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ASYMPTOTIC EXTREMAL GROWTH OF QUASISYMMETRIC FUNCTIONS

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1. Introduction

The purpose of this paper is to determine the asymptotic behaviour of the functions $M_0(x, K)$ and $m_0(x, K)$, defined below, that describe the maximal and minimal growth of K-quasisymmetric functions. The work is based on an earlier paper [5] of the author, which can be regarded as a sequel to the papers [3, 4] of W. K. Hayman and the author.

An increasing homeomorphism f of the real axis $\boldsymbol{R}$ onto itself is called K-quasisymmetric ($K-qs$), where $1 \leqq K < \infty$, if

$$\frac{1}{K} \leqq \frac{f(x+t)-f(x)}{f(x)-f(x-t)} \leqq K \tag{1.1}$$

for all $x \in \boldsymbol{R}$ and $t>0$. The function f is quasisymmetric (qs) if it is $K-qs$ for some K. The condition (1.1) was formulated by Beurling and Ahlfors [1] who proved that qs functions are precisely the boundary values of those quasiconformal maps of the upper half-plane onto itself that fix the point at infinity.

Some results on the growth of qs functions can be found in Kelingos' paper [6], and a more systematic study has been performed in [3, 4, 5]. Following [4], we set

$$N_0(K) = \{f \mid f \text{ is } K-qs,\ f(1) = 1,\ f(-1) = -1\},$$

$$M_0(x, K) = \max \{f(x) \mid f \in N_0(K)\},$$

$$m_0(x, K) = \min \{f(x) \mid f \in N_0(K)\}.$$

We note that by [1], the class $N_0(K)$ is compact.

The class $N_0(1)$ consists of the identity map only, so that $M_0(x, 1) = m_0(x, 1) = x$ for all x. Let K be fixed, $K>1$. In [5, Theorems 1, 2] we constructed piecewise linear odd functions f and g belonging to $N_0(K)$, such that f is the largest convex minorant of $M_0(x, K)$ and g is the smallest concave majorant of $m_0(x, K)$ for $x \geqq -1$. Further

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we found infinitely many points z_n and w_n tending to ∞ as $n\to\infty$ such that

$$f(z_n) = M_0(z_n, K),$$
$$g(w_n) = m_0(w_n, K)$$

for all n. It was also shown that if $r=r(K)$ is rational, i.e.

$$r(K) = \frac{\log K}{\log L} = \frac{p}{q}, \quad L = \frac{1}{2}(K+1), \tag{1.2}$$

where p, q are positive relatively prime integers, then the points z_n, w_n occur at bounded distances. More precisely, we have

$$z_{n+1}-z_n \leqq 2^{2^p+p},$$
$$w_{n+1}-w_n \leqq 2^{2^{2p}+2p}$$

by [5, Theorem 4].

By [4, Theorems 5, 6] we have

$$x^{\alpha_1(K)} \leqq M_0(x, K) \leqq c_1(K)x^{\alpha_1(K)}, \tag{1.3}$$

$$c_2(K)x^{\alpha_2(K)} \leqq m_0(x, K) \leqq x^{\alpha_2(K)} \tag{1.4}$$

for $x\geqq 1$, where the constants $\alpha_1, \alpha_2, c_1, c_2$ depend on K only and can be estimated. Hayman [3, Theorem 1] showed that if $r(K)$ is irrational, then the ratios $M_0(x, K)x^{-\alpha_1(K)}$ and $m_0(x, K)x^{-\alpha_2(K)}$ tend to some limits as $x\to\infty$, say $\gamma_1(K)$ and $\gamma_2(K)$. He also proved [3, Theorem 5] that if $r(K)$ is rational, then these ratios are asymptotic to some periodic functions of $\log x$ (for example $M_0(x, K)\sim x^{\alpha_1(K)}\varphi(\log x)$ where φ is periodic) but left open the question whether or not φ is constant.

In this paper we use the explicit formulas for the above functions f and g together with the fact that f and $M_0(x, K)$ as well as g and $m_0(x, K)$ have the same asymptotic behaviour, to determine the above functions φ and the limits $\gamma_1(K)$ and $\gamma_2(K)$. This will be done in Sections 4 and 6. In Sections 5 and 7 we study the properties of the functions φ, γ_1 and γ_2 to describe the behaviour of M_0 and m_0 more precisely. The proofs are based on difference equations arising from the definitions of f and g, and these will be considered in Sections 2 and 3. In Section 8 we study the asymptotic oscillation properties of an individual $K-qs$ function. In the final Section 9 we prove a technical result used in Section 6. As stating our results precisely requires some preparation, this will be done in the appropriate sections.

2. A difference equation

If $K>1$, we set $L=L(K)=(1/2)(K+1)$, $\Lambda=\Lambda(K)=(1/2)(1+K^{-1})$,

$$r = r(K) = \frac{\log K}{\log L}, \tag{2.1}$$

$$s = s(K) = \frac{\log K^{-1}}{\log \Lambda}. \tag{2.2}$$

We have $1<r<2$, $s>2$, $1/r+1/s=1$. If $r\in Q$, say $r=p/q$ where p, q are positive, relatively prime integers, then $s=p/(p-q)$, $q<p<2q$, $1\leqq p-q<p$, and $p\geqq 3$, $q\geqq 2$. The numbers r and s are rational or irrational simultaneously.

Consider pairs (m, n) of integers $m\geqq 1$, $n\geqq 0$. Following [5], we order these pairs so that

$$K^{m_1}L^{n_1} \leqq K^{m_2}L^{n_2} \leqq \dots. \tag{2.3}$$

The ordering is unique if and only if $r\notin Q$. If $r\in Q$ and if $K^{m_k}L^{n_k}$ has the same value for $M\leqq k\leqq N$, we order these pairs (m_k, n_k) so that $m_M>m_{M+1}>\dots>m_N$.

There is a unique odd piecewise linear continuous function f such that $f(x)=x$ for $0\leqq x\leqq 1$ and such that the slope of f is $K^{m_k}L^{n_k}$ on the interval $[X_{k-1}, X_k]$, where

$$X_k-X_{k-1} = 2^{n_k+1}\binom{m_k+n_k-1}{n_k} \quad \text{for} \quad k\geqq 1, \quad \text{and} \quad X_0 = 1.$$

It was shown in [5, Theorem 1] that $f\in N_0(K)$ and that $f(z)=M_0(z, K)$ whenever

$$z = X_{k-1}+j2^{n_k+1}, \quad 0\leqq j\leqq \binom{m_k+n_k-1}{n_k}, \quad k\geqq 1.$$

Similarly, there is a unique odd function g such that $g(x)=x$ for $0\leqq x\leqq 1$ and such that the slope of g is $K^{-M_k}\Lambda^{N_k}$ on $[Y_{k-1}, Y_k]$, where $M_k\geqq 1$, $N_k\geqq 0$,

$$K^{-M_1}\Lambda^{N_1} \geqq K^{-M_2}\Lambda^{N_2} \geqq \dots, \quad Y_k-Y_{k-1} = 2^{N_k+1}\binom{M_k+N_k-1}{N_k} \quad \text{for}$$
$$k\geqq 1, \quad \text{and} \quad Y_0 = 1.$$

By [5, Theorem 1], we have $g\in N_0(K)$, and $g(w)=m_0(w, K)$ for

$$w = Y_{k-1}+j2^{N_k+1}, \quad 0\leqq j\leqq \binom{M_k+N_k-1}{N_k}, \quad k\geqq 1.$$

Suppose now that $r(K)$ is rational, say $r(K)=p/q$ as before. Then every slope of f can be written as $K^mL^n=L^{(mp+nq)/q}$ since $K^q=L^p$. So the distinct values of the slope of f are given by $L^{m/q}$, where m runs through all positive integers of the form $m=ap+bq$ where $a\geqq 1$ and $b\geqq 0$, and in particular through all integers $m>(p-1)q$. Let the interval of the positive axis where f has the slope $L^{m/q}$ be (x_{m-1}, x_m)

where $m \geqq 1$ (so $x_{m-1}=x_m$ for finitely many small values of m). We set $A_n=x_n-x_{n-1}$. By the definition of f, we have

$$A_n = \sum_{(a,b)\in F_n} \binom{a+b-1}{b} 2^{b+1},$$

where

$$F_n = \{(a, b) | a \geqq 1,\ b \geqq 0,\ ap+bq = n\}.$$

Clearly $A_p=2$, $A_{p+q}=4$, and $A_n=0$ if $1 \leqq n \leqq 2p-1$ and $p \neq n \neq p+q$. We shall show that

(2.4) $$A_n = A_{n-p}+2A_{n-q}, n > p.$$

If $(a, b)\in F_n$, then $(a-1, b)\in F_{n-p}$ if $a \geqq 2$, and $(a, b-1)\in F_{n-q}$ if $b \geqq 1$. Further,

$$\binom{a+b-1}{b} 2^{b+1} = \binom{(a-1)+b-1}{b} 2^{b+1}+2\binom{a+(b-1)-1}{b-1} 2^{(b-1)+1}, \text{ while}$$

$$\binom{(a-1)+b-1}{b} = 0 \text{ if } a = 1 \text{ and } \binom{a+(b-1)-1}{b-1} = 0 \text{ if } b = 0.$$

Also if $(a, b)\in F_{n-p}$, then $(a+1, b)\in F_n$, and if $(a, b)\in F_{n-q}$, then $(a, b+1)\in F_n$. These results imply (2.4).

The equation (2.4) and the values of A_n for $1 \leqq n \leqq p$, determine the numbers A_n uniquely. By the standard results on difference equations, we can write

(2.5) $$A_n = \sum_{i=1}^{p} \beta_i \lambda_i^n$$

for some complex numbers β_i, where the λ_i are the zeros of the polynomial

(2.6) $$P(z) = z^p-2z^{p-q}-1.$$

We shall prove that these zeros are simple.

Before studying the polynomial P more closely, we list the corresponding results for the function g. Roughly speaking, the role of q is taken by $\mu=p-q\in[1, p/2)$. The slopes of g are of the form $A^{n/\mu}$ where $n \geqq p$. If g has the slope $A^{n/\mu}$ on (y_{n-1}, y_n), and $A_n'=y_n-y_{n-1}$, then

$$A_n' = \sum_{(a,b)\in F_n'} \binom{a+b-1}{b} 2^{b+1},$$

where

$$F_n' = \{(a, b) | a \geqq 1,\ b \geqq 0,\ ap+b\mu = n\}.$$

We have

(2.7) $$A_n' = A_{n-p}'+2A_{n-\mu}'.$$

Hence

(2.8) $$A_n' = \sum_{i=1}^{p} \beta_i' (\lambda_i')^n,$$

where the λ_i' are the zeros of

$$Q(z) = z^p-2z^{p-\mu}-1 = z^p-2z^q-1,$$

all of them simple zeros.

3. On the polynomials P and Q

We study the polynomial

$$P(z) = z^p - 2z^m - 1, \tag{3.1}$$

where p and m are relatively prime (in particular, p and m cannot both be even) and $1 \leqq m < p$. We shall apply the results to $m = p - q$ and to $m = q$.

Lemma 1. *The polynomial P has p simple zeros $\lambda_1, \dots, \lambda_p$. One of them, say λ_1, is the unique positive zero of P and*

$$|\lambda_i| < \lambda_1, \quad 2 \leqq i \leqq p.$$

Further, $1 < \lambda_1 < 3$, $\lambda_1^{p-m} < 3$, *and* $\lambda_1 < \sqrt{3}$ *if* $p - m \geqq 2$, *while* $2 < \lambda_1 < 2 + 2^{1-p}$ *if* $m = p - 1$.

We have

$$P'(z) = pz^{p-1} - 2mz^{m-1} = 0$$

if $z=0$ (which is not a zero of P) or if $z^{p-m} = 2m/p$. Hence if $P(z) = P'(z) = 0$ (so $z \neq 0$), we have $z^m = -(2(1-m/p))^{-1}$ and $z^p = -m/(p-m)$. This implies that with $\varrho = m/p \in (0, 1)$, we have

$$2\varrho^\varrho (1-\varrho)^{1-\varrho} = 1.$$

This is satisfied only if $\varrho = 1/2$, i.e. $p = 2m$, which is against our assumption. Hence all the zeros of P are simple.

We have $P(0) = -1 < 0$. For real z, $P(z)$ is real, and for $z > 0$, we have $P'(z) < 0$ for $0 < z < (2m/p)^{1/(p-m)} = A_0$ and $P'(z) > 0$ for $z > A_0$. Hence P has a unique positive zero λ_1. We have $\lambda_1 > 1$ since $P(1) = -2 < 0$. If $\lambda_1^p \geqq 3\lambda_1^m$, then $P(\lambda_1) \geqq \lambda_1^m - 1 > 0$. Hence $\lambda_1 \leqq \lambda_1^{p-m} < 3$, and consequently $\lambda_1 < \sqrt{3}$ if $p - m \geqq 2$. If $m = p - 1$, then $P(2) = (2-2)2^{p-1} - 1 < 0$, so that $\lambda_1 > 2$, while $P(2 + 2^{1-p}) > 2^{1-p}2^{p-1} - 1 = 0$, so that $\lambda_1 < 2 + 2^{1-p}$.

Suppose that $2 \leqq i \leqq p$. Then $|\lambda_i|^p = |2\lambda_i^m + 1| < 2|\lambda_i|^m + 1$ unless $\lambda_i^m > 0$. So $P(|\lambda_i|) < 0$ and hence $|\lambda_i| < \lambda_1$. If $|\lambda_i| \leqq 1$, then $|\lambda_i| < \lambda_1$. Suppose then that $|\lambda_i| > 1$ and that $\lambda_i^m > 0$. Then $\lambda_i^p = 2\lambda_i^m + 1 > 0$ and $(2\pi)^{-1} \arg \lambda_i = k/m = l/p$ for some integers k, l with $0 \leqq k < m$, $0 \leqq l < p$. Since $kp = lm$, we have $m|k$, so $k/m = 0$ and $\lambda_i > 0$. But then $\lambda_i = \lambda_1$, which is impossible. Hence $|\lambda_i| < \lambda_1$ for $2 \leqq i \leqq p$. Lemma 1 is proved.

3.1. We make a few remarks concerning the case $m = p - q$, where $q < p < 2q$.

Remark 1. A more careful analysis shows that for $2 \leqq i \leqq p$, we have

$$|\lambda_i| \leqq \lambda_1 (1 - Ap^{-3}),$$

where A is a positive absolute constant. This seems to be best possible apart from the best value of A.

Remark 2. Applying Rouche's theorem to z^p and $2z^m+1$ on the unit circle we see that P has m zeros in the unit disk. The other $p-m$ zeros lie in $\{|z|>1\}$, except if p and m are odd, in which case $P(-1)=0$. Also all zeros z_0 of P satisfy $|z_0|\geqq B$, where $B\in(0,1)$ is the unique positive zero of z^p+2z^m-1.

Remark 3. In Section 5 we will consider what happens to various quantities as $K_n\to K$, $r(K_n)=p_n/q_n$, $r(K_n)\to r\in(1,2)$, where r is irrational. It might be of interest to study what happens to the zeros of

$$P_n(z)=z^p-2z^m-1,\quad p=p_n,\quad m=p_n-q_n$$

as $n\to\infty$. It seems plausible that for a portion $1/r$ of the zeros (whose number $\to\infty$), the q_n-th powers of the zeros cluster towards the circle $\{|z|=C\}$, where $C>1$ is the unique positive zero of $x^r-2x^{r-1}-1$, while for a portion $1-r^{-1}$ of the zeros, the q_n-th powers cluster towards the circle $\{|z|=B\}$, where $0<B<1$ and B is the unique positive zero of $x^r+2x^{r-1}-1$. However, any useful information would have to be more precise.

3.2. Now we can determine the numbers β_i in (2.5) and β_i' in (2.8).

Lemma 2. *Let* λ_i, $1\leqq i\leqq p$, *be the zeros of* P *given by* (3.1) *with* $m=p-q$. *Then*

$$\beta_i=\frac{2}{\lambda_iP'(\lambda_i)}=\frac{2}{p+2q\lambda_i^{p-q}}\neq 0 \tag{3.2}$$

in (2.5). *Let* λ_i', $1\leqq i\leqq p$, *be the zeros of* P *given by* (3.1) *with* $m=q$. *Then*

$$\beta_i'=\frac{2}{\lambda_i'P'(\lambda_i')}=\frac{2}{p+2(p-q)\lambda_i'^{q}}\neq 0 \tag{3.3}$$

in (2.8).

It suffices to prove that with β_i given by (3.2), (2.5) is true for $1\leqq n\leqq p$. Recall that $A_p=2$ and $A_n=0$ for $1\leqq n<p$. Let R be so large that the disk $\{|z|<R\}$ contains all zeros of P. Then the residue theorem gives

$$\frac{1}{2\pi i}\int_{|z|=R}\frac{z^n\,dz}{zP(z)}=\sum_{i=1}^{p}\frac{\lambda_i^n}{\lambda_iP'(\lambda_i)},\quad n\geqq 1,$$

while for all large R we also have, with $z=Re^{i\theta}$,

$$\frac{1}{2\pi i}\int_{|z|=R}\frac{z^{n-1}\,dz}{P(z)}=\frac{1}{2\pi}\int_0^{2\pi}\frac{z^{n-p}\,d\theta}{1-(2z^m+1)z^{-p}}$$

$$=\frac{1}{2\pi}\int_0^{2\pi}d\theta\, z^{n-p}\sum_{k=0}^{\infty}(2z^{m-p}+z^{-p})^k$$

$$=\frac{1}{2\pi}\int_0^{2\pi}d\theta\, z^{n-p}\Big(1+2z^{m-p}+z^{-p}+\sum_{k=p+1}^{\infty}a_kz^{-k}\Big)$$

for some numbers a_k. Hence this integral is equal to 1 if $n=p$, and equal to zero if $1\leqq n<p$. This proves the claim concerning the β_i, and for the β_i' the proof is the same. It is routine to verify that the second equality holds in (3.2) and (3.3). Lemma 2 is proved.

Remark. Since $|\lambda_i|^{p-q}\leqq\lambda_1^q=\lambda_1^{p-m}<3$, we have $|p+2q\lambda_1^{p-q}|<p+6q<7p$, hence $|\beta_i|>2/(7p)$. Since $\lambda_1>1$, we have $\beta_1<1/p$. One can show that $|\beta_i|\leqq A_1(r)/p$, where $A_1(r)$ depends only on the ratio $r=p/q$, $A_1(r)$ remains bounded as $r\to 1$ (i.e. as $K\to\infty$), but $A_1(r)$ might tend to ∞ as $r\to 2$ (i.e. as $K\to 1$).

Further, one can show that $0<A_2(r)\leqq p|\beta_i'|\leqq A_3(r)$, where the function $A_2(r)\to\infty$ only when $r\to 1$ and $A_3(r)\to\infty$ only when $r\to 2$.

4. Asymptotic behaviour of M_0 and m_0 for rational $r(K)$

Let $r(K)$ be rational, say $r(K)=p/q$, where p, q are positive and coprime. As mentioned in Section 1, the asymptotic behaviour of $M_0(x, K)$ and $m_0(x, K)$ is the same as that of f and g, respectively. The piecewise linear functions f and g are determined by their slopes $L^{n/q}$ and $\Lambda^{n/\mu}$, where $\mu=p-q$, and by the numbers A_n, A_n', which by Lemmas 1 and 2 are asymptotically given by $\beta_1\lambda_1^n$ and $\beta_1'(\lambda_1')^n$. This allows us to determine the asymptotic behaviour of M_0 and m_0. Furthermore, we shall show how β_1, λ_1, β_1', λ_1' are connected to quantities studied in [4].

Let us recall [4, Theorems 5, 6] that if $K>1$, then we have

$$x^{\alpha_1}\leqq M_0(x, K)\leqq c_1(K)x^{\alpha_1},$$
$$c_2(K)x^{\alpha_2}\leqq m_0(x, K)\leqq x^{\alpha_2}$$

for $x\geqq 1$, where $\alpha_1=\alpha_1(K)$ and $\alpha_2=\alpha_2(K)$ are obtained as follows (see [1] or [4, Lemma 1]).

Lemma A. *If $\alpha>0$, then the function*

$$g_\alpha(x)=|x|^\alpha \operatorname{sign} x$$

is K_α–qs, where the best possible K_α is determined as follows. Let t_α be the solution of

$$(t+1)^{1-\alpha}+(t-1)^{1-\alpha}=2, \tag{4.1}$$

so that $1<t_\alpha<2$. Further set

$$\begin{aligned} q_\alpha &= [(t_\alpha+1)^\alpha-1][(t_\alpha-1)^\alpha+1]^{-1} \\ &= [(t_\alpha+1)/(t_\alpha-1)]^{\alpha-1} = 2(t_\alpha+1)^{\alpha-1}-1. \end{aligned}\tag{4.2}$$

Then $K_\alpha=q_\alpha$ for $\alpha>1$, $K_\alpha=1/q_\alpha$ for $0<\alpha<1$, and $K_1=1$.

The quantity q_α is a continuous strictly increasing function of α. Hence for any given $K>1$, there are unique numbers $\alpha_1(K)>1$ and $\alpha_2(K)\in(0, 1)$ such that $K=K_\alpha$ for $\alpha=\alpha_1(K)$ and for $\alpha=\alpha_2(K)$.

We define $D_0=(\lambda_1-1)\beta_1^{-1}$,

$$D_1 = \beta_1 D_0^{\alpha_1}\{(\lambda_1^{\alpha_1}-1)^{-1}-(\lambda_1-1)^{-1}\} < 0,$$

$D_2=D_0^{\alpha_1-1}$, and we define D_0', D_1', D_2', by the same formulas, replacing β_1, λ_1, α_1 by β_1', λ_1', α_2. We prove the following result.

Theorem 1. *Let $r(K)$ be rational. Then as $x\to\infty$, we have*

$$M_0(x, K)x^{-\alpha_1} \sim \varphi_1(\log x+\log D_0), \tag{4.3}$$

$$m_0(x, K)x^{-\alpha_2} \sim \varphi_2(\log x+\log D_0'), \tag{4.4}$$

where φ_1 and φ_2 are continuous periodic functions, φ_1 has period $\log \lambda_1$, φ_2 *has period* $\log \lambda_1'$, *and*

$$\varphi_1(v) = D_1 \exp(-v\alpha_1)+D_2 \exp\big(v(1-\alpha_1)\big), \quad 0 \leqq v < \log \lambda_1, \tag{4.5}$$

$$\varphi_2(v) = D_1' \exp(-v\alpha_2)+D_2' \exp\big(v(1-\alpha_2)\big), \quad 0 \leqq v < \log \lambda_1'. \tag{4.6}$$

Furthermore,

$$\log \lambda_1 = (\log L)\big(q(\alpha-1)\big)^{-1} = q^{-1}\log(t_\alpha+1), \quad \alpha = \alpha_1(K), \tag{4.7}$$

$$\beta_1 = 2q^{-1}\big(r+2(t_\alpha+1)^{r-1}\big)^{-1}, \quad \alpha = \alpha_1(K), \tag{4.8}$$

$$\log \lambda_1' = (\log 1/A)\big((p-q)(1-\alpha)\big)^{-1} = (p-q)^{-1}\log(t_\alpha+1), \quad \alpha = \alpha_2(K), \tag{4.9}$$

$$\beta_1' = 2(p-q)^{-1}\big(s+2(t_\alpha+1)^{s-1}\big)^{-1}, \quad \alpha = \alpha_2(K), \quad s = s(K). \tag{4.10}$$

4.1. We prove (4.3), (4.5), (4.7) and (4.8). The proof of (4.4), (4.6), (4.9) and (4.10) is similar.

Clearly (4.8) follows from (3.2) and (4.7). Next let P be given by (3.1) with $m=p-q$. By Lemma 1, λ_1 is the unique positive zero of P. So to prove (4.7), it suffices to show that $P(\delta)=0$, where $\delta=(t_\alpha+1)^{1/q}$. By Lemma A and (4.2) we have $K=[(t+1)/(t-1)]^{\alpha_1-1}$, where $t=t_\alpha$, $\alpha=\alpha_1(K)$, and $L=(t+1)^{\alpha-1}$, which proves the second equality in (4.7). We get

$$\begin{aligned} P(\delta) &= \delta^p(1-2\delta^{-q})-1 = (\delta^q)^{r-1}(\delta^q-2)-1 \\ &= (t_\alpha+1)^r(t_\alpha-1)(t_\alpha+1)^{-1}-1 \\ &= L^{r/(\alpha-1)}K^{-1/(\alpha-1)}-1 = 0 \end{aligned}$$

since $L^r=K$ by the definition of r. This proves (4.7).

4.2. It remains to consider the asymptotic behaviour of $M_0(x, K)$. By (2.5) and Lemmas 1 and 2 we have

$$A_n = \beta_1\lambda_1^n\big(1+\textstyle\sum_{i=2}^p(\beta_i/\beta_1)(\lambda_i/\lambda_1)^n\big) = \beta_1\lambda_1^n(1+E_n),$$

where $|E_n|\leqq E\sigma^n$ for some σ, $0<\sigma<1$, and some positive E. The function f has the slope $L^{n/q}$ on (x_{n-1}, x_n), and

$$x_n = 1+\textstyle\sum_{i=1}^n A_i.$$

Suppose that $x_n \leqq x < x_{n+1}$. Then

$$f(x) = 1+\sum_{i=1}^{n} A_i L^{i/q}+(x-x_n)L^{(n+1)/q}. \tag{4.11}$$

By [5, Theorem 1] we have $f(x_n)=M_0(x_n, K)$. By [5, Theorem 4], there are points z_1, z_2 such that $x_n \leqq z_1 \leqq x \leqq z_2 \leqq x_{n+1}$, $z_2-z_1 \leqq 2^{2^p+p}$, and $f(z_i)=M_0(z_i, K)$ for $i=1, 2$. Since f and $M_0(x, K)$ are increasing and $f \leqq M_0$ (since $f \in N_0(K)$), we have

$$0 \leqq M_0(x, K)-f(x) \leqq f(z_2)-f(x) = (z_2-x)L^{(n+1)/q} \leqq BL^{n/q}, \tag{4.12}$$

where $B=L^{1/q}2^{2^p+p}$.

Next we estimate x_n and $f(x_n)$. We have

$$x_n = 1+\sum_{i=1}^{n} \beta_1\lambda_1^i(1+E_i) = \beta_1\lambda_1^{n+1}(\lambda_1-1)^{-1}(1+S_n) \tag{4.13}$$

where $|S_n| \leqq S\sigma^n$ for some positive S. Further, we have

$$f(x_n) = 1+\sum_{i=1}^{n} \beta_1(\lambda_1 L^{1/q})^i(1+E_i) = \beta_1\lambda_1^{\alpha(n+1)}(\lambda_1^\alpha-1)(1+T_n) \tag{4.14}$$

where $\alpha=\alpha_1(K)$ and $|T_n| \leqq T\sigma^n$ for some positive T. Note that $\lambda_1 L^{1/q}=\lambda_1^\alpha$ by (4.7).

We write $x=x_n e^v$ and deduce from (4.11), (4.12), (4.13) and (4.14) that

$$\begin{aligned} M_0(x, K)x^{-\alpha} = {} & \beta_1^{1-\alpha}(\lambda_1-1)^{-\alpha}e^{-v\alpha}(1+S_n)^{-\alpha} \\ & \cdot\{(\lambda_1^\alpha-1)^{-1}(1+T_n)+(e^v-1)(\lambda_1-1)^{-1}(1+S_n)\}, \end{aligned} \tag{4.15}$$

where we have included the effect of $M_0(x, K)-f(x)$ in the T_n-term, as we may, and divided through by $\lambda_1^{\alpha(n+1)}$. As $x\to\infty$, we have $n\to\infty$ and so S_n, $T_n\to 0$. We have $0 \leqq v \leqq \log(x_{n+1}/x_n)=\log\lambda_1+o(1)$ and

$$\begin{aligned} v = \log x-\log x_n &= \log x+\log D_0+(n+1)\log\lambda_1+\log(1+S_n) \\ &\equiv \log x+\log D_0 +o(1) \pmod{\log\lambda_1}. \end{aligned}$$

In view of the definition (4.5), (4.15) now implies (4.3).

We claimed that φ_1 is continuous, i.e. $\varphi_1(0)=\lim\varphi_1(v)$ as $v\to\log\lambda_1-$, which reads

$$D_1+D_2 = D_1\lambda_1^{-\alpha}+D_2\lambda_1^{1-\alpha},$$

i.e. $(\lambda_1^\alpha-1)D_1+(\lambda_1^\alpha-\lambda_1)D_2=0$. This follows straight from the definitions of D_1 and D_2. Theorem 1 is proved.

5. On the functions $\varphi_1(v)$ and $\varphi_2(v)$

In this section we study the functions $\varphi_i(v)=\varphi_i(v, K)$ for $i=1, 2$, given by (4.5) and (4.6). This will show more precisely how much $M_0(x, K)x^{-\alpha_1(K)}$ and $m_0(x, K)x^{-\alpha_2(K)}$ can oscillate.

First we consider $\varphi_1(v)$. We may assume that $0 \leqq v \leqq \log\lambda_1$. We have $e^{v\alpha_1}\varphi'(v)=-\alpha_1 D_1-(\alpha_1-1)D_2e^v$ which vanishes at only one point v_0. Since φ_1 is not constant and $\varphi_1(0)=\varphi_1(\log\lambda_1)$, we have $0<v_0<\log\lambda_1$. Since $\varphi_1'(0+)>0$, as one can check, φ_1 takes its maximum value at $v=v_0$ and its minimum value at $v=0$. We write

$$\varrho=\varrho(K)=\frac{\max\varphi_1}{\min\varphi_1}=\frac{\varphi_1(v_0)}{\varphi_1(0)}>1,$$

so that

$$\varrho(K)=\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\frac{\lambda^\alpha-1}{\lambda-1}\left(1-\frac{\lambda-1}{\lambda^\alpha-1}\right)^{1-\alpha}, \tag{5.1}$$

where $\alpha=\alpha_1(K)$ and $\lambda=\lambda_1$. If $r(K)$ is irrational, we set $\varrho(K)=1$. We prove the following result, which shows that $\varrho(K)$ is bounded, that $\varrho(K)$ is continuous at $K=K_0$ if $r(K_0)$ is irrational and that $\varrho(K)$ does not tend to a limit as $K\to\infty$.

Theorem 2. *We have* $\varrho(K)<((3+\sqrt{5})/2)^{1/3}<1.38$ *for all K. If* $K_i\to K$ *and* $r=r(K)$ *is irrational, then* $\varrho(K_i)\to 1$, *and*

$$\varphi_1(0,K_i)\to\gamma_1(K)=\alpha^{-1}2^{1-\alpha}[\log(t+1)]^{\alpha-1}[r+2(t+1)^{r-1}]^{\alpha-1}, \tag{5.2}$$

where $t=t_\alpha$ *and* $\alpha=\alpha_1(K)$. *If* $K_i\to 1$, *then* $\varrho(K_i)\to 1$. *If* $K_i\to\infty$ *and* $r(K_i)=p_i/q_i$, *then*

$$\limsup_{i\to\infty}\varrho(K_i)=e^{-1}\sigma_1/\log\sigma_1\leqq 2/(e\log 2)<1.0615, \tag{5.3}$$

$$\liminf_{i\to\infty}\varrho(K_i)=e^{-1}\sigma_2/\log\sigma_2, \tag{5.4}$$

where

$$\sigma_i=2^{1/\tau_i},\quad \tau_i=M_i(1-2^{-1/M_i}),\quad i=1,2, \tag{5.5}$$

$$M_1=\liminf_{i\to\infty}p_i-q_i, \tag{5.6}$$

$$M_2=\limsup_{i\to\infty}p_i-q_i. \tag{5.7}$$

If $M_i=\infty$, then $\tau_i=1/\log 2$, $\sigma_i=e$, and $e^{-1}\sigma_i/\log\sigma_i=1$. Otherwise $e^{-1}\sigma_i/\log\sigma_i>1$. We have $M_i\geqq 1$. Thus $\sigma_i\leqq 4$ and $e^{-1}\sigma_i/\log\sigma_i\leqq 2/(e\log 2)$ since $M(1-2^{-1/M})$ increases from 1/2 to log 2 as M increases from 1 to ∞. The theorem shows that even for large K, it is possible to have $\varrho(K)$ bounded away from 1. The upper bound $((3+\sqrt{5})/2)^{1/3}$ for $\varrho(K)$ is not best possible, and it may be that $\varrho(K)<2/(e\log 2)$ for all K. In Theorem 5, Section 6, the function $\gamma_1(K)$ will be identified as the limit of $M_0(x,K)x^{-\alpha_1(K)}$ as $x\to\infty$ when $r(K)$ is irrational.

Hayman [3, proof of Theorem 5] showed that $\varrho(K)\leqq\lambda_1^{\alpha_1(K)}$, which also remains below an absolute constant, for example $2\sqrt{3}$.

5.1. To prove Theorem 2, suppose that $r(K)=p/q$ and note that

$$\varphi_1(0,K)=D_1+D_2=\left(\frac{\lambda_1-1}{\beta_1}\right)^{\alpha_1-1}\frac{\lambda_1-1}{\lambda_1^{\alpha_1}-1}.$$

Further, by (4.7), $\lambda_1=(t+1)^{1/q}$ where $t=t_\alpha$ and $\alpha=\alpha_1(K)$, and by Lemma 2 we have $2\beta_1^{-1}=p+2q(t+1)^{r-1}$. Hence if we take here $K=K_i\to K_0$ where $p=p_i$, $q=q_i\to\infty$ and $p_i/q_i\to r(K_0)$, we get (5.2), since $\alpha_1(K)$ and t_α are continuous functions of K.

If $K_i\to K$ and $r(K)$ is irrational, then $q_i\to\infty$, $\lambda_1-1\sim q_i^{-1}\log(t+1)$ and $\lambda_1^\alpha-1\sim q_i^{-1}\,\alpha\log(t+1)$. Hence an analysis of (5.1) shows that $\varrho(K_i)\to 1$.

Similarly we see that $\varrho(K_i)\to 1$ if $K_i\to 1$.

Suppose next that $K_i\to\infty$ in such a way that $p_i-q_i=m$ is a constant. Then by [4, Theorem 6] we have

$$r(K_i)=\frac{\log K_i}{\log L_i}=\frac{q_i+m}{q_i}$$

and

$$q_i=m\frac{\log K_i}{r(K_i)\log(K_i/L_i)}\sim\frac{m\alpha\log 3}{\log 2},\quad \alpha=\alpha_1(K_i).$$

Hence $\lambda^\alpha=(t_\alpha+1)^{\alpha/q_i}\to 2^{1/m}$ since $t_\alpha\to 2$. Further $\lambda-1\sim(\log 2)/(m\alpha)$. It follows that

$$\left(1-\frac{\lambda-1}{\lambda^\alpha-1}\right)^{1-\alpha}\to 2^{1/\tau},\quad \tau=m(2^{1/m}-1).$$

Since

$$\frac{(\alpha-1)^{\alpha-1}}{\alpha^\alpha}\frac{\lambda^\alpha-1}{\lambda-1}\to e^{-1}\tau/\log 2,$$

it follows from (5.1) that $\varrho(K_i)\to e^{-1}\sigma/\log\sigma$, where $\sigma=2^{1/\tau}$. Now (5.3) and (5.4) follow from this result.

5.2. It remains to find an upper bound for $\varrho(K)$. Suppose that $r(K)=p/q$ and that $1<\alpha<2$. Note that $q=2$ only if $r(K)=3/2$, i.e. if $\alpha_1(K)=2$. Now $\alpha<(C^\alpha-1)/(C-1)<\alpha C$ for any $C>1$. Hence

$$\varrho(K)=\left(1-\frac{1}{\alpha}\right)^\alpha\Big/\left\{\left(1-\frac{\lambda-1}{\lambda^\alpha-1}\right)^{\alpha-1}(\alpha-1)\frac{\lambda-1}{\lambda^\alpha-1}\right\}$$

$$<\frac{\lambda^\alpha-1}{\alpha(\lambda-1)}<\lambda=(t+1)^{1/q}\leqq(t+1)^{1/3}.$$

Since t_α is an increasing function of α and $t_\alpha=(1+\sqrt{5})/2$ when $\alpha=2$, we have $\varrho(K)<((3+\sqrt{5})/2)^{1/3}<1.38$ for $1<\alpha_1(K)<2$.

If $\alpha\geqq 2$, we could use the inequality $\alpha<(C^\alpha-1)/(C-1)<\alpha C^{\alpha-1}$ valid for $\alpha>1$, $C>1$, to get $\varrho(K)<\lambda^{\alpha-1}=L^{1/q}$. Since further $1/q\leqq(p-q)/q=\log(K/L)/\log L$ and $K\leqq 2L$, this gives $\varrho(K)<2$ for all $\alpha>1$. We get a better upper bound by observing that for any fixed $\alpha\geqq 2$, the right hand side of (5.1) is an increasing function of λ for $\lambda>1$. Hence using the bound $\lambda\leqq 2^{1/(\alpha-1)}$ obtained above we get

$$\varrho(K)<\frac{(\alpha-1)^{\alpha-1}}{2\alpha^\alpha}\frac{(2^{\alpha/(\alpha-1)}-1)^\alpha}{2^{1/(\alpha-1)}-1}=V(\alpha),$$

say. When α increases from 2 to ∞, $V(\alpha)$ decreases from 9/8 to $2/(e\log 2)$, so that $\varrho(K)<9/8=1.125$ for $\alpha\geqq 2$. This proves Theorem 2.

We remark that if $\alpha=2$, then $K=2+\sqrt{5}$, $r(K)=3/2$, and $\varrho(K)=(2+\sqrt{5})/4<1.05902$.

5.3. One can obtain similar results for the function $\varphi_2(v, K)$. There is a unique point $v_0\in(0, \log\lambda_1')$ such that $\varphi_2'(v_0)=0$. The function φ_2 takes its maximum at $v=0$ and its minimum at $v=v_0$. We set

$$\tilde{\varrho}(K)=\frac{\max\varphi_2}{\min\varphi_2}=(1-\alpha)^{1-\alpha}\alpha^\alpha\left(\frac{\lambda-1}{\lambda^\alpha-1}\right)^\alpha\left(1-\frac{\lambda^\alpha-1}{\lambda-1}\right)^{\alpha-1},$$

where $\alpha=\alpha_2(K)$ and $\lambda=\lambda_1'$. If $r(K)$ is irrational, we set $\tilde{\varrho}(K)=1$. The following result is proved in the same way as Theorem 2.

Theorem 3. *We have* $\tilde{\varrho}(K)<4$ *for all* K. *If* $K_i\to 1$ *or* $K_i\to\infty$, *we have* $\tilde{\varrho}(K_i)\to 1$. *If* $K_i\to K$ *and* $r(K)$ *is irrational, we have* $\tilde{\varrho}(K_i)\to 1$ *and* $\varphi_2(0, K_i)\to\gamma_2(K)=\alpha^{-1}2^{1-\alpha}[\log(t+1)]^{\alpha-1}[s+2(t+1)^{s-1}]^{\alpha-1}$, *where* $\alpha=\alpha_2(K)$, $t=t_\alpha$, *and* $s=s(K)$.

6. Asymptotic behaviour of M_0 and m_0 for irrational $r(K)$

Suppose that $r(K)$ is irrational. Since we have estimates for $M_0(x, K)$ and $m_0(x, K)$ when $r(K)$ is rational, which could be made even more precise as is shown by the remarks in Section 3, one could suggest that we choose a sequence $K_i\to K$ such that $r(K_i)$ rational, and try to obtain the asymptotic behaviour of $M_0(x, K)x^{-\alpha_1(K)}$ from the estimates for $M_0(x, K_i)x^{-\alpha_1(K_i)}$. However, it seems to me that even the best information mentioned in the remarks is far too imprecise for doing this. Therefore we shall consider $M_0(x, K)$ and $m_0(x, K)$ directly.

We start with the following technical result, which will be proved in Section 9 and which forms the basis for our estimates.

Theorem 4. *Let* a, b *be positive numbers such that* $a+b>1$, *suppose that* $r>1$, *and let* C *be the unique positive zero of the function*

$$P(x)=x^r-bx^{r-1}-a. \tag{6.1}$$

Then $C>\max(1, b, a^{1/r})$, $P'(C)>0$, *and the function*

$$S(X)=\sum_{\substack{p,q\geqq 0\\ pr+q\leqq X}}\binom{p+q}{q}a^p b^q \tag{6.2}$$

satisfies

$$S(X)=\frac{C^{X+r}}{CP'(C)\log C}(1+O(X^{-\eta})) \tag{6.3}$$

as $X\to\infty$, *for some positive constant* η.

Note that $CP'(C)=ar+bC^{r-1}$.

Now we can prove the following result.

Theorem 5. *If* $r=r(K)$ *is irrational, then*

(6.4) $$\lim_{x\to\infty} M_0(x, K)x^{-\alpha_1(K)} = \gamma_1(K)$$

and

(6.5) $$\lim_{x\to\infty} m_0(x, K)x^{-\alpha_2(K)} = \gamma_2(K),$$

where

(6.6) $$\gamma_1(K) = \alpha^{-1}2^{1-\alpha}[\log(t+1)]^{\alpha-1}[r+2(t+1)^{r-1}]^{\alpha-1}, \quad t = t_\alpha, \quad \alpha = \alpha_1(K),$$

and

(6.7)
$$\gamma_2(K) = \alpha^{-1}2^{1-\alpha}[\log(t+1)]^{\alpha-1}[s+2(t+1)^{s-1}]^{\alpha-1}, \quad t = t_\alpha, \quad \alpha = \alpha_2(K), \quad s = s(K).$$

Recall that $(r-1)(s-1)=1$.

6.1. Hayman [3, Theorem 1] proved that the limits (6.4) and (6.5) exist. So it suffices to evaluate them. We have no essentially new proof for the existence of the limits. We consider $M_0(x, K)$ only, since the argument for $m_0(x, K)$ is similar.

Now we apply Theorem 4 (so we make no further use of the irrationality of $r(K)$) with $r=r(K)$. Take first $a=1$, $b=2$. Then by [5, Theorem 1], for each $X\geqq 0$, $1+2S(X)$ is equal to a point X_n used in the definition of the function $f\in N_0(K)$ (note that the index p in Theorem 4 corresponds to some p_k-1 in [5, Theorem 1, (1.3)]). Taking then $a=K$, $b=2L$ and denoting the resulting $S(X)$ by $T(X)$, we deduce from [5, Theorem 1, (1.4)] that if $X_n=1+2S(X)$, then $f(X_n)=1+2KT(X)$. As $X\to\infty$, we have $X_n\to\infty$. Further, we have $f(X_n)=M_0(X_n, K)$ for all n by [5, Theorem 1].

The limit (6.4) is therefore equal to

(6.8) $$\lim_{X\to\infty} 2^{1-\alpha}KT(X)S(X)^{-\alpha}, \quad \alpha = \alpha_1(K).$$

If $P(X)$ is given by (6.1) with $a=1$, $b=2$, then $C=t_\alpha+1$. To prove this, it suffices to show that $P(t_\alpha+1)=0$, which is a consequence of (4.1) and (4.2) (cf. the argument in subsection 4.1). If P is given by (6.1) with $a=K$, $b=2L$, let us denote P by P_0 and the corresponding C by C_0. We have $C_0=CL$, since $P_0(CL)=0$. Namely, $P_0(CL)=C^rL^r-2LC^{r-1}L^{r-1}-K=0$ since $L^r=K$ and since $P(C)=0$.

We conclude from Theorem 4 that the limit (6.4) is equal to

$$\lim_{X\to\infty} 2^{1-\alpha}K\frac{(CL)^{X+r}C^\alpha(\log C)^\alpha P'(C)^\alpha}{CL(\log CL)P_0'(CL)C^{\alpha(X+r)}}.$$

Taking into account that $CL=C^\alpha$ by (4.2), we obtain (6.4) after some calculations. Theorem 5 is proved.

Remark. To prove (6.5) we apply Theorem 4 with $a=K^{-1}$, $b=1+K^{-1}$, so that $a+b=1+2K^{-1}$. Therefore it is essential to have the assumption $a+b>1$ instead of, for example, $a\geqq 1$, $b\geqq 1$ in Theorem 4, even though this makes the proof of Theorem 4 more complicated.

7. On $\gamma_1(K)$ and $\gamma_2(K)$

It may be of some interest to see how $\gamma_i(K)$ behaves as $K\to 1$ or $K\to\infty$, for $i=1,2$. This gives a better idea of the order of magnitude of $M_0(x,K)$ and $m_0(x,K)$.

Theorem 6. *We have $\gamma_i(K)\to 1$ for $i=1,2$, as $K\to 1$. As $K\to\infty$, we have $\gamma_2(K)\to 1$ while*

$$\gamma_1(K)\sim BK^A/\log K \tag{7.1}$$

where $A=(\log 3)^{-1}\log[(3\log 3)/2]=0.454676\ldots$ *and*

$$B=(\log 3)\exp\{(\log 2)(3\log 3)^{-1}(\log 4-2\log\log 3)\}=1.41346\ldots.$$

This should be compared to the estimate [4, Theorem 7]

$$\log 4\leqq \liminf_{K\to\infty} c_3(K)\leqq \limsup_{K\to\infty} c_3(K)\leqq \log 9,$$

where $c_3(K)=c_1(K)K^{-1}\log\log K$ and

$$c_1(K)=\sup_{x>1} M_0(x,K)x^{-\alpha_1(K)}.$$

The quantity $c_1(K)$ is much larger than $\gamma_1(K)$ since $c_1(K)$ is affected by $M_0(x,K)$ for x close to one.

Consider now $\gamma_1(K)$. We use (6.6) together with [4, (5.4)], which reads

$$\alpha_1(K)-1=\log K/\log 3-\log 4/\log 27+O((\log K)^{-1}), \tag{7.2}$$

and the result in [4, Section 3] that $t_\alpha\to 2$ as $K\to\infty$, where $\alpha=\alpha_1(K)$. This gives

$$\log\gamma_1(K)=-\log\alpha+(1-\alpha)\log 2+(\alpha-1)\log\log(t+1)+$$
$$+(\alpha-1)\log[r+2(t+1)^{r-1}],$$

where $\alpha=\alpha_1(K)$, $t=t_\alpha$, $r=r(K)$. Hence we see after some calculations that

$$\log\gamma_1(K)=-\log\log K+\log\log 3-(\log K)(\log 2)/\log 3+$$
$$+(\log 2)(\log 4)/\log 27+(\log K)(\log\log(t+1))/\log 3-(\log 4)(\log\log(t+1))/\log 27+$$
$$+(\log K)(\log G)/\log 3-(\log 4)(\log G)/\log 27+O(1/\log K),$$

where $G=r+2(t+1)^{r-1}\to 3$ as $K\to\infty$ and $r\to 1$. Since $(\log K)(r-1)=\log 2+O(1/\log K)$, we have

$$\log G=\log 3+(3\log K)^{-1}(\log 2)\log 9e+o(1/\log K)$$

as $K \to \infty$. Further, by [4, Lemma 1], we have

$$\log\log(t+1) = \log\log L - \log(\alpha-1)$$
$$= \log\log 3 - (\log 2)/(3\log K) + O((\log K)^{-2}),$$

in view of (7.2). Combining these formulas we obtain (7.1).

In fact one can show that

$$\gamma_1(K) = BK^A(\log K)^{-1}(1+O((\log K)^{-1})).$$

As $K \to 1$, we have $\alpha_i(K) \to 1$ for $i=1,2$. Thus $\gamma_i(K) \to 1$ for $i=1,2$, by (6.6) and (6.7).

Since $c_2(K) \leqq \gamma_2(K) \leqq 1$ by (1.4) and since $c_2(K) \to 1$ as $K \to \infty$ by [4, Theorem 5], we have $\gamma_2(K) \to 1$ as $K \to \infty$. This proves Theorem 6.

Remark. One can ask if $\gamma_1(K)$ is strictly increasing for $K \geqq 1$. We can show that this is the case at least when $\alpha_1(K) > 4.54$. Further, we have $\gamma_2(K) < 1$ for $1 < K < \infty$, and $\gamma_2(K) \to 1$ as $K \to 1$ or $K \to \infty$. It might be of some interest to determine the minimum value of $\gamma_2(K)$.

8. Asymptotic behaviour of K-qs functions

Let f be $K-qs$. Hayman [3, Theorem 4] showed that if

$$\limsup_{x\to\infty} f(x)x^{-\alpha_1(K)} > 0,$$

then

$$\liminf_{x\to\infty} f(x)x^{-\alpha_1(K)} > 0,$$

and that if

$$\liminf_{x\to\infty} f(x)x^{-\alpha_2(K)} < \infty,$$

then also

$$\limsup_{x\to\infty} f(x)x^{-\alpha_2(K)} < \infty.$$

In fact Hayman's results are more precise, particularly when $r(K)$ is irrational (see [3, Theorems 3, 4]). The above shows that if a $K-qs$ function grows at least sometimes as fast or as slowly as possible, then the function cannot oscillate too much. However, f can oscillate between two powers close to α_1 and α_2.

Theorem 7. *If $K>1$ and $0<\varepsilon<(\alpha_1(K)-\alpha_2(K))/2$, we set $\delta_1=(\alpha_1+\alpha_2)/2$ and $\delta_2=\alpha_1-\varepsilon-\delta_1$. The odd function f given by*

$$f(x) = \exp\{\delta_1\log x + \delta_2 \log x \cos(\eta \log\log(x+e))\} \tag{8.1}$$

for $x>0$ belongs to $N_0(K)$ and satisfies

$$\limsup_{x\to\infty} f(x)x^{-\alpha_1(K)+\varepsilon} \geqq 1 \tag{8.2}$$

and

$$\liminf_{x\to\infty} f(x)x^{-\alpha_2(K)-\varepsilon} \leqq 1 \tag{8.3}$$

provided that $0<\eta<\eta_0$, *where* η_0 *depends on K and* ε *only.*

Remark. Suppose that $h(x)$ is defined for $x>x_0$ and that $h(x)\to\infty$ as $x\to\infty$. One can ask if there is $f\in N_0(K)$ such that

$$\limsup_{x\to\infty} f(x)\left(x^{\alpha_1(K)}/h(x)\right)^{-1} > 0 \tag{8.4}$$

and

$$\liminf_{x\to\infty} f(x)\left(x^{\alpha_2(K)}h(x)\right)^{-1} < \infty. \tag{8.5}$$

One could try to find such an odd function f given by

$$f(x) = \exp\{E_1\log x+[E_2\log x-\log h(x)]\cos\eta\psi(x)\}$$

for $x>0$, where $E_1=(\alpha_1+\alpha_2)/2$, $E_2=(\alpha_1-\alpha_2)/2$, η is a small positive number depending on K and h, the function h is assumed to satisfy regularity conditions not essentially affecting its rate of growth, and $\psi(x)\to\infty$ slowly as $x\to\infty$, the rate of growth of ψ depending on that of h.

It seems to me that such a construction of f *might* work for some functions h growing more slowly than the powers $h(x)=x^{\varepsilon}$, but the case of an arbitrary h remains open.

8.1. We proceed to prove Theorem 7. To show that the function f given by (8.1) satisfies (1.1), we may assume that $x>0$ and write xt instead of t. Then (1.1) is equivalent to

$$f(x(1+t)) \leqq (K+1)f(x)-Kf(x(1-t)) \tag{8.6}$$

and

$$f(x(1+t)) \geqq (1+K^{-1})f(x)-K^{-1}f(x(1-t)), \tag{8.7}$$

which are to be proved for all positive x and t. Further, we must show that f is strictly increasing for $x>0$ if $\eta\leqq\eta_0$.

For brevity, we write $\psi(x)=\log\log(x+e)$, so that $\psi'(x)=[(x+e)\log(x+e)]^{-1}$. We have

$$xf'(x)/f(x) = \delta_1+\delta_2\cos\eta\psi(x)-\delta_2\eta(\sin\eta\psi(x))(x\log x)\psi'(x)$$

$$\geqq \delta_1-\delta_2-\delta_2\eta = \alpha_2(K)+\varepsilon-\delta_2\eta > 0$$

if η is small enough, so that then f is strictly increasing and defines a homeomorphism of the real axis onto itself. Note that $\log x<\log(x+e)$ for $x\geqq 1$ and that $|x\log x|< 1<e\leqq|\psi'(x)|^{-1}$ for $0\leqq x\leqq 1$.

8.2. We shall prove (8.6). The proof of (8.7) is similar. In view of (8.1) we can write (8.6) in the form

$$(1+t)^{\theta_1} e^{\theta_2} \leqq (K+1)+K(t-1)^{\theta_3} e^{\theta_4}, \quad t > 1, \tag{8.8}$$

$$(1+t)^{\theta_1} e^{\theta_2}+K(1-t)^{\theta_5} e^{\theta_6} \leqq K+1, \quad 0 < t \leqq 1, \tag{8.9}$$

where

$$\begin{aligned}
\theta_1 &= \delta_1+\delta_2 \cos \eta\psi(xt+x),\\
\theta_2 &= \delta_2 \log x(\cos \eta\psi(xt+x)-\cos \eta\psi(x)),\\
\theta_3 &= \delta_1+\delta_2 \cos \eta\psi(xt-x),\\
\theta_4 &= \delta_2 \log x(\cos \eta\psi(xt-x)-\cos \eta\psi(x)),\\
\theta_5 &= \delta_1+\delta_2 \cos \eta\psi(x-xt),\\
\theta_6 &= \delta_2 \log x(\cos \eta\psi(x-xt)-\cos \eta\psi(x)).
\end{aligned}$$

Note that $\alpha_2(K)+\varepsilon \leqq \theta_1,\ \theta_3, \theta_5 \leqq \alpha_1(K)-\varepsilon$. Further since for $0<y<z$ we have

$$|\cos \eta\psi(y)-\cos \eta\psi(z)| \leqq \eta|\psi(y)-\psi(z)| \tag{8.10}$$

$$= \eta \int_y^z \frac{du}{(u+e)\log(u+e)} \leqq \frac{\eta(z-y)}{(y+e)\log(y+e)},$$

we obtain

$$|\theta_2| \leqq \eta\delta_2 xt|\log x|[(x+e)\log(x+e)]^{-1} \leqq B\eta t,$$

$$|\theta_4| \leqq \eta\delta_2 x(t-2)|\log x|\psi'(x) \leqq B\eta t \quad \text{if} \quad t \geqq 2, \tag{8.11}$$

$$|\theta_4| \leqq \eta\delta_2 x(2-t)|\log x|\psi'(xt-x) \quad \text{if} \quad 1 \leqq t < 2,$$

$$|\theta_6| \leqq \eta\delta_2 xt|\log x|\psi'(x-xt),$$

where B depends on K only.

8.3. Recall ([1] or [4, Lemma 1] or Lemma A) that if $\alpha=\alpha_1(K)$ then for $1 \leqq \theta \leqq \alpha-\varepsilon$, we have

$$(1+t)^{\theta} \leqq K+1+K(t-1)^{\theta}-\sigma(\varepsilon, K)\big(1+(t-1)^{\theta}\big) \tag{8.12}$$

for $t \geqq 1$, where $\sigma(\varepsilon, K)>0$, and

$$(1+t)^{\theta}+K(1-t)^{\theta} \leqq K+1-\sigma(K) \tag{8.13}$$

for $0<t\leqq 1$, where $\sigma(K)>0$. We deduce from these inequalities and from (8.11) that if $t_0=t_0(\varepsilon, K)$ is large enough and $\eta=\eta(\varepsilon, K)$, $\tau=\tau(\varepsilon, K)$ and $\omega=\omega(\varepsilon, K)$ are small enough, then

(i) (8.9) holds for $0<t\leqq\tau$ if

$$\theta_5 \log(1-t)+\theta_6 \leqq \omega \tag{8.14}$$

for $0<t\leqq\tau$;

(ii) (8.9) holds for $\tau<t\leqq 1-\tau$ if $|\theta_1-\theta_5|\leqq\omega$, since $|\theta_6|\leqq B\eta C(\tau)$, where $C(\tau)$ depends on τ only;
(iii) (8.9) holds for $1-\tau<t<1$ and hence by continuity for $t=1$ if

$$\theta_5 \log(1-t)+\theta_6 \leqq \omega \log(1-t) \tag{8.15}$$

for $1-\tau<t<1$;
(iv) (8.8) holds for $1<t\leqq 1+\tau$;
(v) (8.8) holds for $1+\tau<t<t_0$ if $|\theta_1-\theta_3|\leqq\omega$ and $|\theta_4|\leqq\omega$; and
(vi) (8.8) holds for $t\geqq t_0$ if $t^{\theta_1-\theta_3}\leqq(K-\omega)e^{\theta_4-\theta_2}$, i.e.

$$\delta_2(\log t)(c_2-c_1) \leqq \log(K-\omega)+\delta_2(\log x)(c_1-c_2) \tag{8.16}$$

for $x>0$ and $t\geqq t_0$, where $c_1=\cos\eta\psi(xt-x)$ and $c_2=\cos\eta\psi(xt+x)$, and $K-\omega>1$.

8.4. It remains to verify that the conditions (i) to (vi) can be satisfied. We shall denote by B or B_1 any constant depending only on K and ε, not necessarily the same every time.

If $0<t\leqq\tau$, then by (8.11) we have

$$\begin{aligned}\theta_5 \log(1-t)+\theta_6 &\leqq (\alpha_2+\varepsilon)\log(1-t)+\delta_2\eta xt\,|\log x|\,\psi'(x-xt)\\ &\leqq -\alpha_2 t+B\eta t<0\end{aligned}$$

if η is small enough, so that (8.14) holds.

If $\tau<t\leqq 1-\tau$, then

$$|\theta_1-\theta_5| \leqq 2\eta\delta_2 xt\psi'(x-xt) \leqq B\eta\tau^{-1} \leqq \omega$$

if η is small enough.

If $1-\tau<t<1$, then by (8.11), we have

$$\theta_5 \log(1-t)+\theta_6 \leqq (\alpha_2+\varepsilon)\log(1-t)+\eta\delta_2 x\,|\log x|\,\psi'(x-xt),$$

which does not exceed

$$(\alpha_2+\varepsilon)\log(1-t)+B\eta \leqq \alpha_2\log(1-t) \quad \text{if} \quad 0<x\leqq B_1,$$

where $B_1\geqq 2$. Suppose that $x>B_1$, and choose a positive number $\delta=\delta(K)$ such that $\gamma=(\delta_1+\delta_2)/(2\delta_2)-\delta>1$. If $x\leqq(1-t)^{-\gamma}$, we have

$$\begin{aligned}\theta_5 \log(1-t)+\theta_6 &\leqq \delta_1\log(1-t)-\delta_2(\log x)\cos\eta\psi(x)+\\ &\quad +\delta_2\log\big(x(1-t)\big)\cos\eta\psi(x-xt)\\ &\leqq \big(\delta_1-\gamma\delta_2+(1-\gamma)\delta_2\big)\log(1-t)\\ &= 2\delta\delta_2\log(1-t) \leqq \omega\log(1-t)\end{aligned}$$

as required. If $x>(1-t)^{-\gamma}$, then $x(1-t)\geqq x^{1-1/\gamma}\geqq B_1^{1-1/\gamma}$. Thus

$$\begin{aligned}\psi(x)-\psi(x-xt) &= \log\log(x+e)-\log\log\big(x(1-t)+e\big)\\ &\leqq -B\log\big[1-(\log(1-t)^{-1})/(\log x)\big]\\ &\leqq B\big(\log(1-t)^{-1}\big)/\log x.\end{aligned}$$

It follows that $|\theta_6|\leqq B\eta\log(1-t)^{-1}$, so that (8.15) holds in all cases.

8.5. If $1+\tau<t<t_0$, then $|\theta_4|\leqq\omega$ by (8.11), and $|\theta_1-\theta_3|\leqq B\eta x\psi'(xt-x)\leqq\omega$ if η is small enough, depending on τ and t_0.

Finally if $t\geqq t_0$ we assume first that $x(t_0-1)\geqq 1$. We have

$$|c_1-c_2| \leqq 2\eta x\psi'(xt-x) \leqq 2\eta(t-1)^{-1}\big(\log(xt-x+e)\big)^{-1},$$

so that $|\delta_2(\log x)(c_1-c_2)|\leqq B\eta$ and $|\delta_2(\log t)(c_2-c_1)|\leqq B\eta$. Hence (8.16) holds for some positive ω if η is small enough.

If $x(t_0-1)<1$, we consider separately the possibilities $xt<1$ and $xt\geqq 1$. In each case we have to prove that

$$(\log xt)(c_2-c_1) \leqq \omega_1 = \delta_2^{-1}\log(K-\omega), \tag{8.17}$$

which implies (8.16).

If $xt\leqq 1$, then $x<1$ and

$$|(\log xt)(c_2-c_1)| \leqq B\eta x\,|\log xt| \leqq B\eta/t \leqq B\eta \leqq \omega_1.$$

If $xt\geqq 1$, then we obtain

$$|(\log xt)(c_2-c_1)|\leqq 2\eta x(\log xt)\big[\big(x(t-1)+e\big)\log\big(x(t-1)+e\big)\big]^{-1}\leqq B\eta\leqq\omega_1,$$

considering, for example, the cases $x(t-1)\geqq 2$, $1\leqq x(t-1)\leqq 2$ and $x(t-1)\leqq 1$. Hence (8.17) and thus (8.16) holds in all cases.

This completes the proof of Theorem 7.

9. Proof of Theorem 4

In this section we prove Theorem 4, stated in Section 6.

First we note that if $N\geqq 1$, then

$$1-\frac{B}{N} \leqq N!\left(\frac{N^N}{e^N}(2\pi N)^{1/2}\right)^{-1} \leqq 1+\frac{B}{N} \tag{9.1}$$

where $B=1/11$. We shall denote any positive constant by B, not necessarily the same every time. We have

$$\sum\nolimits_{q\leqq X}\binom{q}{q}b^q\leqq Bb^X=o\left(\frac{C^X}{\sqrt{X}}\right), \tag{9.2}$$

$$\sum\nolimits_{pr\leqq X}\binom{p}{0}a^p\leqq Ba^{X/r}=o\left(\frac{C^X}{\sqrt{X}}\right). \tag{9.3}$$

Hence we may assume that $p\geqq 1$ and $q\geqq 1$ in (6.2).

So we have by (9.1),

$$(9.4)\qquad \binom{p+q}{q} \leqq B\frac{(p+q)^{p+q}}{p^p q^q \sqrt{2p}}\left(\frac{1}{p}+\frac{1}{q}\right)^{1/2} \leqq B\left(1+\frac{q}{p}\right)^p\left(1+\frac{p}{q}\right)^q,$$

where instead of B we first have

$$(1+(11(p+q))^{-1})(1-(11p)^{-1})^{-1}(1-(11q)^{-1})^{-1} \leqq 1.265.$$

We write

$$(9.5)\qquad x = q/X,\quad y = rp/(X-q).$$

Since $p, q \geqq 1$ and $pr+q \leqq X$, we have

$$0 < \frac{1}{X} \leqq x \leqq 1-\frac{r}{X} < 1$$

and

$$0 < \frac{r}{X-1} \leqq y \leqq 1.$$

We define for $0<x<1$, $0<y\leqq 1$,

$$h(x, y) = a^{y(1-x)/r} b^x \left(1+\frac{y(1-x)}{rx}\right)^x \left(1+\frac{rx}{y(1-x)}\right)^{y(1-x)/r},$$

$$h(x) = h(x, 1) = a^{(1-x)/r} b^x \left(1+\frac{1-x}{rx}\right)^x \left(1+\frac{rx}{1-x}\right)^{(1-x)/r},$$

$$T(x, y) = (1-x)\left(\frac{1}{x}+\frac{r}{y(1-x)}\right)^{1/2},$$

$$T(x) = T(x, 1) = (1-x)\left(\frac{1}{x}+\frac{r}{1-x}\right)^{1/2}.$$

Using the values given by (9.5) for x and y, we obtain

$$(9.6)\qquad \left(1+\frac{q}{p}\right)^p \left(1+\frac{p}{q}\right)^q a^p b^q = h(x, y)^X,$$

$$(9.7)\qquad \frac{(p+q)^{p+q}}{p^p q^q}\left(\frac{1}{p}+\frac{1}{q}\right)^{1/2} a^p b^q = h(x, y)^X T(x, y)\frac{\sqrt{X}}{X-q}.$$

9.1. We proceed to find some properties of $H(x, y) = \log h(x, y)$ and $H(x) = H(x, 1)$. We have

$$rH''(x) = \frac{(r-1)^2}{rx+1-x} - \frac{r}{x} - \frac{1}{1-x} = \frac{-r}{x(1-x)((r-1)x+1)} < 0,$$

so that

$$(9.8)\qquad rH'(x) = \log\frac{b^r((r-1)x+1)^{r-1}(1-x)}{a(rx)^r}$$

is strictly decreasing for $0<x<1$, and there is a unique number $x_0\in(0, 1)$ such that $H'(x_0)=0$. Then $H''(x_0)<0$, and $H(x)$ has its unique global maximum at $x=x_0$. Further, $H(x)$ is strictly increasing for $0<x\leqq x_0$ and strictly decreasing for $x_0\leqq x<1$.

Using the fact that $H'(x_0)=0$, we see after a lengthy but routine calculation that $P(h(x_0))=0(h(x_0)=e^{H(x_0)})$. Hence $h(x_0)=C$ and $H(x_0)>0$. A calculation also shows that the number γ defined by

$$(9.9)\qquad \gamma = \left(\frac{1}{x_0}+\frac{r}{1-x_0}\right)^{1/2}\left\{\left[\log\left(1+\frac{rx_0}{1-x_0}\right)\right](-H''(x_0))^{1/2}\right\}^{-1}$$

satisfies

$$(9.10)\qquad \gamma = C^{r-1}/(P'(C)\log C),$$

and that

$$(9.11)\qquad 1 < C^r = a\left(1+\frac{rx_0}{1-x_0}\right).$$

We omit the details.

9.2. Our next aim is to show that $H(x_0)=H(x_0, 1)$ is the unique maximum of $H(x, y)$ for $0<x<1$, $0<y\leqq 1$. We have

$$(9.12)\qquad r\frac{\partial}{\partial y}H(x, y) = (1-x)\log\left[a\left(1+\frac{rx}{y(1-x)}\right)\right].$$

For a fixed x, either this is positive for $0<x<1$, in which case $H(x, y)$ is strictly increasing, or there is y_0 such that $\partial_y H(x, y)>0$ for $0<y<y_0$ and $\partial_y H(x, y)<0$ for $y_0<y<1$. In the former case $H(x, y)<H(x)<H(x_0)$ for $0<y<1$ and $x\neq x_0$, and $H(x_0, y)<H(x_0)$ for $0<y<1$. In the latter case we have $0<a<1$ and $x<x_0-\varepsilon_1$ for some ε_1. Namely, if $x\geqq x_0-\varepsilon_1$ and $0<y\leqq 1$, then since $x/(1-x)$ is an increasing function of x, we have

$$\log a\left(1+\frac{rx}{y(1-x)}\right) \geqq \log a\left(1+\frac{r(x_0-\varepsilon_1)}{y(1-x_0+\varepsilon_1)}\right)$$
$$\geqq \log a\left(1+\frac{r(x_0-\varepsilon_1)}{1-x_0+\varepsilon_1}\right) > 0$$

by (9.11) if ε_1 is small enough.

If there are any x, y with $\partial_y H(x, y)=0$, then clearly there is $x_1\in(0, x_0-\varepsilon_1)$ with the following property. For every $x\in(0, x_1)$ there is $y=y_0(x)\in(0, 1)$ such that $\partial_y H(x, y)=0$ i.e. $a(1+rx/[y_0(x)(1-x)])=1$, while for $x\in(x_1, 1)$, $H(x, y)$ is a strictly increasing function of y. Suppose that this is the case. Since $h(0, y)=1<h(x_0)$, it suffices to show that

$$\sup_{0<x\leqq x_1} H(x, y_0(x)) < H(x_0).$$

As we must have $y_0(x_1)=1$, we have $H(x, y_0(x))\leqq H(x_0)-\varepsilon_2$ by continuity for some positive ε_2 and for $x_1-\varepsilon_2\leqq x\leqq x_1$. If $H(x, y_0(x))>1$ for any $x\leqq x_1-\varepsilon_2$, then

$H(x, y_0(x))$ attains a maximum on $(0, x_1-\varepsilon_2]$. So it suffices to prove that $H(x, y_0(x))<H(x_0)$ for these x, i.e. that $D=e^{H(x, y_0(x))}<C$. Using the fact that $\partial_y H(x, y)=0$ for $y=y_0(x)$, we obtain

$$1 < D = \left(\frac{b}{1-a}\right)^x < \left(\frac{b}{1-a}\right)^{x_1} = D_0 < \frac{b}{1-a}$$

since $b>1-a$. Further since $y_0(x_1)=1$, we have

$$x_1 = \frac{1-a}{ar+1-a}.$$

As $P(D_0)\leqq P(C)=0$ implies that $D_0\leqq C$, we shall show that for any fixed $a\in(0, 1)$ and $r>1$, the expression $P(D_0)$ is $\leqq 0$ for any $b>1-a$, where D_0 and x_1 are as above. This reads

$$h_1(b) = -A_1 b^{e_1}+a+A_2 b^{e_1+e_2} \geqq 0,$$

where $A_1=(1-a)^{-rx_1}$, $A_2=(1-a)^{(1-r)x_1}$, $e_1=rx_1$, $e_2=1-x_1>0$, and $h_1(1-a)=0$, while $h_1'(b)>0$ for $b>1-a$. Thus $h_1(b)>0$ for $b>1-a$. This proves that $H(x_0)$ is the unique maximum of $H(x, y)$.

9.3. Now we can estimate $S(X)$. Suppose that $0<\delta<1/2$, $\varepsilon>0$, $0<x_0-\varepsilon< x_0+\varepsilon<1$, and set

$$V(\delta, \varepsilon) = \{(p, q)|p, q \geqq 1, pr+q \leqq X, |x-x_0| \geqq \varepsilon \text{ or } y \leqq 1-\delta\}$$

where x and y are given by (9.5). With

$$U(p, q) = \binom{p+q}{q} a^p b^q,$$

we have by (9.4) and (9.6),

$$\sum\nolimits_{V(\delta,\varepsilon)} U(p, q) \leqq B \sum\nolimits_{V(\delta,\varepsilon)} h(x, y)^X \leqq BX^2(\eta C)^X = o\left(\frac{C^X}{\sqrt{X}}\right), \tag{9.13}$$

where $\eta=\eta(\delta, \varepsilon, r, a, b)\in(0, 1)$ is such that $h(x, y)\leqq\eta C$ if $|x-x_0|\geqq\varepsilon$ or $y\leqq 1-\delta$. Note that there are at most BX^2 terms in the sum.

We set

$$W = W(\delta, \varepsilon) = \{(p, q)|p, q \geqq 1, pr+q \leqq X, |x-x_0| < \varepsilon \text{ and } y > 1-\delta\}.$$

If $(p, q)\in W$, then with

$$R(p, q) = \frac{(p+q)^{p+q}}{p^p q^q \sqrt{2\pi}} \left(\frac{1}{p}+\frac{1}{q}\right)^{1/2},$$

we have

$$\left|\binom{p+q}{q} - R(p, q)\right| \leqq BR(p, q)X^{-1}. \tag{9.14}$$

Hence it suffices to prove the asymptotic formula (6.3) for

$$F(X) = \sum\nolimits_W R(p, q) a^p b^q$$

instead of $S(X)$. That together with our earlier estimates (9.2), (9.3), (9.13) and (9.14) then proves Theorem 4.

By (9.7) we have

$$F(X) = \frac{X\sqrt{X}}{r\sqrt{2\pi}} \sum\nolimits_W e^{XH(x,y)} T(x, y) \frac{1}{X} \frac{r}{X-q}, \tag{9.15}$$

where x, y are given by (9.5). Here q runs from $X(x_0-\varepsilon)$ to $X(x_0+\varepsilon)$ and for each q, p runs from $(1-\delta)(X-q)/r$ to $(X-q)/r$. Hence any successive values of x and y, say x_1 and x_2, or y_1 and y_2, satisfy

$$|x_1-x_2| = 1/X, \quad |y_1-y_2| \leqq B/X,$$

where B depends on r and x_0. Using this and the monotonicity properties of $H(x, y)$, one can show that

$$\left|F(X) - \frac{X^{3/2}}{r\sqrt{2\pi}} I(X)\right| \leqq \frac{BC^X}{\sqrt{X}}, \tag{9.16}$$

where

$$I(X) = \int_{x_0-\varepsilon}^{x_0+\varepsilon} \int_{1-\delta}^{1} e^{XH(x,y)} T(x, y)\, dx\, dy,$$

provided that δ and ε are small enough and X is large enough. We omit the details.

9.4. It remains to estimate $I(X)$ for small but fixed δ and ε as $X\to\infty$. In particular, we make sure that $x_1<x_0-\varepsilon$, if there exists x_1 as above. We note that

$$0 < H(x)-H(x, y) = (1-y)\partial_y H(x, \xi)$$

for some $\xi\in(y, 1)$. If $y\leqq 1-BX^{-1}\log X$ for a suitable B then

$$\begin{aligned} H(x, y) &\leqq H(x_0)-(1-y) \min_{y\leqq\xi\leqq 1} \partial_y H(x, \xi) \\ &= \log C-(1-y)(1-x)r^{-1}\log a\left(1+\frac{rx}{1-x}\right) \\ &\leqq \log C - 5X^{-1}\log X \end{aligned}$$

and $\exp XH(x, y)\leqq C^X X^{-5}$. Hence

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon} \int_{1-\delta}^{1-BX^{-1}\log X} T(x, y) e^{XH(x,y)}\, dx\, dy \leqq BC^X X^{-5}.$$

Consider next

$$E(x, y) = H(x)-H(x, y)-(1-y)\partial_y H(x, 1)$$

for $|x-x_0|\leqq\varepsilon$, $1-BX^{-1}\log X\leqq y\leqq 1$. Let x be fixed. We have $E(x, 1)=0$. Hence $-E(x, y)=(1-y)\partial_y E(x, \xi)$ for some $\xi\in(y, 1)$. We have

$$\partial_y E(x, y) = -\partial_y H(x, y)+\partial_y H(x, 1) = (1-y)\partial_{yy} H(x, \xi_1)$$

for some $\xi_1 \in (y, 1)$. So

$$-E(x, y) = (1-y)(1-\xi)\partial_{yy}H(x, \xi_1)$$

where $y \leqq \xi \leqq \xi_1 \leqq 1$. We have

$$\partial_{yy}H(x, y) = \frac{-x(1-x)}{y(y(1-x)+rx)} < 0,$$

and $|\partial_{yy}H(x, y)| \leqq B$ for $|x-x_0| \leqq \varepsilon$, $1-\delta \leqq y \leqq 1$. Thus

$$0 \leqq E(x, y) \leqq B(1-y)^2 \leqq BX^{-2}(\log X)^2.$$

It follows that

$$e^{XH(x,y)} = e^{XH(x)}e^{-X(1-y)\partial_y H(x,1)}e^{-XE(x,y)}$$

$$= e^{XH(x)} \exp\left\{-X(1-y)(1-x)r^{-1}\log\left(1+\frac{rx}{1-x}\right)\right\}(1+E_1)$$

$$= e^{XH(x)}(\exp g(x, y))(1+E_1),$$

say, where

$$E_1 = E_1(x, y) = 1-\exp(-XE(x, y))$$

and so

$$|E_1| \leqq BX^{-1}(\log X)^2.$$

Hence

$$\int_{x_0-\varepsilon}^{x_0+\varepsilon}\int_{1-BX^{-1}\log X}^{1} e^{XH(x,y)}T(x, y)\,dx\,dy$$

$$= \iint e^{XH(x)}T(x, y)e^{g(x,y)}\,dx\,dy\left(1+O(1)X^{-1}(\log X)^2\right).$$

A similar argument shows that

$$T(x, y) = T(x, 1)+O(1)X^{-1}\log X,$$

so that we can replace $T(x, y)$ by $T(x)$ in the above integral. The resulting integral depends on y only through $1-y$ in the exponent, so that integrating with respect to y we obtain

$$\text{(9.17)} \qquad \int_{x_0-\varepsilon}^{x_0+\varepsilon} \frac{e^{XH(x)}T(x)r\,dx}{X(1-x)\log(1+rx/(1-x))}\left(1-\exp\{-B(\log X)B_1(x)\}\right),$$

where $B_1(x)=(1-x)\log(1+rx/(1-x))$. The second term in brackets is $O(X^{-\eta})$ for some positive $\eta<1$. To deal with the first term we use the following standard result (see e.g. [2, Theorem 2, p. 19]).

Lemma B. *Let $H(x)$ and $q(x)$ be analytic functions of x, regular on (c, d) and continuous on $[c, d]$. Let H be real and suppose that H attains its maximum at $x=x_0 \in (c, d)$ only, while $H''(x_0)<0$. Then as $X \to \infty$, we have*

$$\int_c^d e^{XH(x)}q(x)\,dx = e^{XH(x_0)}q(x_0)\left(\frac{2\pi}{X|H''(x_0)|}\right)^{1/2}\left(1+\frac{O(1)}{X}\right).$$

Applying Lemma B to our present $H(x)$ and to

$$q(x) = rT(x)[X(1-x)\log(1+rx/(1-x))]^{-1},$$

as we may in view of what we have proved about $H(x)$, we see that the integral (9.17) is equal to

$$\left(\frac{2\pi}{X}\right)^{1/2} \frac{re^{XH(x_0)}T(x_0)(-H''(x_0))^{-1/2}}{X(1-x_0)\log(1+rx_0/(1-x_0))}(1+O(X^{-1})).$$

Using this, the definition of $T(x)$, (9.16), and our earlier estimates we get

$$S(X) = \gamma C^X(1+O(X^{-\eta})).$$

In view of (9.10), this proves Theorem 4.

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