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THE COMPOSITION OF HARMONIC MAPPINGS

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0. Introduction

The function $f(z)$ is said to be a harmonic mapping if $f(z)$ is complex-valued harmonic:

$$f_{\bar{z}z} = 0.$$

Special properties of two-dimensional harmonic vectors were first considered by Kneser and Radó [3]. There is now a large literature on the subject.[2] Analytic functions and affine mappings, $f(z)=\mu z+\nu\bar{z}$, (μ, ν complex constants) constitute the simplest examples. Evidently, $f(z)$ is a harmonic mapping if and only if

$$f(z) = A(z)+\overline{B(z)},$$

where $A(z)$, $B(z)$ are analytic.

The current contribution is devoted to some elementary facts about compositions which seem to have escaped attention. It is easy to verify that if f and g are harmonic mappings, with domain $g\supset \text{range} f$, then $g\circ f$ is not "in general" harmonic. Trivial exceptions occur when f (or its conjugate) is analytic and g an arbitrary harmonic mapping, and when f is an arbitrary harmonic mapping and g is affine. That, however, there also exist *non*-trivial exceptions follows from an example given by Choquet [1] for which

$$g(f(z)) \equiv z, \tag{0.1}$$

even though neither f nor g are analytic, anti-analytic, or affine. Choquet [1, pp.164—165] credits J. Deny with proving that Choquet's example of harmonic mappings f, g satisfying (0.1) is essentially unique.

Our object (Theorem 1 and its corollary) is to obtain local descriptions of all harmonic mappings f with the property that $g\circ f$ is harmonic for some non-affine harmonic g. The descriptions are local in the sense that they concern sufficiently

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[2]) See [2] for a recent bibliography.

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small simply-connected neighborhoods of points where whatever non-constant analytic functions are involved are non-zero. It is perhaps surprising that very explicit descriptions can be obtained. For completeness we have also included a proof of the Choquet—Deny theorem, as [1] does not contain one.

1. Characterization of harmonic decompositions

We can assume, without loss of generality, that $f(z)$ has the form

$$f(z) = z+\overline{B(z)}, \tag{1.1}$$

where $B(z)$ is analytic, and that $g(w)$ has the form,

$$g(w) = C(w)+\overline{D(w)}, \tag{1.2}$$

where $C(w)$, $D(w)$ are analytic functions of w. The most general harmonic map has the form $f\circ\Phi$, where Φ is analytic, and f has form (1.1). The problem is to characterize those functions $B(z)$ for which there exist $C(w)$, $D(w)$, with $C''(w)$, $D''(w)$ not both vanishing, such that $g\circ f\circ\Phi$ is harmonic. Whether or not B has this property is of course independent of Φ.

Let

$$G(z) = B'(z).$$

Calculating the laplacian of $g\big(f(z)\big)$ we find that a necessary and sufficient condition that $g\big(f(z)\big)$ is harmonic is that

$$\overline{G(z)}\,C''(w) = -G(z)\overline{D''(w)} \quad \big(w = f(z) = z+\overline{B(z)}\big). \tag{1.3}$$

Our basic problem is to characterize the solutions of this functional equation. One solution is obtained when $G(z)$, $C''(w)$, and $D''(w)$ are appropriately related constants.

From now on assume that $G(z)$ is not a constant[3]. Since (1.3) implies that $|C''(w)|=|D''(w)|$, a necessary condition is that $D''(w)=\exp(is)C''(w)$ for some real constant s, or

$$\overline{G(z)}\,\Psi(w) = G(z)\overline{\Psi(w)}, \quad \big(w = f(z) = z+\overline{B(z)}\big), \tag{1.4}$$

where

$$C''(w) = -ie^{-(is/2)}\Psi(w), \quad D''(w) = -ie^{is/2}\Psi(w).$$

The condition that there exists a non-zero analytic function $\Psi(w)$ satisfying (1.4) is necessary and sufficient on $B(z)$.

[3]) In that case there exists a neighborhood in which f is one-to-one.

To transform (1.4) into a condition involving only G, let

$$\gamma(w) = G(f^{-1}(w)), \quad (w = f(z) = z + \overline{B(z)}). \tag{1.5}$$

We observe the following: A necessary and sufficient condition that there exists a non-vanishing analytic function $\Psi(w)$ satisfying (1.4) is that

$$R(w) = \frac{\gamma_{w\bar{w}}}{\gamma} - \frac{\gamma_w \gamma_{\bar{w}}}{\gamma^2} = \text{real function of } w. \tag{1.6}$$

The condition (1.6) is necessary, because, by (1.4),

$$\log \overline{\gamma(w)} + \log \Psi(w) = \log \gamma(w) + \log \overline{\Psi(w)},$$

and if we operate with $\partial^2/\partial\bar{w}\partial w$ on both sides we obtain $R(w) = \overline{R(w)}$. On the other hand, if (1.5) holds then

$$\frac{\partial}{\partial \bar{w}}\left[\frac{\gamma_w}{\gamma} - \frac{\overline{\gamma_{\bar{w}}}}{\bar{\gamma}}\right] = 0.$$

Therefore,

$$\frac{\partial}{\partial w}[\log \gamma - \log \bar{\gamma}] = F_1(w) = \text{analytic function of } w,$$

and, hence,

$$\log \gamma - \log \bar{\gamma} = \int F_1(w)\,dw - \overline{F_2(w)},$$

where $F_2(w)$ is also an analytic function of w. Thus,

$$\frac{\gamma(w)}{\overline{\gamma(w)}} = \frac{L(w)}{\overline{M(w)}}, \quad \text{with} \quad L(w) = \exp \int F_1(w)\,dw, \quad M(w) = \exp F_2(w).$$

Since it follows that $|L(w)| = |M(w)|$, we have $M(w) = \exp(it) L(w)$ for some real constant t. Setting

$$\Psi(w) = e^{it/2} L(w),$$

we obtain $\gamma(w)\overline{\Psi(w)} = \overline{\gamma(w)}\Psi(w)$. This shows that (1.6) is also sufficient for (1.4).

The next step is to employ (1.5) to compute $R(f(z))$, making use of the relations

$$\frac{\partial z}{\partial w} = \frac{\overline{f_z}}{J} = \frac{1}{J}, \quad \frac{\partial z}{\partial \bar{w}} = -\frac{f_{\bar{z}}}{J} = -\frac{\bar{G}}{J}, \tag{1.7}$$

$$J = |f_z|^2 - |f_{\bar{z}}|^2 = 1 - |G|^2, \quad G = B'.$$

We find that

$$\gamma_w \circ f = G'/J, \quad \gamma_{\bar{w}} \circ f = -\bar{G}G'/J,$$

$$\gamma_{w\bar{w}} \circ f = (-\bar{G}^2 G'^2 + G|G'|^2 - J\bar{G}G'')/J^3.$$

Therefore,

$$J^3 R(f(z)) = -\frac{\bar{G}^2 G'^2}{G} + |G'|^2 - J\frac{\bar{G}G''}{G} + J\frac{\bar{G}G'^2}{G^2}. \tag{1.8}$$

This suggests defining

$$R_1(z) = \frac{J^3 R(f(z)) - |G'|^2}{|G|^2},$$

which is real if and only if $R(f(z))$ is real. By (1.8),

$$R_1(z) = S(z) - T(z)\overline{G(z)}, \tag{1.9}$$

where S, T are the following analytic functions:

$$S(z) = \frac{G'(z)^2 - G(z)G''(z)}{G(z)^3}, \quad T(z) = \frac{2G'(z)^2 - G(z)G''(z)}{G(z)^2}. \tag{1.10}$$

Since R_1 is real we have

$$S - \bar{S} = \bar{G}T - G\bar{T}, \tag{1.11}$$

and this condition is equivalent to the basic requirement (1.6). Applying $\partial^2/\partial\bar{z}\partial z$ to both sides, one sees that (1.11) holds if and only if

$$S(z) = a - \bar{\varkappa}G(z), \quad T(z) = cG(z) + \varkappa, \tag{1.12}$$

where a and c are real constants and $\varkappa$ is a complex constant.

Relations (1.10) and (1.12) yield the pair of simultaneous differential equations

$$G'^2 - GG'' = aG^3 - \bar{\varkappa}G^4, \quad 2G'^2 - GG'' = cG^3 + \varkappa G^2$$

which constitute necessary and sufficient conditions on G. Subtracting the first equation from the second we obtain the equivalent pair

$$G'^2 = \bar{\varkappa}G^4 + (c-a)G^3 + \varkappa G^2, \quad G'^2 - GG'' = aG^3 - \bar{\varkappa}G^4. \tag{1.13}$$

The key to the situation is that the two equations (1.13) are highly dependent. The first implies

$$\frac{d}{dz}\left(\frac{G'}{G}\right)^2 = [2\bar{\varkappa}G + (c-a)]G'.$$

On the other hand, the second equation of (1.13) implies that

$$\frac{d}{dz}\left(\frac{G'}{G}\right)^2 = \frac{2G'}{G}(\bar{\varkappa}G^2 - aG) = 2(\bar{\varkappa}G - a)G'.$$

So we must have $a = -c$. Conversely, if we set $a = -c$, then we see, as above, that the first of equations (1.13) implies the second.

We note that if $\varkappa = c = a = 0$, then (1.13) also holds when G is constant. Thus we have proved the following:

Theorem 1. *Suppose* $f(z) = z + \overline{B(z)}$, $G(z) = B'(z)$. *A necessary and sufficient condition that there locally exists a non-affine complex harmonic function* $g(w)$, *such* *that* $g(f(z))$ *is harmonic is that* $G(z)$ *satisfies*

$$G'^2 = \alpha^2 G^4 + 2cG^3 + \bar{\alpha}^2 G^2 \tag{1.14}$$

for some complex constant α *and some real constant* c.

$G(z)$ and $B(z)$ can be expressed in terms of elementary functions. We distinguish the following separate cases; the functions $G(z)$ are given up to a translation of the z plane.

(I) $$\alpha = c = 0, \quad G(z) = \text{const.}$$

(II) $$\alpha = 0, \quad c \neq 0, \quad G(z) = \frac{2}{cz^2}.$$

(III) $$\alpha \neq 0, \quad c = r|\alpha|^2, \quad (r = \text{real const}, \ r^2 \neq 0, 1),$$

$$G(z) = -\frac{\bar{\alpha}/\alpha}{r + \sqrt{1-r^2}\sinh(\bar{\alpha}z)}.$$

(IV) $$\alpha \neq 0, \quad c = 0, \quad G(z) = -\frac{\bar{\alpha}/\alpha}{\sinh(\bar{\alpha}z)}.$$

(V) $$\alpha \neq 0, \quad c = |\alpha|^2, \quad G(z) = -\frac{\bar{\alpha}}{\alpha}\frac{e^{\bar{\alpha}z}}{e^{\bar{\alpha}z}-1}.$$

The above expressions are obtained by integrating the differential equation (1.14) and solving for $G(z)$. Integrating the functions $G(z)$, the following classification of harmonic mappings $f(z)$ is obtained.

Corollary. *Suppose* $f(z)$ *is complex harmonic. A necessary and sufficient condition that there locally exists a non-affine complex harmonic function* $g(w)$, *such that* $g(f(z))$ *is harmonic, is that* $f(z)$ *is*[4] *one of the following types, where* v *is a complex constant, and* r *a real constant.*

Type 0: $f(z) = \Phi(z)$, *where* $\Phi(z)$ *is analytic.*

Type 1: $f(z) = \Phi(z) + v\overline{\Phi(z)}, \quad v \neq 0.$

Type 2: $f(z) = \Phi(z) + \dfrac{r}{\overline{\Phi(z)}}, \quad r \neq 0.$

Type 3: $f(z) = \Phi(z) + \overline{v \tanh^{-1}\left(\sqrt{1-r^2} - r\tanh\dfrac{\Phi(z)}{v}\right)}, \quad r^2 \neq 0, 1.$

Type 4: $f(z) = \Phi(z) - \overline{v \log\tanh\dfrac{\Phi(z)}{2v}}.$

Type 5: $f(z) = \Phi(z) - \overline{v \log\left[\exp\left(\dfrac{\Phi(z)}{v}\right) - 1\right]}.$

[4] Up to translations and conjugation of z and $f(z)$.

The associated non-affine harmonic mappings $g(w)$ can be determined in principle by integrating (1.6). When $f(z)$ is of type 1, there is an especially simple solution; namely (up to an additive affine function)

$$g(w) = \bar{v}w^2 - v\bar{w}^2.$$

2. Harmonic decompositions of the identity

Choquet—Deny Theorem. *Suppose f is a sense-preserving harmonic homeomorphism and is neither analytic nor affine. A necessary and sufficient condition that f^{-1} is also harmonic is that*

$$f(z) = \frac{\sigma}{\bar{\alpha}} z + \frac{1}{\bar{\alpha}} \log \left[\frac{\mu - e^{-\sigma z}}{\bar{\mu} - e^{-\bar{\sigma}\bar{z}}} \right] + \text{const}, \tag{2.1}$$

where σ, α, μ are non-zero complex constants, $|\mu| > \sup_z |e^{-\sigma z}|$.

Proof. While one could proceed to test each of the types 1 to 5 of the corollary of Theorem 1 to see which fulfilled the requirement that f^{-1} is also one of these types, it is more efficient to take advantage of the symmetry of the relation

$$g(f(z)) = z, \tag{2.2}$$

by means of an independent proof. Assuming

$$w = f(z) = A(z) + \overline{B(z)}, \tag{2.3}$$

where $A(z)$, $B(z)$ are analytic, and where

$$J = |A'|^2 - |B'|^2 > 0, \quad B' \neq 0, \tag{2.4}$$

in the domain of definition of f, we have, in place of (1.7),

$$g_w = \overline{A'}/J, \quad g_{\bar{w}} = -\overline{B'}/J. \tag{2.5}$$

By (2.3), (2.4), (2.5) one finds that $g_{\bar{w}w} = 0$ if and only if

$$A'|B'|\overline{A''} - |A'|^2 B'\overline{B''} = \overline{A'}^2\overline{B'}A'' - \overline{A'}\,\overline{B'}^2 B''. \tag{2.6}$$

Dividing by $|A'|^2|B'|^2$, (2.6) becomes

$$\frac{\overline{A''}}{\overline{A'}} - \frac{\overline{B''}}{\overline{B'}} = \alpha\overline{A'} - \beta\overline{B'}, \quad \text{where} \quad \alpha(z) = \frac{A''}{A'B'}, \quad \beta(z) = \frac{B''}{A'B'}, \tag{2.7}$$

or, equivalently,

$$(\bar{\alpha} + \beta)\overline{B'} = (\alpha + \bar{\beta})\overline{A'}.$$

By (2.4), therefore,

$$\beta = -\bar{\alpha}.$$

Since α, β are both analytic, they must therefore both be constants,

$$A'' = \alpha A'B', \quad B'' = -\bar{\alpha} A'B'. \tag{2.8}$$

Since affine f are excluded, α cannot be 0. Eliminating B between the differential equations (2.8),

$$\frac{B''}{B'} = -\bar{\alpha} A' = \frac{\alpha A'B''}{A''} = \frac{A'A''' - A''^2}{A'A''} = \frac{A'''}{A''} - \frac{A''}{A'}.$$

Integrating, we obtain

$$A' = \text{const}\, e^{-\bar{\alpha} A} + \text{const}.$$

This is a separable differential equation. For the solution we have

$$A(z) = \frac{\sigma}{\bar{\alpha}} z + \frac{1}{\bar{\alpha}} \log(\mu - e^{-\sigma z}) + \text{const}, \tag{2.9}$$

$$A'(z) = \frac{\mu\sigma}{\bar{\alpha}} (\mu - e^{-\sigma z})^{-1}, \quad A''(z) = -\frac{\mu\sigma^2}{\bar{\alpha}} (\mu - e^{-\sigma z})^{-2} e^{-\sigma z}.$$

Therefore, by the first equation of (2.8),

$$B'(z) = -\frac{\sigma}{\alpha} \frac{e^{-\sigma z}}{\mu - e^{-\sigma z}}$$

and therefore,

$$B(z) = -\frac{1}{\alpha} \log(\mu - e^{-\sigma z}) + \text{const}. \tag{2.10}$$

Conversely, if (2.9), (2.10) hold then (2.7) is satisfied.

The restriction on μ in the statement of the theorem is to insure that $J > 0$. We note that the harmonic mapping f of the Choquet—Deny Theorem falls within type 5 of Section 1.

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