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CONVEX INCREASING FUNCTIONS PRESERVE THE SUB-F-EXTREMALITY

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1. Introduction

Suppose that $u\colon G\to(a,b)$ is a subharmonic function and that $f\colon(a,b)\to\mathbf{R}$ is an increasing convex function. It is relatively easy to check that $v=f\circ u$ satisfies the mean-value inequality and hence v is again subharmonic in G. The purpose of this note is to give a generalization of this fact in a non-linear potential theory developed by S. Granlund, P. Lindqvist and O. Martio in [GLM 1—3]; see also [LM 1—2].

1.1. Theorem. *Suppose that G is an open set in $\mathbf{R}^n$ and that $u\colon G\to(a,b)$ is a sub-F-extremal in G. If $f\colon(a,b)\to\mathbf{R}$ is an increasing convex function, then $f\circ u$ is a sub-F-extremal in G.*

The cases $a=-\infty$ and $b=+\infty$ are not excluded.

In the classical potential theory Theorem 1.1 is due to M. Brelot [B, p. 16]. He makes use of an approximation method. There is another proof for this result, based on integral means, in the book of T. Radó [R, p. 15]; see also [HK, p. 46]. However, such a method is not available in our case. In a non-linear potential theory P. Lindqvist and O. Martio have recently proved a special case of Theorem 1.1 in [LM 2].

2. Sub-F-extremals

Suppose that G is an open set in $\mathbf{R}^n$ and that $F\colon G\times\mathbf{R}^n\to\mathbf{R}$ satisfies the following assumptions:

(a) For each open set $D\subset\subset G$ and $\varepsilon>0$, there is a compact set $C\subset D$ with $m(D\setminus C)<\varepsilon$ and $F|C\times\mathbf{R}^n$ is continuous.

(b) For a.e. $x\in G$ the function $h\mapsto F(x,h)$ is strictly convex and differentiable in $\mathbf{R}^n$.

(c) There are $0<\alpha\leqq\beta<\infty$ such that for a.e. $x\in G$

$$\alpha|h|^n\leqq F(x,h)\leqq\beta|h|^n,\quad h\in\mathbf{R}^n.$$

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(d) For a.e. $x \in G$

$$F(x, \lambda h) = |\lambda|^n F(x, h), \quad \lambda \in \mathbf{R}, \quad h \in \mathbf{R}^n.$$

An alternative characterization of the functions F satisfying (a)—(d) is given in [K]. Note that the exponent n in (c) and (d) is essential for applications in conformal geometry; cf. [GLM 1—2]. It is not essential for Theorem 1.1; see Remark 4.4.

A function $u \in C(G) \cap \mathrm{loc}\, W_n^1(G)$, i.e., u is ACL^n in G, is called *an F-extremal in G* if for all domains $D \subset\subset G$

$$I_F(u, D) = \inf_{v \in F_u} I_F(v, D),$$

where

$$I_F(v, D) = \int_D F(x, \nabla v(x))\, dm(x)$$

is the variational integral with the kernel F and

$$F_u = \{v \in C(\bar{D}) \cap W_n^1(D) \mid v = u \text{ in } \partial D\}.$$

A function u is an F-extremal if and only if $u \in C(G) \cap \mathrm{loc}\, W_n^1(G)$ is a weak solution of the Euler equation

$$\nabla \cdot \nabla_h F(x, \nabla u(x)) = 0.$$

An upper semi-continuous function $u\colon G \to \mathbf{R} \cup \{-\infty\}$ is called *a sub-F-extremal* if u satisfies *the F-comparison principle* in G, i.e., if $D \subset\subset G$ is a domain and $h \in C(\bar{D})$ is an F-extremal in D, then $h \geqq u$ in ∂D implies $h \geqq u$ in D. A function $u\colon G \to \mathbf{R} \cup \{\infty\}$ is a *super-F-extremal* if $-u$ is a sub-F-extremal. For basic properties of F- extremals and sub-F-extremals we refer to [GLM 2].

A sub-F-extremal $u\colon G \to \mathbf{R}$ is called *a regular sub-F-extremal* if u is ACL^n, i.e., $u \in C(G) \cap \mathrm{loc}\, W_n^1(G)$. Note that it follows from [GLM 3, Theorem 4.1] that a sub-F-extremal $u\colon G \to \mathbf{R}$ is regular if $u \in C(G)$. In what follows we shall make use of the following lemma.

2.1. Lemma. *Let u be an ACL^n-function in G. Then u is a regular sub-F-extremal in G if and only if for all non-negative* $\varphi \in C_0(G) \cap W_{n,0}^1(G)$

$$\int_G \nabla_h F(x, \nabla u) \cdot \nabla \varphi \, dm \leqq 0.$$

Proof. The result follows from [GLM 2, Theorem 5.17] via an easy approximation procedure.

2.2. Lemma. *Suppose that $u\colon G \to \mathbf{R} \cup \{-\infty\}$ is a sub-F-extremal in G and that $D \subset\subset G$ is a domain. Then there exists a decreasing sequence of regular sub-F-extremals u_i in D which are continuous in $\bar{D}$ such that* $\lim_{i \to \infty} u_i = u$ *in D. Moreover, for each $\varepsilon > 0$ the sequence u_i can be chosen such that* $\sup_D u_1 \leqq \sup_D u + \varepsilon$.

Proof. This follows from [GLM 3, Section 4].

2.3. Lemma. *Suppose* $u\colon G\to\mathbf{R}\cup\{-\infty\}$ *to be a function such that for each domain* $D\subset\subset G$ *there exist a decreasing sequence of sub-F-extremals* u_i *in* D *with* $\lim_{i\to\infty}=u$ *in* D. *Then* u *is a sub-F-extremal in* G.

Proof. Clearly u is upper semi-continuous in G. Let $D\subset\subset G$ be a domain and $h\in C(\bar{D})$ an F-extremal in D such that $h\geqq u$ in ∂D. Choose a domain $D'\subset\subset G$ and a decreasing sequence u_i of sub-F-extremals in D' such that $D\subset\subset D'$ and $\lim_{i\to\infty} u_i=u$ in D'. Fix $\varepsilon>0$. Since for every $x\in\partial D$ there is $i_x\in\mathbf{N}$ such that

$$u_{i_x}(x)-h(x)-\varepsilon<0,$$

the upper semi-continuity of $u_i-h-\varepsilon$, together with the compactness of ∂D, implies

$$u_i\leqq u_{i_0}\leqq h+\varepsilon$$

in ∂D for $i\geqq i_0$. Thus the sub-F-extremality of u_i yields

$$u_i\leqq h+\varepsilon$$

in D for $i\geqq i_0$. Hence

$$u\leqq h+\varepsilon$$

in D. Letting $\varepsilon\to 0$ we obtain the desired result.

We close this section with the following lemma, leaving its easy proof to the reader.

2.4. Lemma. *Suppose that* $u_i\colon G\to\mathbf{R}\cup\{-\infty\}$ *is a sequence of sub-F-extremals in* G *with* $u_i\to u$ *uniformly on compact subsets of* G. *Then* u *is a sub-F-extremal in* G.

3. Approximation lemmas

Let $a,b\in[-\infty,\infty]$. We say that a function $f\colon(a,b)\to\mathbf{R}$ *preserves the (regular) sub-F-extremality in G* if for each domain $D\subset G$ $f\circ u$ is a (regular) sub-F-extremal in D whenever $u\colon D\to(a,b)$ is a (regular) sub-F-extremal.

3.1. Lemma. *Let* $f\colon(a,b)\to\mathbf{R}$ *be an increasing continuous function. Suppose that for each* $\delta>0$ *there is a sequence of functions* $f_i\colon(a+\delta,b-\delta)\to\mathbf{R}$, f_i *preserving the regular sub-F-extremality in G for each i and* $f_i\to f$ *uniformly on compact subsets of* $(a+\delta,b-\delta)$. *Then f preserves the sub-F-extremality in G.*

Proof. Let $G'\subset G$ be a domain. Let $u_0\colon G'\to(a,b)$ be a sub-F-extremal. Clearly $f\circ u_0$ is upper semi-continuous in G'. Let $D\subset\subset G'$ be a domain, and write

$$d_0=\max_{x\in\bar{D}} u_0(x)<b.$$

Let $0<3\varepsilon<b-d_0$. Then $u=u_0+\varepsilon$ is again a sub-F-extremal in G' and $u(x)=$

$u_0(x)+\varepsilon > a+\varepsilon/2$ in G'. By Lemma 2.2 we may choose a decreasing sequence of sub-F-extremals $u_i \in C(\bar{D})$ in D such that $\lim_{i\to\infty} u_i = u$ in D and that

$$\sup_{x\in\bar{D}} u_1(x) \leqq d_0+2\varepsilon < b-\varepsilon.$$

Observe that u_i: $D\to(a+\varepsilon/2, b-\varepsilon/2)$ is a regular sub-F-extremal in D and choose a sequence f_k such that $f_k\to f$ uniformly on compact subsets of $(a+\varepsilon/2, b-\varepsilon/2)$ and that $f_k\circ u_i$ is a sub-F-extremal in D for each k and i. Fix i. Then $f_k\circ u_i \to f\circ u_i$ uniformly on compact subsets of D as $k\to\infty$. Now Lemma 2.4 implies that $f\circ u_i$ is a sub-F-extremal in D and $f\circ u_i$ is a decreasing sequence of sub-F-extremals in D which tends to $f\circ u$. Then $f\circ u = f\circ(u_0+\varepsilon)$ is a sub-F-extremal in D. Letting $\varepsilon\to 0$ Lemma 2.3 yields that $f\circ u_0$ is a sub-F-extremal in G'.

Next we shall consider smooth convex functions f.

3.2. Lemma. *Suppose f: $(a, b)\to\mathbf{R}$ to be a convex increasing function such that $f\in C^2(a, b)$. Then f preserves the regular sub-F-extremality in G.*

Proof. Let $D\subset G$ be a domain. Suppose that u: $D\to(a, b)$ is a regular sub-F-extremal. Clearly $v=f\circ u \in C(D)\cap \operatorname{loc} W_n^1(D)$. Choose $\varphi\in C_0^\infty(D)$, $\varphi\geqq 0$. It suffices to show that

$$\int_D \nabla_h F(x, \nabla v)\cdot\nabla\varphi\, dm \leqq 0;$$

see Lemma 2.1. Let $\psi(x)=f'(u(x))^{n-1}\varphi(x)$. Then $\psi\in C_0(D)\cap W_{n,0}^1(D)$ and $\psi\geqq 0$. Furthermore,

$$\nabla\psi(x) = (n-1)f''(u(x))f'(u(x))^{n-2}\varphi(x)\nabla u(x)+f'(u(x))^{n-1}\nabla\varphi(x) \tag{3.3}$$

a.e. in D. Since the homogeneity assumption (d) of F implies for a.e. $x\in G$

$$\nabla_h F(x, \lambda h) = |\lambda|^{n-2}\lambda\nabla_h F(x, h)$$

for $\lambda\in\mathbf{R}$ and $h\in\mathbf{R}^n$, we obtain

$$\nabla_h F(x, \nabla v(x)) = \nabla_h F(x, f'(u(x))\nabla u(x)) = f'(u(x))^{n-1}\nabla_h F(x, \nabla u(x)) \tag{3.4}$$

a.e. in D. Now (3.4) and (3.3), together with the regular sub-F-extremality of u, yield

$$0 \geqq \int_D \nabla_h F(x, \nabla u)\cdot\nabla\psi\, dm = (n-1)\int_D f''(u)f'(u)^{n-2}\varphi\nabla_h F(x, \nabla u)\cdot\nabla u\, dm$$
$$+\int_D f'(u)^{n-1}\nabla_h F(x, \nabla u)\cdot\nabla\varphi\, dm \geqq \int_D \nabla_h F(x, \nabla v)\cdot\nabla\varphi\, dm,$$

since $\nabla_h F(x, \nabla u)\cdot\nabla u\geqq 0$ a.e. in D by the assumptions (b) and (c).

4. Final steps

4.1. *Proof for Theorem 1.1.* We make a convolution approximation. Choose $\eta_j \in C_0^\infty(\mathbf{R})$, $\eta_j \geqq 0$ such that $\operatorname{spt} \eta_j = [-1/j, 1/j]$ and $\int_{\mathbf{R}} \eta_j(y)\, dy = 1$. For $x \in (a+1/j, b-1/j)$ define

$$f_j(x) = \int_{\mathbf{R}} f(x-y)\eta_j(y)\, dy,$$

where $f(x-y)\eta_j(y)=0$ if $y \notin \operatorname{spt} \eta_j$. In the light of Lemma 3.2, f_j preserves the regular sub-F-extremality in G. Hence Lemma 3.1 implies the desired result.

4.2. Remark. Suppose that f: $[a, b) \to [-\infty, \infty)$ is convex and increasing. If $f(a) = -\infty$, then $f(t) = -\infty$ for each $t \in [a, b)$ or $f(t) > -\infty$ for each $t \in (a, b)$. Thus we can use Theorem 1.1 to prove the following slight extension: *Suppose that f: $[a, b) \to [-\infty, \infty)$ is convex and increasing and that u: $G \to \mathbf{R} \cup \{-\infty\}$ is a sub-F-extremal in G with $u(G) \subset [a, b)$. Then $f \circ u$ is a sub-F-extremal in G.*

4.3. Remark. It is clear that $f \circ u$ is a super-F-extremal if u is and if f is concave and increasing. Also, it is evident that if v is a sub-F-extremal and if f is concave and decreasing, then $f \circ v$ is a super-F-extremal.

4.4. Remark. Theorem 1.1 holds also when the exponent n in (c) and (d) is replaced by a more general constant $p \in (1, \infty)$. For $p \neq n$ the proof is similar to that above. Observe that the continuity of the solution to an obstacle problem, needed in the proofs of 2.1 and 2.2, is established in [MZ].

4.5. Corollary. *Suppose that u: $G \to \mathbf{R}$ is an F-extremal in G and $p \in [1, \infty)$. Then $|u|^p$ is a sub-F-extremal in G.*

Proof. Note that $|u| = \max(u, -u)$ is a sub-F-extremal and $t \mapsto t^p$, $t \geqq 0$, is convex and increasing. The result follows from Remark 4.2.

4.6. Corollary. *Suppose that u is a non-negative super-F-extremal in G. Then u^p is a super-F-extremal for $p \in [0, 1]$.*

We close with a remark on quasiregular mappings, which have an important role in a non-linear potential theory; see [GLM 2, Sections 6 and 7]. Suppose that G and G' are domains in $\mathbf{R}^n$ and f: $G \to G'$ is a quasiregular mapping. Then

$$F(x, h) = \begin{cases} J(x, f)\,|f'(x)^{-1*}h|^n & \text{if } J(x, f) \neq 0, \\ |h|^n & \text{otherwise,} \end{cases}$$

satisfies (a)—(d) in G; see [GLM 2, 6.4]. Here $J(x, f)$ stands for the Jacobian determinant of f at x and A^* denotes the adjoint of the linear map A: $\mathbf{R}^n \to \mathbf{R}^n$. Now $\log |f|$ is a sub-F-extremal in G [GLM 2, Theorem 7.10]. Thus, using the convex increasing function e^{tp} for $p \geqq 0$, we obtain

4.7. Corollary. *Suppose that f and F are as above. Then $|f|^p$ is a sub-F-extremal in G for $p \geqq 0$.*

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