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THE CAPACITY METRIC ON RIEMANN SURFACES

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1. Introduction

Let $c_\beta(\zeta)|d\zeta|$ denote the capacity metric on a Riemann surface. In this paper two basic facts dealing with this metric are established. First, if $f\colon X\to Y$ is an analytic mapping of Riemann surfaces, then f is distance decreasing relative to the capacity metric. For $X, Y\notin O_G$, a necessary and sufficient condition that $f\colon X\to Y$ be an isometry is given: f must be injective and $Y\setminus f(X)$ must be a closed set of capacity zero. This condition is obtained from an analogous property of the Green's function. Let g_X, g_Y denote the Green's function on X, Y, respectively. If $g_X=g_Y\circ f$, then f must be as before. The second property of the capacity metric that we derive is an interpretation of this metric in terms of the reduced modulus of a path family of cycles homologous to a point. This naturally leads to the question of an analogous interpretation of other metrics. This question is answered for the Hahn metric by using the family of closed curves homotopic to a point.

2. Definition of the capacity metric

Let X be a Riemann surface. The customary notation for the capacity metric on X is $c_\beta(\zeta)|d\zeta|$, where β represents the ideal boundary of X. This notation is not convenient because we have need to consider the capacity metric on several surfaces simultaneously. In order to clearly indicate the dependence on the surface X, we shall employ the notation $c_X(\zeta)|d\zeta|$ for the capacity metric on X. Now we define the capacity metric. Suppose $\zeta\in X$ and t is a local parameter in a neighborhood of ζ such that $t(\zeta)=0$. Let $\mathscr{A}_\zeta^*(X)$ denote the family of all multiple-valued analytic functions F defined on X such that $|F|$ is single-valued, $F(\zeta)=0$ and $F'(\zeta)=1$ for one of the branches. Here $F'(\zeta)$ represents the derivative of $F\circ t^{-1}$ at the origin. Set

$$M[F]=\sup_{z\in X}|F(z)|.$$

Then ([7], [8, pp. 177—178])

$$\frac{1}{c_X(\zeta)}=\min_{F\in\mathscr{A}_\zeta^*(X)}M[F].$$

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If $c_X(\zeta)>0$, then the unique minimizing function is $F_X=\exp(p_X+ip_X^*)$, where p_X is the capacity function with pole at ζ. The usual notation for this capacity function is p_β, but we need to indicate the dependence on X.

The condition $c_X(\zeta)=0$ is independent of the point $\zeta\in X$ [8, p. 178]. The identical vanishing of the capacity metric $c_X(\zeta)|d\zeta|$ is equivalent to $X\in O_G$ [7]. Recall that O_G is the class of Riemann surfaces which do not possess a Green's function. For $c_X(\zeta)>0$,

$$p_X(z,\zeta)=k_X(\zeta)-g_X(z,\zeta),$$

where $g_X(z,\zeta)$ is the Green's function for X with logarithmic singularity at ζ and $k_X(\zeta)=k_\beta(\zeta)$ is the Robin constant defined by

$$k_X(\zeta)=\lim_{t\to 0}[g_X(z,\zeta)+\log|t|]$$

[8, p. 55]. Thus, if $c_X(\zeta)>0$, then

$$|F_X|=\exp(p_X)\leqq k_X(\zeta)$$

and $M[F_X]=k_X(\zeta)$. Consequently,

$$c_X(\zeta)=\exp(-k_X(\zeta)).$$

If X is a hyperbolic simply connected Riemann surface, then it is elementary to show that $c_X(\zeta)|d\zeta|=\lambda_X(\zeta)|d\zeta|$, where $\lambda_X(\zeta)|d\zeta|$ denotes the hyperbolic metric on X with constant curvature -4. An explicit formula for the capacity metric of an annulus is given in [9].

3. Green's function

In the nontrivial case the definition of the capacity metric involves the Green's function and the Robin constant. In this section we derive a property of the Green's function that will be basic in establishing a result for the capacity metric in Section 4. Moreover, this property of the Green's function might be of independent interest.

We begin by recalling a precise form of the Lindelöf principle that was established by Heins [2]. Suppose $X, Y\notin O_G$ and $f\colon X\to Y$ is an analytic function. Then for any $\omega\in Y$,

$$g_Y(f(z),\omega)=\sum_{f(\zeta)=\omega} n(\zeta,f)\,g_X(z,\zeta)+u_\omega(z),$$

where u_ω is a nonnegative harmonic function on X and $n(\zeta,f)$ is the order of f at the point ζ. Furthermore, u_ω has the canonical decomposition $u_\omega=q_\omega+s_\omega$, where q_ω is quasibounded and s_ω is singular. The following dichotomy holds: either $q_\omega>0$ for all $\omega\in Y$ or else $q_\omega=0$ for all $\omega\in Y$. The function f is said to belong to the class Bl if $q_\omega=0$ for all $\omega\in Y$. Also, $s_\omega=0$ except possibly for a set of ω of capacity zero. If $u_\omega=0$ for all $\omega\in Y$, then f is said to be of type Bl_1. Set

$$v_f(w) = \sum_{f(z)=w} n(z,f).$$

If f is of type Bl, then either $v=\sup\{v_f(w)\colon w\in Y\}$ is finite and $\{w\colon v_f(w)<v\}$ is a closed set of capacity zero or else $\{w\colon v_f(w)<\infty\}$ is an F_σ set of capacity zero. Thus f covers Y exactly the same number of times (possibly infinite) except for an F_σ set of capacity zero. If the valence of f is finite, then the exceptional set is closed. If f is of type Bl_1, then $v_f(w)$ is constant (possibly infinite).

Recall that the Green's function is a conformal invariant. This means that if $X, Y\notin O_G$ and $f\colon X\to Y$ is a conformal mapping, then $g_Y(f(z),f(\zeta))=g_X(z,\zeta)$ for all $(z,\zeta)\in X\times X$. The following theorem is sort of a converse.

Theorem 1. *Suppose* $X, Y\notin O_G$ *and* $f\colon X\to Y$ *is an analytic function. If there exist distinct points,* $p, q\in X$ *such that*

$$g_Y(f(p),f(q)) = g_X(p,q), \tag{1}$$

then f is injective and $Y\setminus f(X)$ *is a closed set of capacity zero.*

Proof. We begin by showing that if (1) holds, then $f(\zeta)\neq f(q)$ for all $\zeta\in X\setminus\{q\}$, $n(q,f)=1$ and $g_Y(f(\zeta),f(q))=g_X(\zeta,q)$ for all $\zeta\in X$. The sharp form of the Lindelöf principle gives

$$g_Y(f(\zeta),f(q)) = \sum_{f(z)=f(q)} n(z,f)\,g_X(\zeta,z)+u(\zeta), \tag{2}$$

for $\zeta\in X$, where $u=u_{f(q)}$ is a nonnegative harmonic function. For $\zeta=p$ equation (2) yields

$$\begin{aligned} g_Y(f(p),f(q)) &= \sum_{f(z)=f(q)} n(z,f)\,g_X(p,z)+u(p)\\ &\geqq n(q,f)\,g_X(p,q)+u(p)\\ &\geqq g_X(p,q). \end{aligned}$$

Equation (1) implies that equality holds throughout, so $n(q,f)=1$, $u=0$ and $f(\zeta)\neq f(q)$ for all $\zeta\in X\setminus\{q\}$. Thus, equation (2) becomes

$$g_Y(f(\zeta),f(q)) = g_X(\zeta,q) \tag{3}$$

for all $\zeta\in X$. This establishes the claim made at the beginning of the paragraph.

Now, we complete the proof. From (3) and the symmetry of the Green's function, we obtain $g_Y(f(q),f(\zeta))=g_X(q,\zeta)$ for all $\zeta\in X$. The argument given in the preceding paragraph immediately implies that $f(z)\neq f(\zeta)$ for all $z\in X\setminus\{\zeta\}$, $n(z,f)=1$ and

$$g_Y(f(z),f(\zeta)) = g_X(z,\zeta) \tag{4}$$

for all $z\in X$. But this also holds for all $\zeta\in X$. It follows directly that f is injective. The remainder of the theorem is obtained from the sharp form of the Lindelöf principle. From equation (4) we conclude that $u_\omega=0$ for all $\omega\in f(X)$. In particular,

$q_\omega=0$ for all $\omega\in f(X)$. The dichotomy given in the Lindelöf principle implies that we actually have $q_\omega=0$ for all $\omega\in Y$. Hence, f is of type Bl. Because f is injective, $\sup\{v_f(w)\colon w\in Y\}=1$. Then $Y\setminus f(X)=\{w\in Y\colon v_f(w)<1\}$ is a closed set of capacity zero.

It is elementary to show that if $f\colon X\to Y$ is injective and $Y\setminus f(X)$ has capacity zero, then equality holds in (1) for all $p, q\in X$. Just note that Y and $Y\setminus f(X)$ possess the same Green's function in this situation.

4. Properties of the capacity metric

First, we establish the elementary result that the hyperbolic metric dominates the capacity metric.

Theorem 2. *Let X be a hyperbolic Riemann surface. Then $c_X(\zeta)|d\zeta|\leqq\lambda_X(\zeta)|d\zeta|$. If equality holds at a single point, then X is simply connected.*

Proof. There is nothing to prove if $X\in O_G$, so we assume $X\notin O_G$. Fix $\zeta\in X$ and a local coordinate t at ζ with $t(\zeta)=0$. Suppose $F_X\in\mathscr{A}_\zeta^*(X)$ is the unique extremal function relative to the local coordinate t. Let $\mathbf{D}$ denote the open unit disk and $\pi\colon \mathbf{D}\to X$ an analytic universal covering such that $\pi(0)=\zeta$. Then $c_X(\zeta)F_X\circ\pi$ is a single-valued analytic mapping of $\mathbf{D}$ into itself that fixes the origin. Also, this function vanishes at each point of the set $\pi^{-1}(\zeta)$. Schwarz' lemma gives

$$c_X(\zeta)|F_X'(\zeta)||\pi'(0)|\leqq 1,$$

where $\pi'(0)$ denotes the derivative of $t\circ\pi$ at the origin. Thus,

$$c_X(\zeta)\leqq 1/|\pi'(0)|=\lambda_X(\zeta).$$

Equality implies that $c_X(\zeta)F_X\circ\pi$ is a rotation of $\mathbf{D}$ about the origin. In particular, it vanishes just once, so $\pi^{-1}(\zeta)=\{0\}$. This implies that π is univalent, so X must be simply connected.

Next, we demonstrate that an analytic function is distance decreasing relative to the capacity metric.

Theorem 3. *Suppose X and Y are Riemann surfaces and $f\colon X\to Y$ is an analytic function. Then*

$$f^*\big(c_Y(\zeta)|d\zeta|\big)\leqq c_X(\zeta)|d\zeta|,$$

where $f^*\big(c_Y(\zeta)|d\zeta|\big)$ denotes the pull-back to X via f of the capacity metric on Y. If $X\notin O_G$ and equality holds at a point, then f is injective and $Y\setminus f(X)$ is a closed set of capacity zero.

Proof. Fix $\zeta\in X$ and set $\omega=f(\zeta)$. Let u be a local coordinate at ω with $u(\omega)=0$. If $n(\zeta,f)\geqq 2$, then the pull-back of any metric via f vanishes at ζ. There is nothing to prove in this case, so we may assume that $n(\zeta,f)=1$. In this situation f is univalent in a neighborhood of ζ so that $t=u\circ f$ is a local coordinate at ζ with $t(\zeta)=0$. Let $f'(\zeta)$ denote the derivative of $u\circ f\circ t^{-1}$ at the origin. Again, there is nothing to prove if $Y\in O_G$, so we assume $c_Y(\omega)>0$. Let $F_Y\in\mathscr{A}_\omega^*(Y)$ be the unique extremal for $c_Y(\omega)$ relative to the local coordinate u. Then $(F_Y\circ f)/f'(\zeta)\in A_\zeta^*(X)$ so that

$$\frac{1}{c_X(\zeta)}\leqq M[(F_Y\circ f)/f'(\zeta)]\leqq\frac{M[F_Y]}{|f'(\zeta)|}=\frac{1}{c_Y(\omega)|f'(\zeta)|}$$

or

$$c_Y(\omega)\,|f'(\zeta)|\leqq c_X(\zeta).$$

This establishes the inequality in the theorem.

Next, assume that $X\notin O_G$ and that equality holds at ζ. Then $Y\notin O_G$ and $f'(\zeta)\neq 0$. Since equality holds at ζ,

$$\exp(-k_Y(\omega))\,|f'(\zeta)|=\exp(-k_X(\zeta))$$

or

$$k_Y(\omega)-\log|f'(\zeta)|=k_X(\zeta).$$

Also, when equality holds at ζ the work in the preceding paragraph shows that $(F_Y\circ f)/f'(\zeta)\in A_\zeta^*(X)$ is a minimizing function, so it equals F_X. This gives $p_X=p_Y\circ f-\log|f'(\zeta)|$, or

$$k_X(\zeta)-g_X(z,\zeta)=k_Y(\omega)-g_Y(f(z),f(\zeta))-\log|f'(\zeta)|,$$

$$g_X(z,\zeta)=g_Y(f(z),f(\zeta)).$$

By applying Theorem 1, we obtain the desired conclusion.

Observe that if $X\in O_G$ and $Y\notin O_G$, then Theorem 3 implies that every analytic function $f\colon X\to Y$ must be constant. Also, if $X\notin O_G$, $f\colon X\to Y$ is injective and $Y\setminus f(X)$ is a closed set of capacity zero, then it is not difficult to show that equality holds in Theorem 3 at every point of X.

5. Reduced modulus interpretation of the capacity metric

We start by defining the reduced modulus of a special type of family of paths on a Riemann surface that always leads to a metric on the surface.

Let X be a Riemann surface and $\mathscr{F}$ a family of paths on X. Suppose $\zeta\in X$ and t is a local coordinate at ζ such that $t(\zeta)=0$. Assume that the range of this local coordinate contains the disk of radius R centered at the origin. For $0<r<s<R$ assume that the family $\mathscr{A}(r,s)$ of closed Jordan curves in $A(r,s)=\{z\in X\colon r<|t(z)|<s\}$ which separate the boundary components is a subset of $\mathscr{F}$. The symbol $\mathscr{F}_\zeta(r)$ denotes the set of paths in $\mathscr{F}$ which lie in $X\setminus\{z\in X\colon |t(z)|<r\}$. We show that $M(\mathscr{F}_\zeta(r))+(1/2\pi)\log r$ increases as r decreases, where $M(\mathscr{F}_\zeta(r))$ denotes the modulus of the path family $\mathscr{F}_\zeta(r)$. Note that $\mathscr{F}_\zeta(s)\cup\mathscr{A}(r,s)\subset\mathscr{F}_\zeta(r)$ for $0<r<$

$s<R$. Because the families $\mathscr{F}_\zeta(s)$ and $\mathscr{A}(r, s)$ have disjoint support [8, p. 321]

$$M(\mathscr{F}_\zeta(s))+M(\mathscr{A}(r, s)) \leqq M(\mathscr{F}_\zeta(r)).$$

Now, $M(\mathscr{A}(r, s))=(1/2\pi)\log(s/r)$ [8, p. 325], so that

$$M(\mathscr{F}_\zeta(s))+\frac{1}{2\pi}\log s \leqq M(\mathscr{F}_\zeta(r))+\frac{1}{2\pi}\log r.$$

Define

$$\tilde{M}(\mathscr{F}_\zeta) = \lim_{r\downarrow 0} M(\mathscr{F}_\zeta(r))+\frac{1}{2\pi}\log r.$$

This quantity is called the reduced modulus of the family $\mathscr{F}$ at the point ζ. The value of the reduced modulus does depend on the choice of the local coordinate at ζ. It is not difficult to verify that $\exp(-2\pi\tilde{M}(\mathscr{F}_\zeta))|d\zeta|$ is an invariant form, or metric, on X; see [4] for analogous results.

Our goal is to express the capacity metric in terms of the reduced modulus of a path family.

Definition. Let X be a Riemann surface and $\zeta\in X$. A 1-cycle c on $X\setminus\{\zeta\}$ is said to be homologous to ζ if for every neighborhood U of ζ, c is homologous to a closed Jordan curve in $U\setminus\{\zeta\}$ which winds around ζ once in the positive direction. Let $\mathscr{H}_\zeta$ denote the family of all 1-cycles on $X\setminus\{\zeta\}$ that are homologous to ζ.

Theorem 4. *Let* $X\notin O_G$. *Then*

$$c_X(\zeta)\,|d\zeta| = \exp(-2\pi\tilde{M}(\mathscr{H}_\zeta))\,|d\zeta|.$$

Proof. Fix $\zeta\in X$. It suffices to demonstrate equality at ζ. In fact, it is enough to show equality relative to some fixed local coordinate at ζ. We begin by selecting a local coordinate that will make it easy to demonstrate equality. Let $g(z)=g_X(z, \zeta)$ be the Green's function for X with logarithmic singularity at ζ. In a deleted neighborhood of ζ let g^* denote a harmonic conjugate for g. Of course, g^* is not single-valued; it is only determined up to an additive multiple of 2π. However, in a small neighborhood of ζ the function $t(z)=\exp(-g(z)-ig^*(z))$ is a local coordinate that satisfies $t(\zeta)=0$. We shall establish equality in terms of this special local coordinate at ζ.

Next, assume that X is the interior of a compact bordered Riemann surface $\bar{X}$. Let $B(\zeta, r)=\{z\in X\colon |t(z)|<r\}$, where $r>0$ is sufficiently small. Assume that $\partial B(\zeta, r)$ is positively oriented. Then g is harmonic on $Y=X\setminus\overline{B(\zeta, r)}$, has the constant value 0 on ∂X and the constant value $\log(1/r)$ on $\partial B(\zeta, r)$. Thus, $\omega=g/\log(1/r)$ is the harmonic measure of $\partial B(\zeta, r)$ with respect to the surface Y. It follows that $d\omega$ is the Γ_h-reproducing differential for any 1-cycle c on Y which is homologous to

$\partial B(q, r)$ [6, p. 135]. That is,

$$\int_c \sigma = (\sigma, d\omega)_Y = \iint_Y \sigma \wedge {}^*d\omega$$

for any square integrable harmonic differential σ on Y, where c is as above. The set of such 1-cycles is simply $\mathscr{H}_\zeta(r)$. A result of Accola [1] implies that

$$M(\mathscr{H}_\zeta(r)) = \|d\omega\|_Y^{-2}.$$

By making use of Stokes' theorem, we find that

$$\|d\omega\|_Y^2 = \iint_Y (\omega_x^2+\omega_y^2)\, dx\, dy = \iint_{\partial Y} \omega\, {}^*d\omega$$

$$= -\int_{\partial B(\zeta, r)} {}^*d\omega = \frac{2\pi}{\log(1/r)}.$$

Consequently, $M(\mathscr{H}_\zeta(r)) = -(1/2\pi)\log r$, so that $\tilde{M}_\zeta(\mathscr{H}_\zeta)=0$ and

$$\exp\left(-2\pi \tilde{M}(\mathscr{H}_\zeta)\right) = 1.$$

All that remains is to show that the capacity metric at ζ also has the value 1 relative to the local coordinate t. Now,

$$k_X(\zeta) = \lim_{t\to 0}\left(g(z, \zeta)+\log|t|\right) = 0$$

since $g(z, \zeta) = -\log|t|$, where $t=t(z)$. Thus, $c_X(\zeta)=\exp\left(-k_X(\zeta)\right)=1$. This completes the proof in case X is the interior of a compact bordered Riemann surface.

The general case follows by making use of an exhaustion of X by compact bordered surfaces. This method has been employed frequently, see [4] and other references mentioned there. For this reason we omit all details in the general case.

6. Reduced modulus interpretation of other metrics

It is possible to give a similar reduced modulus interpretation of the Hahn metric. For basic properties of this metric on a Riemann surface, see [5].

Definition. Let X be a Riemann surface and $\zeta \in X$. A closed path c on $X \setminus \{p\}$ is said to be homotopic to ζ if for every neighborhood U of ζ, c is freely homotopic to a closed Jordan curve in $U \setminus \{\zeta\}$ which winds around ζ once in the positive direction. Let $\mathscr{H}_\zeta$ denote the family of all closed paths on $X \setminus \{\zeta\}$ that are homotopic to ζ.

Theorem 5. *Let X be a Riemann surface. Then* $S_X(\zeta)|d\zeta| = \exp\left(-2\pi \tilde{M}(\mathscr{H}_\zeta)\right)$, *where $S_X(\zeta)|d\zeta|$ denotes the Hahn metric on X.*

Proof. In [5] it was shown that $S_X(\zeta)|d\zeta| = \exp\left(-\tilde{M}_X(\zeta)\,|d\zeta|\right)$, where $\tilde{M}_X(\zeta)$ denotes the following extremal value. For any hyperbolic simply connected region

Ω on X that contains ζ, $\tilde{M}_\Omega(\zeta)$ denotes the reduced modulus of Ω at ζ. Then

$$\tilde{M}_X(\zeta) = \sup \tilde{M}_\Omega(\zeta),$$

where the supremum is taken over all hyperbolic simply connected regions Ω on X that contain ζ. In [5] the modulus of the annulus $\{w\colon r_1<|w|<r_2\}$ was taken to be $\log(r_2/r_1)$; note the absence of the factor $1/2\pi$ in front of the logarithm. Now, it is known that $\tilde{M}_X(\zeta)=2\pi\tilde{M}(\mathscr{K}_\zeta)$ [3], so Theorem 5 is established.

If X is a hyperbolic Riemann surface, then

$$c_X(\zeta)\,|d\zeta| \leqq \lambda_X(\zeta)\,|d\zeta| \leqq S_X(\zeta)\,|d\zeta|.$$

Theorems 4 and 5 give reduced modulus interpretations for both the capacity metric and the Hahn metric. The preceding inequality naturally suggests the following question: Is there a reduced modulus interpretation of the hyperbolic metric? Of course, the same question can be asked for other metrics on a Riemann surface.

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