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INJECTIVITY, QUASICONFORMAL REFLECTIONS AND THE LOGARITHMIC DERIVATIVE

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1. Introduction

Let C be a K-quasicircle in the complete complex plane $\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$ (i.e., C is the image of the real axis $\hat{\mathbf{R}}$ under a K-quasiconformal automorphism of $\hat{\mathbf{C}}$, with $K\geqq 1$), and D_1, D_2 to be the components of $\hat{\mathbf{C}}\setminus C$, such that $\infty\in\bar{D}_2=D_2\cup C$. Let $\varrho_i(z)|dz|$ be the Poincaré metric in D_i, $i=1,2$, with curvature -4. In [2] Ahlfors proved the following injectivity criterion in terms of the Schwarzian derivative

$$S_f(z)=\left[(f''/f')'-\tfrac{1}{2}(f''/f')^2\right](z),\quad z\in D_1.$$

Theorem A. *If C is a K-quasicircle, then there exists a positive constant $\varepsilon=\varepsilon(K)$, depending only on K, such that f is univalent in D_1 whenever it is meromorphic there, with a nonvanishing derivative and*

$$\|S_f\|_{2,D_1}=\sup_{z\in D_1}\varrho_1(z)^{-2}|S_f(z)|\leqq\varepsilon. \tag{1.1}$$

Moreover, strict inequality in (1.1) *implies the existence of a quasiconformal extension of f to $\hat{\mathbf{C}}$.*

Writing

$$S_f=T_f'-\tfrac{1}{2}T_f^2,\quad T_f=f''/f'$$

and applying Lemma 3 of [9], one can easily show that the *correspondence* $\pi\colon T_f\to S_f$ is continuous with respect to the norms $\|S_f\|_{2,D_1}$, defined in (1.1), and

$$\|T_f\|_{1,D_1}=\sup_{z\in D_1}\varrho_1(z)^{-1}|T_f(z)|. \tag{1.2}$$

This observation yields the following analogue for Theorem A.

Theorem B. *If C is a K-quasicircle, then there exists a positive constant $\varepsilon_1=\varepsilon_1(K)$, depending only on K, such that every analytic function f in D_1, with $f'\neq 0$, is univalent whenever*

$$\|T_f\|_{1,D_1}\leqq\varepsilon_1, \tag{1.3}$$

and if there is a strict inequality in (1.3), *f has a quasiconformal extension to $\hat{\mathbf{C}}$.*

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Remark. Martio and Sarvas [11] proved the injectivity part of Theorems A and B, using the uniformity property of quasidiscs. On the other hand, Gehring [8] and Astala—Gehring [5] proved that these injectivity criteria are valid only on quasidiscs.

In this paper, a different quantitative proof for Theorem B is provided, based on Ahlfors' idea of the proof of Theorem A but utilizing a slight modification of his quasiconformal extension formula. First we show that two properties of quasiconformal reflections, which have been proven in [2] under the assumption $\infty\in C$, remain valid also if $\infty\in\bar{D}_2$. Then Theorem B is proved and in the last section it is applied to the universal Teichmüller space.

2. Quasiconformal reflections

An automorphism h of $\hat{\mathbf{C}}$ is a K-quasiconformal reflection at C if

(i) $h|_C=\mathrm{id}$ and $h\circ h=\mathrm{id}$ on $\hat{\mathbf{C}}$.

(ii) The mapping $z\to\overline{h(z)}$ is a (sense-preserving) K-quasiconformal mapping, i.e.,

$$\left|\frac{\partial h}{\partial z}\right|\leqq\frac{K-1}{K+1}\left|\frac{\partial h}{\partial\bar{z}}\right|<\left|\frac{\partial h}{\partial\bar{z}}\right|.$$

Properties (i) and (ii) yield a third one:

(iii) $h(D_i)=D_j,\quad i\neq j,\quad i,j=1,2.$

Lemma 1 (Ahlfors [2]). *If* $\mathbf{C}$ *is a K-quasicircle,* $\mathbf{C}$ *admits a* K^2*-quasiconformal reflection.*

Lemma 2 (cf [2]). *Let* $h\colon\hat{\mathbf{C}}\to\hat{\mathbf{C}}$ *be a K-quasiconformal reflection at C. If* $\infty\in\bar{D}_2$ *and* $z_0\in C$, *then*

$$|h(z)-z_0|\leqq\lambda(K)|z-z_0|,\quad\text{for all}\quad z\in D_2,\tag{2.1}$$

where

$$\lambda(K)=\left[\mu^{-1}\left(\frac{\pi K}{2}\right)\right]^{-2}-1$$

and $\mu(r)$ *is the conformal module of the Grötzsch extremal ring domain separating the exterior of the unit disc B from the interval* $(0,r)$ *for* $0\leqq r<1$ (see [10], pp. 53, 81).

Proof. Let f be a conformal mapping of the upper half plane U onto D_2, with $f(\infty)=\infty$ if $\infty\in C$, or $f(\infty)=z_0\in C$ if $\infty\in D_2$. Define

$$\omega(\zeta)=\begin{cases}f(\zeta) & \text{for}\quad\zeta\in U,\\ h(f(\bar{\zeta})) & \text{for}\quad\zeta\in L=\mathbf{C}\setminus\bar{U}.\end{cases}$$

Then $C=\omega(\mathbf{R})$ and $\omega\colon \hat{\mathbf{C}}\to\hat{\mathbf{C}}$ is K-quasiconformal. Thus, if $z\in D_2$, then

$$z=\omega(\zeta)=f(\zeta) \quad \text{for some} \quad \zeta\in U$$

and

$$h(z)=h(f(\zeta))=\omega(\bar{\zeta}), \quad \bar{\zeta}\in L.$$

First, if $\infty\in C$, take $\zeta_0=\omega^{-1}(z_0)\in\mathbf{R}$. Then the cross-ratio

$$|(\bar{\zeta},\zeta,\zeta_0,\infty)|=\left|\frac{\bar{\zeta}-\zeta_0}{\zeta-\zeta_0}\right|=1,$$

and (2.1) follows by the quasi-invariance of the cross-ratio under quasiconformal mappings (see [1]), and since ω maps $\{\bar{\zeta},\zeta,\zeta_0,\infty\}$ onto $\{h(z),z,z_0,\infty\}$, respectively.

Next, if $\infty\in D_2$, let $\omega^{-1}(\infty)=\alpha\in U$. Then

$$|(\bar{\zeta},\zeta,\infty,\alpha)|=\left|\frac{\zeta-\alpha}{\bar{\zeta}-\alpha}\right|<1 \quad \text{for} \quad \zeta\in U,$$

and (2.1) is obtained again since $\omega\colon \{\bar{\zeta},\zeta,\infty,\alpha\}\to\{h(z),z,z_0,\infty\}$, respectively.

Q.e.d.

Lemma 3 (cf. [2]). *If C admits a K-quasiconformal reflection, then there exists a $\mathscr{K}$-quasiconformal reflection H at C satisfying*

$$\varrho_j(H(z))\,|dH(z)|\leqq\mathscr{L}\cdot\varrho_i(z)\,|dz|,\quad z\in D_i,\quad i\neq j,\quad i,j=1,2 \tag{2.2}$$

for some constants $\mathscr{K}=\mathscr{K}(K)$ and $\mathscr{L}=\mathscr{L}(K)$ depending only on K.

The proof of Lemma 3 is given in [2]. The assumption $\infty\in C$ in it might be dropped, since it was needed only for deriving the corresponding inequality in the Euclidean metric. From Ahlfors' proof in [2] and the estimates in [3] and [7] one gets the following estimates:

$$\mathscr{K}(K)\leqq M^2,\quad \mathscr{L}(K)\leqq 4M^2(M+1),\quad \text{where} \quad M=\lambda(K). \tag{2.3}$$

3. Proof of Theorem B

Assume first that f has the following property:

(∗) *f is locally conformal on some domain containing $\bar{D}_1=D_1\cup C$.*

By Lemma 1, C admits a K^2-quasiconformal reflection, and hence, by Lemma 3, there exists another $\mathscr{K}$-quasiconformal reflection H at C, with $\mathscr{K}\leqq M^2=\lambda(K^2)^2$, which satisfies Condition (2.2) with $\mathscr{L}\leqq 4M^2(M+1)$.

Now, extend f into D_2 as follows:

$$\tilde{f}(\zeta)=f(z)+(\zeta-z)f'(z),\quad \zeta\in D_2,\quad z=H(\zeta)\in D_1. \tag{3.1}$$

This extension has the complex derivatives

$$\frac{\partial \tilde{f}}{\partial \bar{\zeta}} = (\zeta - z)\frac{\partial z}{\partial \bar{\zeta}} f''(z) \quad \text{and} \quad \frac{\partial \tilde{f}}{\partial \zeta} = f'(z) + (\zeta - z)\frac{\partial z}{\partial \zeta} f''(z), \quad z = H(\zeta) \in D_1,$$

and hence it has the complex dilatation

$$\tilde{\mu}(\zeta) = \frac{(\zeta - z)\dfrac{\partial z}{\partial \bar{\zeta}} T_f(z)}{1 + (\zeta - z)\dfrac{\partial z}{\partial \zeta} T_f(z)}, \quad \zeta \in D_2, \quad z = H(\zeta) \in D_1. \tag{3.2}$$

But by Lemma 2 we have for all $\zeta_0 \in C$

$$|\zeta - H(\zeta)| \leqq |\zeta - \zeta_0| + |\zeta_0 - H(\zeta)| \leqq (1 + \lambda(\mathscr{K}))|\zeta - \zeta_0|, \quad \zeta \in D_2. \tag{3.3}$$

Choose $\zeta_0 \in C$ such that $|\zeta - \zeta_0| = d(\zeta, C)$ and apply the inequality $d(\zeta, C) \leqq \varrho_2(\zeta)^{-1}$, $\zeta \in D_2$ (by Schwarz' lemma). Thus

$$|\zeta - H(\zeta)| \leqq (1 + \lambda(\mathscr{K}))\varrho_2(\zeta)^{-1}, \quad \zeta \in D_2. \tag{3.3'}$$

On the other hand, Property (2.2) of H implies

$$\left|\frac{\partial H(\zeta)}{\partial \bar{\zeta}}\right| \leqq \mathscr{L} \cdot \varrho_2(\zeta)/\varrho_1(H(\zeta)), \quad \zeta \in D_2, \tag{3.4}$$

and since $\zeta \to \overline{H(\zeta)}$ is $\mathscr{K}$-quasiconformal, it follows that

$$\left|\frac{\partial H(\zeta)}{\partial \zeta}\right| \leqq q \left|\frac{\partial H(\zeta)}{\partial \bar{\zeta}}\right| \quad \text{with} \quad q = q(K) = \frac{\mathscr{K} - 1}{\mathscr{K} + 1} < 1. \tag{3.5}$$

From (3.2), (3.3′), (3.4) and (3.5) we conclude

$$|\tilde{\mu}(\zeta)| \leqq \frac{A(K)\varrho_1(z)^{-1}|T_f(z)|}{1 - q(K)A(K)\varrho_1(z)^{-1}|T_f(z)|}, \quad \zeta \in D_2, \quad z = H(\zeta) \in D_1, \tag{3.6}$$

where $A(K) = 4M^2(M+1)(1 + \lambda(M^2))$, $q(K) = (M^2 - 1)/(M^2 + 1)$, $M = \lambda(K^2)$. Thus $\|\tilde{\mu}\|_\infty \leqq k < 1$ whenever f satisfies

$$\|T_f\|_{1, D_1} \leqq A(K)^{-1}\frac{k}{1 + q(K)k} < \varepsilon_1 = \frac{A(K)^{-1}}{1 + q(K)} = \varepsilon_1(K). \tag{3.7}$$

Hence, the mapping

$$F(z) = \begin{cases} f(z) & \text{for} \quad z \in \bar{D}_1 = D_1 \cup C, \\ \tilde{f}(z) & \text{for} \quad z \in D_2 \end{cases} \tag{3.8}$$

is a local homeomorphism all over $\hat{\mathbf{C}}$, and therefore it is an automorphism of $\hat{\mathbf{C}}$, thus proving the theorem whenever f satisfies $(*)$.

Now, if f does not satisfy $(*)$, let $\{G_n\}$ be any decreasing sequence — $\bar{G}_{n+1}\subset G_n$ — of simply-connected hyperbolic domains, such that $D_1=\bigcap_{n=1}^{\infty} G_n$. Let $g_n\colon G_n\to D_1$ be the conformal mapping with $g_n(z_0)=z_0$ and $g_n'(z_0)>0$ for some fixed $z_0\in D_1\subset G_n$. Then $\{g_n\}$ converges to the identity mapping locally uniformly on D_1. Define

$$\psi_n=(T_f\circ g_n)g_n',\quad n\geqq 1.$$

Then $\{\psi_n\}$ converges to T_f locally uniformly on D_1. Note that each ψ_n is analytic on $\bar{D}_1\subset G_n$, and by

$$\varrho_{G_n}(z)\leqq \varrho_1(z),\quad z\in D_1\subset G_n$$

(which is a consequence of Schwarz' lemma) it follows that

$$\varrho_1(z)^{-1}|\psi_n(z)|\leqq \varrho_{G_n}(z)^{-1}|g_n'(z)|\left|T_f(g_n(z))\right|$$
$$=\varrho_1(g_n(z))^{-1}\left|T_f(g_n(z))\right|\leqq \|T_f\|_{1,D_1},\quad z\in D_1.$$

Thus if f_n is any solution of the equation

(3.9) $$y''-\psi_n y'=0\quad \text{in } G_n,$$

then f_n satisfies property $(*)$ in addition to the assumptions of the theorem and $T_{f_n}=\psi_n$. Furthermore, since the general solution of (3.9) is of the form af_n+b, we can choose the solution f_n that fixes two given points on C. Hence, the sequence $\{F_n\}$ of mappings defined by

$$F_n(\zeta)=\begin{cases} f_n(\zeta) & \text{for}\quad \zeta\in\bar{D}_1,\\ f_n(z)+(\zeta-z)f_n'(z) & \text{for}\quad \zeta\in D_2,\quad z=H(\zeta)\in D_1\end{cases}$$

form a normal family of quasiconformal automorphisms of $\hat{C}$. We conclude that $\{F_n\}$ contains a subsequence converging locally uniformly on $\hat{C}$ to

$$F_0(\zeta)=\begin{cases} f_0(\zeta) & \text{for}\quad \zeta\in\bar{D}_1,\\ \tilde{f}_0(\zeta) & \text{for}\quad \zeta\in D_2,\end{cases}$$

where $f_0=\lim_{n\to\infty}f_n$ is conformal and univalent in D_1, with $T_{f_0}=\lim_{n\to\infty}T_{f_n}=T_f$, and

$$\tilde{f}_0(\zeta)=\lim_{n\to\infty}F_n(\zeta)=f_0(\zeta)+(\zeta-z)f_0'(z),\quad \zeta\in D_2,\quad z=H(\zeta)\in D_1$$

has the complex dilatation $\tilde{\mu}(\zeta)$ given in (3.2).

Finally, $T_f=T_{f_0}$ implies $f=af_0+b$ for some constants $a,b\in C$, so that f is also univalent in D_1 and has the quasiconformal extension to $\hat{C}$, given by (3.1).

Q.e.d.

4. Universal Teichmüller space

For $n=1, 2$, let $B_n(D)$ be the Bers' space of analytic functions ψ in a hyperbolic domain D, bounded in the norm

$$\|\psi\|_{n,D} = \|\psi\|_n = \sup_{z\in D} \varrho_D(z)^{-n}|\psi(z)|.$$

Let $M(D)$ be the open unit ball in the Banach space $L^\infty(D)$ of all complex-valued measurable functions μ supported in D and bounded in the norm

$$\|\mu\|_{\infty,D} = \|\mu\|_\infty = \operatorname*{ess\,sup}_{z\in D} |\mu(z)|.$$

Let C, D_1, D_2 be as before. For each $\mu\in M(D_2)$ let f_μ be the unique homeomorphic solution of the Beltrami equation

$$\frac{\partial w}{\partial \bar z} = \mu \frac{\partial w}{\partial z}$$

fixing the points $\{0, 1, \infty\}$. Since $\mu|_{D_1}=0$, f_μ is conformal in D_1, so we may define the two mappings

$$\eta_c^{(1)}\colon M(D_2) \to \mu \to T_\mu = T_{f_\mu}\in B_1(D_1), \quad T_f = f''/f'$$

and

$$\eta_c^{(2)}\colon M(D_2) \to \mu \to S_\mu = T'_\mu - \tfrac{1}{2} T_\mu^2 \in B_2(D_1),$$

the second of which is known (see [6]) as Bers' embedding of the universal Teichmüller space $T(D_2)$ of D_2 in $B_2(D_1)$. The image $T_1(D_2)$ of the first mapping $\eta_c^{(1)}$ in $B_1(D_1)$ may also be considered a representation of $T(D_2)$. Bers proved in [6] the following properties of the embedding $\eta_c^{(2)}$:

Theorem C. *Let C be a quasicircle in $\hat{\mathbf{C}}$. The mapping $\eta_c^{(2)}\colon M(D_2)\to B_2(D_1)$ is holomorphic (as a mapping between two complex Banach spaces). Its derivative at the origin is*

$$(4.1) \qquad D\eta_c^{(2)}(0)\colon \mu \to -\frac{6}{\pi}\iint_{D_2} \frac{\mu(\zeta)}{(\zeta-z)^4}\, d\xi\, d\eta, \quad z\in D_1, \quad \mu\in L^\infty(D_2),$$

and $D\eta_c^{(2)}(0)$ has the right-inverse

$$(4.2) \qquad \Lambda_c^{(2)}\colon \psi \to \frac{1}{2}(\zeta - H(\zeta))^2 \frac{\partial H(\zeta)}{\partial \bar\zeta}\, \psi(H(\zeta)), \quad \zeta\in D_2, \quad \psi\in B_2(D_1),$$

i.e., each $\psi\in B_2(D_1)$ satisfies the reproduction formula

$$(4.3) \qquad \psi(z) = -\frac{3}{\pi}\iint_{D_2} (\zeta - H(\zeta))^2 \frac{\partial H(\zeta)}{\partial \bar\zeta}\, \psi(H(\zeta))(\zeta - z)^{-4}\, d\xi\, d\eta, \quad z\in D_1$$

(cf. Lemma 6 in [9]), *where H is the quasiconformal reflection at C of Lemma 3.*

It appears that the embedding $\eta_c^{(1)}$ has similar properties:

Theorem D. *Let C be a quasicircle in $\hat{\mathbf{C}}$, and D_1 and D_2 its interior and exterior domains, respectively* (i.e., $\infty \in \bar{D}_2$). *The mapping $\eta_c^{(1)}\colon M(D_2) \to B_1(D_1)$ is holomorphic. Its derivative at the origin is the linear mapping*

$$D\eta_c^{(1)}(0)\colon \mu \to -\frac{2}{\pi}\iint_{D_2} \frac{\mu(\zeta)}{(\zeta - z)^3}\, d\xi\, d\eta, \quad z \in D_1, \quad \mu \in L^\infty(D_2), \tag{4.4}$$

and $D\eta_c^{(1)}(0)$ has the right inverse

$$\Lambda_c^{(1)}\colon \psi \to (\zeta - H(\zeta))\frac{\partial H(\zeta)}{\partial \bar{\zeta}}\psi(H(\zeta)), \quad \zeta \in D_2, \quad \psi \in B_1(D_1), \tag{4.5}$$

i.e.,

$$\psi(z) = -\frac{2}{\pi}\iint_{D_2} (\zeta - H(\zeta))\frac{\partial H(\zeta)}{\partial \bar{\zeta}}\psi(H(\zeta))(\zeta - z)^{-3}\, d\xi\, d\eta, \tag{4.6}$$

$$z \in D_1, \quad \psi \in B_1(D_1).$$

Proof. The holomorphicity of $\eta_c^{(1)}$ and Formula (4.4) are proven exactly as the corresponding part of Theorem C (see [6] and also [9]). The rest of the theorem follows from Formula (3.2), which means that the mapping

$$\tilde{\Lambda}_c^{(1)}\colon \psi \to \mu(\zeta) = \frac{(\zeta - H(\zeta))\dfrac{\partial H(\zeta)}{\partial \bar{\zeta}}\psi(H(\zeta))}{1 + (\zeta - H(\zeta))\dfrac{\partial H(\zeta)}{\partial \bar{\zeta}}\psi(H(\zeta))}, \quad \psi \in B_1(D_1), \quad \zeta \in D_2,$$

is a local cross-section for $\eta_c^{(1)}\colon M(D_2) \to B_1(D_1)$ near the origin, i.e.,

$$\eta_c^{(1)} \circ \tilde{\Lambda}_c^{(1)} \psi = \psi \quad \text{for} \quad \psi \in B_1(D_1) \quad \text{with} \quad \|\psi\|_{1, D_1} \quad \text{small.} \tag{4.7}$$

But direct computation easily yields

$$D\tilde{\Lambda}_c^{(1)}(0) = \Lambda_c^{(1)}, \tag{4.8}$$

and (4.6) is derived by applying the chain rule to (4.7). Q.e.d.

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