b_k, h)\geqq 2$ for $k=1, \ldots, m$, and*

$$\sum_{k=1}^m p(g-b_kf) = 2(n+2).$$

(iv) *If $F\not\equiv 0$ is any solution of the same equation* (1.1) *as $f$ and $g$ but is not a constant multiple of $g-b_kf$ for any $k=1, \ldots, m$, then $p(F)=0$.*

Remark. From Corollary 2 (iv) we see that most nontrivial solutions of (1.1) where $A(z)\not\equiv 0$ is a polynomial, will have shortage zero.

*Proof.* Set $\alpha=(n+2)/2$ and $A(z)=a_n z^n+...+a_0$. Since $p(f)=0$ and $N(r, h)=N(r, f, 0)$, we will obtain from (4.8) that

$$N(r, h) = (1+o(1))\frac{2\sqrt{|a_n|}}{\pi\alpha} r^\alpha$$

as $r\to\infty$. Thus $\delta(\infty, \mathrm{h})=0$ from Lemma 3. This proves (ii).

Since $N(r, h, b_k)=N(r, g-b_k f, 0)$, we will obtain from (4.8) that

$$N(r, h, b_k) = (1+o(1))\frac{2\alpha-p(g-b_k f)}{\pi\alpha^2}\sqrt{|a_n|}\, r^\alpha$$

as $r\to\infty$. Combining this with Lemma 3 gives $\delta(b_k, h)=p(g-b_k f)(n+2)^{-1}>0$, and so $p(g-b_k f)\geqq 2$ for $k=1, ..., m$ from Theorem 5. Since $\sum_{k=1}^{m}\delta(b_k, h)=2$ from Lemma 3, part (iii) follows.

We have $m\geqq 2$ from (iii). It is obvious that $m=2$ when $A(z)$ is a nonzero constant. Suppose now that $m=2$ and $A(z)$ is nonconstant. Then from (iii) it follows that $p(g-b_1 f)=p(g-b_2 f)=n+2$. Thus $g-b_1 f$ and $g-b_2 f$ are linearly independent solutions of (1.1), and each has only finitely many zeros. This is impossible from [1, Theorem 1]. This proves (i).

To prove (iv), we write $F=c_1 g+c_2 f$ for constants $c_1, c_2$. If $c_1=0$ then $p(F)=0$. If $c_1\neq 0$, then $F=c_1(g-bf)$ where $b\neq b_k$ for $k=1, ..., m$. Hence $p(F)=0$ from Corollary 1 (iii). This proves (iv), and the proof of Corollary 2 is now complete.

Remarks. Corollary 2 shows that for any given equation (1.1) where $A(z)\not\equiv 0$ is a polynomial, there must exist two nontrivial solutions $f_1$ and $f_2$ such that $p(f_1)\neq p(f_2)$ (we mention that for some nonconstant polynomials $A(z)$, we can explicitly calculate the number $p(f)$ for every solution $f\not\equiv 0$ of equation (1.1) — see Examples 1 and 6 in Section 8). Thus from (4.9), we see that the growth of $T(r, f)$ will depend on both $f$ and $A(z)$. In contrast, the growth of $\log M(r, f)$ depends only on $A(z)$ by the following result, which is essentially due to Valiron.

Theorem 6. *If $f\not\equiv 0$ is any solution of equation* (1.1) *where $A(z)=a_n z^n+...+a_0$ is a polynomial of degree n with $a_n\neq 0$, then as $r\to\infty$,*

$$\log M(r, f) = (1+o(1))\frac{\sqrt{|a_n|}}{\alpha} r^\alpha \tag{4.14}$$

*where $\alpha=(n+2)/2$.*

*Proof.* We will denote by $V(r)$ the central index of $f$. From results of Valiron [17, p. 108], there exist constants $B>0$ and $\ell>0$ such that

$$V(r) = (1+o(1))Br^\ell \tag{4.15}$$

as $r\to\infty$, and

$$\log M(r, f) = (1+o(1))\frac{B}{\ell}r^{\ell} \tag{4.16}$$

as $r\to\infty$. On the other hand, by applying the Wiman—Valiron theory [17, p. 105] to equation (1.1) we obtain that

$$V(r) = (1+o(1))\sqrt{|a_n|}\,r^{\alpha} \tag{4.17}$$

as $r\to\infty$ outside a possible exceptional set of finite logarithmic measure. From (4.17) and (4.15) we obtain $B=\sqrt{|a_n|}$ and $\ell=\alpha$. Now (4.14) follows from (4.16), and Theorem 6 is proven.

We next discuss the frequency of the real zeros of a solution $f\not\equiv 0$ of equation (1.1) where $A(z)$ is a real entire function. The following result is due to Wiman [18] (see [10, p. 472]).

Lemma 4. *Let $A(x)$ be a positive continuous function on an interval $[r_0, +\infty)$, which has a continuous derivative on $(r_0, +\infty)$, and which satisfies*

$$\frac{A'(x)}{(A(x))^{3/2}}\to 0$$

*as $x\to+\infty$. Then, any real-valued solution $f(x)\not\equiv 0$ of the equation $f''+A(x)f=0$ on $[r_0, +\infty)$ has infinitely many zeros, and the number $q(r)$ of zeros of $f$ on $[r_0, r)$ satisfies*

$$q(r) = \frac{(1+o(1))}{\pi}\int_{r_0}^{r}(A(x))^{1/2}\,dx \tag{4.18}$$

*as $r\to\infty$.*

Another proof of Lemma 4 is in [7].

The next result is elementary and useful.

Lemma 5. *Let $A(z)$ be a real entire function. If $f$ is a solution of equation* (1.1) *that possesses a real zero, then $f\equiv Cg$ where $C$ is a constant and $g$ is a real solution of* (1.1).

*Proof.* Since $A(z)$ is real, it follows that $f(z)=c_1f_1(z)+c_2f_2(z)$ where $f_1$ and $f_2$ are linearly independent real solutions of equation (1.1) and $c_1$, $c_2$ are constants. If $c_1c_2=0$ then the assertion is already true.

Suppose $c_1c_2\neq 0$. If $z_0$ is a real zero of $f$ then we have $c_1f_1(z_0)+c_2f_2(z_0)=0$. Since $f_1$ and $f_2$ are real this gives $\overline{c_1}f_1(z_0)+\overline{c_2}f_2(z_0)=0$. It follows that $c_1=bc_2$ where $b$ is real. Then $f=c_2(bf_1+f_2)$ and the assertion is proved.

Corollary 3. *Let $A(z)=a_nz^n+\ldots+a_0$ be a real polynomial of degree $n$ which is not identically zero, and set $\alpha=(n+2)/2$. Suppose that a solution $f\not\equiv 0$ of equation* (1.1) *has a real zero.*

(i) *Let n be odd. Then as* $r\to\infty$,

$$n_R(r, f, 0) = \big(1+o(1)\big)\frac{\sqrt{|a_n|}}{\pi\alpha}\, r^{\alpha}, \tag{4.19}$$

$$N_R(r, f, 0) = \big(1+o(1)\big)\frac{\sqrt{|a_n|}}{\pi\alpha^2}\, r^{\alpha}. \tag{4.20}$$

*Also, if* $a_n>0$ *then f has only finitely many negative zeros, while if* $a_n<0$ *then f has only finitely many positive zeros.*

(ii) *Let n be even. If* $a_n>0$ *then f has an infinite number of positive zeros and also an infinite number of negative zeros, and as* $r\to\infty$,

$$n_R(r, f, 0) = \big(1+o(1)\big)\frac{2\sqrt{a_n}}{\pi\alpha}\, r^{\alpha}, \tag{4.21}$$

$$N_R(r, f, 0) = \big(1+o(1)\big)\frac{2\sqrt{a_n}}{\pi\alpha^2}\, r^{\alpha}. \tag{4.22}$$

*If* $a_n<0$ *then f has only finitely many real zeros.*

*Proof.* It follows from Lemma 5 that $f\equiv Cg$ where $C\neq 0$ is a constant and $g$ is a real solution of equation (1.1). Hence we may assume that $f$ is real.

First suppose that $n$ is odd. If $a_n>0$ then it follows that (4.18) holds. Since $A(x)<0$ for $x<x_1$, we can use a well-known application of the Sturm comparison theorem to conclude that $f$ can have only finitely many negative zeros. Hence (4.19) will follow from (4.18), and then (4.20) follows from (4.19), and thus (i) is proven in the case when $a_n>0$. By considering $f(-x)$ we see that (i) will hold when $a_n<0$.

If $n$ is even, then we can use similar reasoning both on the positive real axis and on the negative real axis to prove (ii).

## 5. Proof of Theorem 2

Set $\alpha=(n+2)/2$. We make the assumption that $f$ has an infinite number of zeros and that $\lambda_{NR}(f)<\alpha$. Since $p(f)$ is an even number (from Theorem 5), it follows that $p(f)=n$ from Lemma 2 and the transformation described with (4.5). Then from (4.8) and (4.9) we obtain

$$\lim_{r\to\infty}\frac{N(r, f, 0)}{T(r,f)} = \frac{4}{n+4}. \tag{5.1}$$

Since $\lambda_{NR}(f)<\alpha$, it follows from the result of Bank and Laine [1, Theorem 1] that $\lambda_R(f)=\alpha$. Let $g$ be the canonical product of the nonreal zeros of $f$, and set $f=hg$. Then $\sigma(g)<\alpha$ and, since $\sigma(f)=\alpha$ from (4.14), $h$ is an entire function such that $\sigma(h)=\alpha$. Since $h$ has only real zeros and $\sigma(h)$ is a positive even integer, it follows from a result of Shea [15, Corollary 2.1] that

$$\underline{\lim}_{r\to\infty} \frac{N(r, h, 0)}{T(r, h)} = 0. \tag{5.2}$$

From $f=hg$, $\sigma(g)<\alpha$, $p(f)=n$, (4.8), and (4.9), we easily obtain that as $r\to\infty$, $N(r, h, 0)=(1+o(1))N(r, f, 0)$ and $T(r, h)=(1+o(1))T(r, f)$. Thus from (5.2) we obtain

$$\underline{\lim}_{r\to\infty} \frac{N(r, f, 0)}{T(r, f)} = 0,$$

which contradicts (5.1). This contradiction proves Theorem 2.

## 6. Proof of Theorem 1

If $n=2+4k$ where $k$ is some nonnegative integer, then the conclusion follows easily, because in the contrary case we would obtain from Theorem 2 that $f_1$ and $f_2$ would each have only finitely many zeros which is impossible from [1, Theorem 1].

Now suppose that $n\neq 2+4k$ for $k=0, 1, 2, \ldots$. We assume that the conclusion is false. If $Q(z)$ is the canonical product of the nonreal zeros of $f_1$ and $f_2$, then $\sigma(Q)<(n+2)/2$. From (4.14) and [1, Theorem 1] we obtain that

$$E = f_1 f_2 = PQ \tag{6.1}$$

where $P$ is an entire function whose zeros are all real, and $\lambda(E)=\sigma(E)=\lambda(P)=\sigma(P)=(n+2)/2$. Then $A(z)$ must be real from Theorem 3.

Now for $n\geqq 3$ we have $2<\lambda(P)<\infty$, and so by Corollary 1.2 of [4] we obtain that $\delta(0, P)>0$. If $n=1$ then $\lambda(P)=3/2$, and so from Corollary 3(i) and [4, Corollary 1.1] we deduce that $\delta(0, P)>0$. Therefore, $\delta(0, P)>0$ in all cases. Hence there exists a constant $b>0$ such that for $r\geqq r_0$,

$$\frac{m(r, P, 0)}{T(r, P)} \geqq b. \tag{6.2}$$

Now we consider the following identity due to Bank and Laine [1, p. 354]:

$$-4A = (c/E)^2-(E'/E)^2+2(E''/E) \tag{6.3}$$

where $c\neq 0$ is a constant. Since $E$ has finite order, it follows from (6.3) and Nevanlinna's fundamental estimate of the logarithmic derivative that

$$m(r, E, 0) = O(\log r) \tag{6.4}$$

as $r\to\infty$. (We mention that it was noted on page 354 of [1] that $\delta(d, E)=0$ for all $d\neq\infty$.)

Choose $\beta$ such that $\sigma(Q)<\beta<(n+2)/2$. From (6.4), (6.2), and (6.1) we obtain that as $r\to\infty$,

$$T(r,P)\leqq O\big(m(r,P,0)\big)\leqq O\big(m(r,E,0)+m(r,Q)\big)=o(r^{\beta}),$$

which contradicts that $P$ has order $(n+2)/2$. This proves Theorem 1.

Remark. We mention that it was not necessary to use Theorem 2 in the above proof, because alternatively we could have used Corollary 3(ii) and [4, Corollary 1.2] to obtain in the above argument that $\delta(0,P)>0$ when $n=2$.

## 7. When $A(z)$ is a polynomial with degree $\neq 2+4k$

We will now prove the following two analogous theorems to Theorem 2 for the cases when $A(z)$ is a nonconstant polynomial with degree $\neq 2, 6, 10, \dots$.

Theorem 7. *Let $f\not\equiv 0$ be a solution of equation* (1.1) *where $A(z)=a_n z^n+\dots+a_0$ is a polynomial of odd degree n, and set $\alpha=(n+2)/2$. Then exactly one of the following two cases must occur:*

(i) $\lambda_{NR}(f)=\alpha$.

(ii) $\lambda_{NR}(f)<\alpha$ *and* $\delta(0,f)=\Delta(0,f)=(n+1)/(n+3)$, *and furthermore as* $r\to\infty$,

$$N(r,f,0)=\big(1+o(1)\big)\frac{\sqrt{|a_n|}}{\pi\alpha^2}\,r^{\alpha},$$

$$T(r,f)=\big(1+o(1)\big)\frac{(n+3)\sqrt{|a_n|}}{2\pi\alpha^2}\,r^{\alpha}.$$

Theorem 8. *Let $f\not\equiv 0$ be a solution of equation* (1.1) *where $A(z)=a_n z^n+\dots+a_0$ is a polynomial of degree $n=4k$ where k is some positive integer, and set $\alpha=(n+2)/2$. Then exactly one of the following three cases must occur:*

(i) *$f$ has only finitely many zeros.*

(ii) $\lambda_{NR}(f)=\alpha$.

(iii) $\lambda_{NR}(f)<\alpha$ *and* $\delta(0,f)=\Delta(0,f)=n/(n+4)$, *and furthermore as* $r\to\infty$,

$$N(r,f,0)=\big(1+o(1)\big)\frac{2\sqrt{|a_n|}}{\pi\alpha^2}\,r^{\alpha},$$

$$T(r,f)=\big(1+o(1)\big)\frac{(n+4)\sqrt{|a_n|}}{2\pi\alpha^2}\,r^{\alpha}.$$

Remark. All of the cases in Theorems 7 and 8 can occur. Regarding Theorem 7(ii), there are examples when $n=1$ where a solution $f$ has all real zeros (see Example 1 in Section 8). Regarding Theorem 8(iii), there are examples when $n=4$ where a solution $f$ has an infinite number of zeros and all of the zeros are real (see Example 2 in

Section 8). From Corollary 2 we have that $p(f)=0$ for most $f$ in the situations of Theorems 7 and 8, and such solutions $f$ belong to the cases Theorem 7(i) and Theorem 8(ii).

*Proof of Theorem* 7. We will assume that $\lambda_{NR}(f)<\alpha$. Since $p(f)$ is an even number (from Theorem 5), it follows that $p(f)=n+1$ from Lemma 2 and the transformation described with (4.5). Then we have the case (ii) from (4.8) and (4.9). This proves Theorem 7.

*Proof of Theorem* 8. We will assume that $f$ has an infinity of zeros and that $\lambda_{NR}(f)<\alpha$. From Theorem 5, Lemma 2, and the transformation described with (4.5), we find that $p(f)=n$. Then we have the case (iii) from (4.8) and (4.9). This proves Theorem 8.

## 8. Examples

We will now give several examples which will both complement the theory and also exhibit the sharpness of our results.

Example 1. The well-known Airy differential equation

$$f''-zf=0 \tag{8.1}$$

possesses a solution $f_0$ such that the zeros of $f_0$ are all real and negative (see [13, pp. 413—415] where $f_0(z)\equiv Ai(z)$). From Theorem 7 we obtain $\delta(0,f_0)=\Delta(0,f_0)=1/2$, and furthermore as $r\to\infty$,

$$N(r, f_0, 0) = \big(1+o(1)\big)\frac{4}{9\pi}\, r^{3/2},$$

$$T(r, f_0) = \big(1+o(1)\big)\frac{8}{9\pi}\, r^{3/2}.$$

We mention that for the function $f_0$, the inequality (2.12) in [15, Theorem 2] is an equality with $\underline{\lim}$ replaced by lim.

We also have

$$\log M(r, f_0) = \big(1+o(1)\big)\frac{2}{3}\, r^{3/2}$$

as $r\to\infty$, from (4.14).

Clearly, $p(f_0)=2$. If $c_1$ and $c_2$ are the two nonreal cube roots of unity, and $f_1(z)=f_0(c_1z)$ and $f_2(z)=f_0(c_2z)$, then $f_1$ and $f_2$ are both solutions of equation (8.1), $p(f_1)=p(f_2)=2$, and the three functions $f_0, f_1, f_2$ are pairwise linearly independent (the last statement follows from the initial conditions on $f_0=Ai$ (see [13, p. 392]). From Corollary 2 it follows that if $f\not\equiv 0$ is a solution of (8.1) such that $f$ is not a constant multiple of either $f_0, f_1$, or $f_2$, then $p(f)=0$.

We mention that it follows from the above and a suitable linear change of independent variable that there exists a solution $F_0$ of the equation

$$f''+(a_1z+a_0)f=0$$

where $a_1\neq 0$ and $a_0$ are real constants, such that all the zeros of $F_0$ are real and $\delta(0, F_0)=\Delta(0, F_0)=1/2$.

This example shows that case (ii) in Theorem 7 can occur.

Example 2. It is well-known [16] that equation (1.1) with $A(z)=\beta-z^4$, $\beta\in\boldsymbol{R}$, has a complete orthonormal set of real eigenfunctions $\{\psi_n(z)\}_{n=0}^{\infty}$ for $L^2(\boldsymbol{R})$ as a real space, with corresponding distinct eigenvalues $\{\beta_n\}_{n=0}^{\infty}$, such that $0<\beta_0<\beta_1<\beta_2<\dots$, and $\beta_n\to+\infty$ as $n\to\infty$. Also, $\psi_n$ is even or odd as $n$ is even or odd, and $\psi_n$ has exactly $n$ real zeros. Titchmarsh [16, pp. 172—173] showed that all the nonreal zeros of $\psi_n$ are purely imaginary and that $\lambda(\psi_n)=3$. Thus $\psi_0$ and $\psi_1$ each have only purely imaginary zeros. If $f_n(z)=\psi_n(iz)$, then for each $n$, $f_n$ satisfies the equation

$$f''+(z^4-\beta_n)f = 0,$$

$f_n$ has only real zeros except possibly for finitely many, and $\lambda(f_n)=\sigma(f_n)=3$. In particular, $f_0$ and $f_1$ have only real zeros.

From Theorem 8 we obtain (for all $n$) that $\delta(0,f_n)=\Delta(0,f_n)=1/2$, and furthermore as $r\to\infty$,

$$N(r, f_n, 0) = \big(1+o(1)\big)\frac{2}{9\pi}\, r^3,$$

$$T(r, f_n) = \big(1+o(1)\big)\frac{4}{9\pi}r^3.$$

We mention that for the functions $f_0$ and $f_1$, the inequality (2.15) in [15, Corollary 2.1] is an equality with $\underline{\lim}$ replaced by lim.

We also have (for each $n$)

$$\log M(r, f_n) = \big(1+o(1)\big)\frac{1}{3}\, r^3$$

as $r\to\infty$, from (4.14).

This example shows that case (iii) in Theorem 8 can occur.

Remark. Some further properties of each $\psi_n(z)$ (and each $\beta_n$) can be found in Chapter III of [6].

Example 3. Given any sequence $z_1, z_2, z_3, \dots$ of distinct complex numbers such that $z_n\to\infty$, there exist entire functions $f(z)$ and $A(z)$ that satisfy $f''+A(z)f=0$ such that $f$ has precisely the zeros $z_1, z_2, \dots$. (In particular we can choose each $z_n$ to be real.)

To prove this statement let $Q(z)$ be a Weierstrass product with exactly the zeros

$z_1, z_2, \ldots$. By the interpolation theorem [14, p. 298] there exists an entire function $g$ such that

$$g(z_n) = -\frac{Q''(z_n)}{2Q'(z_n)} \quad \text{for each } n.$$

Let $h$ be a primitive of $g$. Then $f=Qe^h$ satisfies the equation $f''+A(z)f=0$ where

$$A = -g^2 - g' - \frac{Q''+2Q'g}{Q}$$

is entire.

We next will illustrate how Theorem 1 is sharp. Theorem 1 is not true for $n=0$ since if $a_0 \neq 0$ is a real constant then the equation $f''+a_0 f=0$ possesses of course two linearly independent solutions $f_1, f_2$ neither of which has any nonreal zeros. There are examples (Examples 1 and 2) when $A(z)$ is a polynomial of degree $n=1$ or $n=4$, where equation (1.1) possesses a solution $f$ such that the zeros of $f$ are all real and $\lambda(f)=(n+2)/2$. For any even $n \geqq 0$, there exists a polynomial $A(z)$ of degree $n$ such that equation (1.1) possesses a solution with no zeros.

Concerning the results in Theorems 1 and 3, when $A(z)$ is transcendental we give some possibilities that can occur in the following

Example 4. Let $k$ be a positive integer or $\infty$. Then there exists an entire function $A(z)$ of order $k$ such that equation (1.1) possesses two linearly independent solutions $f_1, f_2$ which each have only real zeros and $\lambda(f_1)=\lambda(f_2)=1$. When $k$ is a positive integer we will construct an $A(z)$ that is nonreal, while when $k=\infty$ we will construct an $A(z)$ that is real and also an $A(z)$ that is nonreal. We will make these constructions by applying the following result of Bank and Laine (see Lemmas $B$ and $C$ of [3]).

Lemma 6. *Suppose that $E(z)$ is an entire function and $c \neq 0$ is a constant such that $E'(z_0)=\pm c$ whenever $E(z_0)=0$. Then the function $A(z)$ defined by* [1, p. 354]

$$A = \frac{(E')^2 - c^2 - 2EE''}{4E^2} \tag{8.2}$$

*is entire, and there exist two linearly independent solutions $f_1, f_2$ of equation* (1.1) *such that* (i) $E=f_1 f_2$, (ii) $c \equiv f_1 f_2' - f_1' f_2$, (iii) $f_1(z_0)=0$ *exactly when* $E(z_0)=0$ *and* $E'(z_0)=-c$, *and* (iv) $f_2(z_0)=0$ *exactly when* $E(z_0)=0$ *and* $E'(z_0)=c$.

We will now make the above mentioned constructions.

First let $k$ be a positive integer. Set

$$E(z) = \sin z \cdot \exp\left(\frac{2iz^k}{\pi^{k-1}}\right).$$

Noticing that $E'(z_0)=\pm 1$ whenever $E(z_0)=0$, we see that the function $A(z)$ in (8.2) (with $c=1$) is entire and is of order $k$. From Lemma 6, it follows that there exist two linearly independent solutions $f_1, f_2$ of equation (1.1) such that $f_1$ and $f_2$ each have only real zeros and $\lambda(f_1)=\lambda(f_2)=1$. A calculation will show that $A(z)$ is nonreal; for example, $A(\pi/4)$ is not real when $k=1$ and $A(\pi/2)$ is not real when $k\geqq 2$.

Now let $k=\infty$. If we set $E(z)=\sin z\cdot\exp(i\sin z)$, then $A(z)$ in (8.2) (with $c=1$) is entire with infinite order. The asserted statement then follows by the above reasoning, and $A(z)$ is nonreal (e.g. $A(\pi/2)$ is not real).

When $k=\infty$, if we set $E(z)=\sin z\cdot\exp(\sin z)$, then the asserted statement follows by the above reasoning, and in this case $A(z)$ is real.

Example 5. In the situation of Theorem 4 we can have $\lambda_R(f)=\sigma(A)=1$ in (1.2) because $f(z)=\exp(-iz/2)\cdot\sin(\pi e^{iz})$ satisfies equation (1.1) where $A(z)=1/4-\pi^2e^{2iz}$. Regarding the estimate (1.3) we have

$$N_R(r, f, 0) = T(r, A)+O(1) = \frac{2r}{\pi}+O(1)$$

as $r\to\infty$, in this particular case.

Example 6. Let $q\geqq 2$ be an integer, and consider the equation

$$f''+A_q(z)f = 0, \tag{8.4}$$

where $A_q(z)=-(q^2/4)z^{2q-2}-(q(q-1)/2)z^{q-2}$. We will now explicitly calculate the number $p(f)$ for every solution $f\not\equiv 0$ of equation (8.4). Equation (8.4) possesses the solution $f_1(z)=\exp(z^q/2)$. If we define $f_2=hf_1$ where $h(z)=\int_0^z\exp(-w^q)\,dw$, then it follows from the reduction of order method that $f_2$ is a solution of (8.4) that is linearly independent with $f_1$. From [12, p. 21], if $z_0, z_1, \dots, z_{q-1}$ are defined by

$$z_k = \exp\left(\frac{2\pi ik}{q}\right)\int_0^\infty \exp(-x^q)\,dx,$$

then as $r\to\infty$,

$$T(r, h) = \big(1+o(1)\big)\frac{r^q}{\pi}, \tag{8.5}$$

and (for $k=0, 1, \dots, q-1$)

$$N(r, h, z_k) = \big(1+o(1)\big)(1-1/q)T(r, h). \tag{8.6}$$

Hence from (8.6) and (8.5) we can deduce that as $r\to\infty$,

$$N(r, f_2-z_kf_1, 0) = \big(1+o(1)\big)(1-1/q)\frac{1}{\pi}r^q \tag{8.7}$$

for each $k$. By comparing (8.7), (8.4), and (4.8) we obtain that

$$p(f_2-z_kf_1) = 2 \tag{8.8}$$

for $k=0, 1, \dots, q-1$. Since the functions $f_1, f_2-z_0f_1, f_2-z_1f_1, \dots, f_2-z_{q-1}f_1$ are pairwise linearly independent, it follows from (8.8), $p(f_1)=2q$, and Corollary 2 that if $f\not\equiv 0$ is any solution of equation (8.4) that is not a multiple of one of $f_1$, $f_2-z_0f_1, \dots, f_2-z_{q-1}f_1$, then $p(f)=0$.

Example 7. It is well-known that for $n \geqq 0$, the Hermite polynomial $H_n(z)$ of degree $n$ has exactly $n$ real zeros and the Hermite function $\varphi_n(z)=H_n(z)\exp(-z^2/2)$ satisfies the Hermite—Weber differential equation $f''+(2n+1-z^2)f=0$.

## 9. When $A(z)$ is transcendental periodic

We now give some results and examples that concern the frequency of the real zeros of solutions $f\not\equiv 0$ of equation (1.1) where $A(z)$ is a transcendental entire periodic function. Since we use Theorem 4 to prove our results, we will state them as corollaries.

Corollary 4. *Let $A(z)$ be a nonconstant entire periodic function with a period that is not purely imaginary, and let $f\not\equiv 0$ be a solution of equation* (1.1).

(i) *If the period is real, then $\lambda_R(f)\leqq 1$.*

(ii) *If the period is not real, then as $r\to\infty$,*

$$N_R(r, f, 0) \leqq 2T(r, A)+O(\log r). \tag{9.1}$$

*Proof.* If the period is real then $A(z)$ is bounded on the real axis. Let $x_0$ be a real zero of $f$, and set $F(z)=f(z)(f'(x_0))^{-1}$. Then $F$ satisfies equation (1.1) and $F(x_0)=0$, $F'(x_0)=1$. By a theorem of Hille [10, Theorem 11.1.1, p. 579] it follows that $f$ cannot have a zero on the open real segment $x_0<x<x_0+r_0$ where $r_0$ is the least positive zero of the solution $v(r)$ of the equation

$$v''(r)+|A(x_0+r)|v(r) = 0 \tag{9.2}$$

that satisfies $v(0)=0$, $v'(0)=1$. Since $|A(x)|<C$ for some $C>0$, for all real $x$, it follows from the Sturm comparison theorem applied to (9.2) and the equation $g''+Cg=0$, that $r_0>\pi/\sqrt{C}$. Clearly this means that $\lambda_R(f)\leqq 1$.

Now suppose that the period of $A(z)$ is not real and not purely imaginary. It follows from Lemma 7 (which follows this proof) that $A(z)$ is not real. Then (9.1) follows from Theorem 4.

The proof of Corollary 4 is complete (once we prove Lemma 7).

Lemma 7. *Let $A(z)$ be a nonconstant entire function with a period $\omega$ that is neither real nor purely imaginary. Then $A(z)$ is not real.*

*Proof.* Suppose $A(z)$ is real. Then we have

$$A(z+\bar{\omega}) \equiv \overline{A(\bar{z}+\omega)} \equiv \overline{A(\bar{z})} \equiv A(z).$$

Since a nonconstant entire function can have only one linearly independent period over $\boldsymbol{R}$, it follows that $\bar{\omega}=c\omega$ for some $c\in\boldsymbol{R}$. Thus $\omega$ is either real or purely imaginary which contradicts the hypothesis. This proves Lemma 7.

Example 8. It follows from formula (4.18) that any real solution $f\not\equiv 0$ of the equation $f''+e^z f=0$ satisfies $\lambda_R(f)=+\infty$. Thus in the hypothesis of Corollary 4, the condition that the period not be purely imaginary is necessary.

Remark. Example 5 in Section 8 shows that we can have $\lambda_R(f)=1$ in the situation of Corollary 4(i).

Corollary 5. *Let $B(\zeta)$ be a rational function which is analytic on $0<|\zeta|<\infty$, and which has poles of odd order at both $\zeta=0$ and $\zeta=\infty$. Let $\mu$ be a nonreal number and set $A(z)=B(e^{\mu z})$. Then every solution $f\not\equiv 0$ of equation* (1.1) *satisfies* $\lambda_{NR}(f)=+\infty$.

*Proof.* From a result of Bank and Laine [2, Theorem 3] we have that $\lambda(f)=+\infty$. Thus Corollary 5 follows from Corollary 4.

Example 9. It follows from Corollaries 4 and 5 that any solution $f\not\equiv 0$ of Mathieu's equation $f''+(a+b\cos 2z)f=0$ where $a$ and $b\neq 0$ are constants, satisfies $\lambda_R(f)\leqq 1$ and $\lambda_{NR}(f)=\infty$.

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Received 3 April 1985