Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 12, 1987, 119—134

ON SUBHARMONIC FUNCTIONS IN STRIPS

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1. Introduction

Various properties of subharmonic functions on a strip $R^n\times(0,1)$ have been studied by many authors (see Hardy, Ingham and Pólya [17] for $n=1$; Brawn [6—12], Armitage [3, 4], Armitage and Fugard [5] for $n\geqq 1$). Recently Yoshida [26—28] extended some of them for subharmonic functions on a cylinder $R^1\times D$, where D is a smooth bounded domain in R^m. The purpose of this paper is to consider several properties of subharmonic functions on a (generalized) strip $R^n\times D$ from a point of view different from Yoshida's. We shall argue especially the Phragmén—Lindelöf principle, the harmonic majorization and the properties of hyperplane mean values of subharmonic functions. In case $m=1$ every bounded domain in R^m is similar to $(0,1)$. However, there arises a regularity problem for bounded domains D in R^m if $m>1$. In this paper we let D be a bounded Lipschitz domain if $m>1$.

We denote by $P=(X,Y)$ a point in $R^{n+m}=R^n\times R^m$, where $X=(x_1,\dots,x_n)\in R^n$ and $Y=(y_1,\dots,y_m)\in R^m$. We write $|P|$, $|X|$ and $|Y|$ for $(\sum_{j=1}^n x_j^2+\sum_{j=1}^m y_j^2)^{1/2}$, $(\sum_{j=1}^n x_j^2)^{1/2}$ and $(\sum_{j=1}^m y_j^2)^{1/2}$, respectively. We identify R^n and R^m with $\{(X,Y);\ Y=0\}$ and $\{(X,Y);\ X=0\}$, respectively. We denote by S^{n-1} the unit sphere $\{\alpha\in R^n;\ |\alpha|=1\}$ with center at the origin in R^n, and by τ the surface measure on S^{n-1}. We write briefly L_D for a strip $R^n\times D$.

Let s be a subharmonic function on L_D. If

$$\text{(PL)}\qquad \limsup_{P\to Q,\, P\in L_D} s(P)\leqq 0 \quad \text{for} \quad Q\in\partial L_D,$$

then we say that s satisfies the Phragmén—Lindelöf boundary condition. We denote by s^+ the positive part of s. In view of the behavior at ∞ of the Bessel function $I_{n/2-1}$ of the third kind of order $n/2-1$,

$$(1)\qquad I_{n/2-1}(r)\sim r^{-1/2}e^r \quad \text{as} \quad r\to\infty,$$

([23; p. 203]), Brawn's result [7; Theorem 2, Corollary] may read as follows:

Theorem A. *Let* $m=1$ *and* $D=(0,1)$. *If s is a subharmonic function in* L_D

doi:10.5186/aasfm.1987.1202

satisfying (PL) *such that*

$$\liminf_{r\to\infty} r^{(n-1)/2} e^{-\pi r} \int_{S^{n-1}} \int_0^1 s^+(r\alpha, y) \sin(\pi y)\, dy\, d\tau(\alpha) = 0, \tag{2}$$

then $s \leqq 0$ *in* L_D.

We note that the function $\sin(\pi y)$ appearing in (2) is related to the eigenvalue problem

$$\begin{aligned} (\Delta_Y + \mu) f &= 0 \quad \text{on } D, \\ f &= 0 \quad \text{on } \partial D, \end{aligned} \tag{3}$$

where $\Delta_Y = \sum_{j=1}^m \partial^2/\partial y_j^2$. If $m=1$ and $D=(0, 1)$, then the constant π^2 and the function $\sin(\pi y)$ are the first positive eigenvalue of (3) and its positive eigenfunction. In general we let λ_D be the square root of the first positive eigenvalue of (3) and let f_D be its positive eigenfunction. In Section 3 we shall prove the following generalization of Theorem A:

Theorem 1. *Let* $m \geqq 1$ *and* $D \subset R^m$. *If* s *is a subharmonic function on* L_D *satisfying* (PL) *such that*

$$\liminf_{r\to\infty} r^{(n-1)/2} \exp(-\lambda_D r) \int_{S^{n-1}} \int_D s^+(r\alpha, Y) f_D(Y)\, dY\, d\tau(\alpha) = 0, \tag{4}$$

then $s \leqq 0$ *on* L_D.

Next we shall deal with subharmonic functions which do not necessarily satisfy (PL). Let $\mathscr{S}$ be the set of all functions s defined on the closure of L_D that are subharmonic in L_D and satisfy

$$\limsup_{P\to Q,\, P\in L_D} s(P) = s(Q) < +\infty \quad \text{for every } Q \in \partial L_D.$$

Let $\mathscr{A}$ be the set of all subharmonic functions s that have nonnegative harmonic majorants (see [11; p. 262]). In case $m=1$ we can easily deduce the following sufficient condition for s to belong to $\mathscr{A}$ in the same line as in Brawn [8; Theorem 2]:

Theorem B. *Let* $m=1$ *and* $D=(0, 1)$. *If* $s \in \mathscr{S}$ *satisfies* (2) *and*

$$\int_{R^n} s^+(X, 0)(1+|X|)^{(1-n)/2} e^{-\pi|X|}\, dX < \infty,$$

$$\int_{R^n} s^+(X, 1)(1+|X|)^{(1-n)/2} e^{-\pi|X|}\, dX < \infty,$$

then $s \in \mathscr{A}$.

In order to generalize Theorem B to the case $m \geqq 1$, we need to define the normal derivative of f_D. Let n_Y be the inward normal at Y with respect to ∂D and let σ be the surface measure on ∂D. It is well known that n_Y exists σ-a.e. on ∂D (see e.g. [22; p. 242]). We shall observe in the next section that for σ-a.e. Y on ∂D the normal derivative of f_D,

$$\frac{\partial}{\partial n_Y} f_D(Y) = \lim_{t \downarrow 0} \frac{\partial}{\partial n_Y} f_D(Y+tn_Y)$$

exists, and that $\partial f_D/\partial n_Y > 0$ σ-a.e. and is square integrable with respect to σ. We note that if $m=1$, $D=(0, 1)$ and $f_D(y)=\sin(\pi y)$, then $\partial f_D/\partial n_Y=\pi$ on $\{0, 1\}$, and $\sigma=\delta_0+\delta_1$, where δ_0 and δ_1 are the Dirac measures at 0 and 1. Our generalization of Theorem B is

Theorem 2. *Let* $m \geqq 1$ *and* $D \subset R^m$. *If* $s \in \mathcal{S}$ *satisfies* (4) *and*

$$(5) \qquad \int_{R^n}\int_{\partial D} s^+(X, Y)(1+|X|)^{(1-n)/2} \exp(-\lambda_D|X|) \frac{\partial f_D}{\partial n_Y}(Y)\, dX\, d\sigma(Y) < \infty,$$

then $\int_{\partial L_D} s^+(Q)\omega(P, dQ)$ *is a nonnegative harmonic majorant of* s *and hence* $s \in \mathcal{A}$ *where* ω *is the harmonic measure.*

Finally we shall consider the mean

$$\mathcal{M}s(Y) = \int_{R^n} s(X, Y)\, dX$$

of a subharmonic function s on L_D. In case $m=1$ and $D=(0, 1)$, there are a number of studies on $\mathcal{M}s$ (see [3, 4], [7—9, 12], [15] and [17]). As a typical example we quote [8; Theorem 3]:

Theorem C. *Let* $m=1$ *and* $D=(0, 1)$. *If* $s \in \mathcal{S}$ *and*

(i) $\int_{R^n} |s(X, y)|\, dX < \infty, \quad 0 \leqq y \leqq 1,$

(ii) $\limsup_{|(X,y)| \to \infty, (X,y) \in L_D} s^+(X, y)|X|^{(n-1)/2} e^{-\pi|X|} = 0,$

then $\mathcal{M}s$ *is a convex function of* $y \in D$.

Since the assumption of Theorem C implies that $s \in \mathcal{A}$ by Theorem B, it may be natural to consider the properties of $\mathcal{M}s$ for $s \in \mathcal{A}$. Noting that the subharmonicity corresponds to the convexity in case $m>1$, we shall prove

Theorem 3. *Let* $m \geqq 1$ *and* $D \subset R^m$. *Let* $s \in \mathcal{A}$ *and let* h *be a nonnegative harmonic majorant of* s. *If*

$$\int_{R^n} h(X, Y)\, dX < \infty \quad \textit{for some} \quad Y \in D,$$

then $\mathcal{M}s(Y)$ *is a subharmonic function on* D *or identically* $-\infty$ *on* D.

From this theorem we shall derive

Corollary 1. *Let* $m \geqq 1$ *and* $D \subset R^m$. *If* $s \in \mathcal{S}$ *satisfies* (4) *and*

$$\int_{\partial D}\int_{R^n} s^+(X, Y) \frac{\partial f_D}{\partial n_Y}(Y)\, dX\, d\sigma(Y) < \infty,$$

then $\mathcal{M}s(Y)$ *is a subharmonic function on D or identically* $-\infty$ *on D.*

In case $m=1$ and $D=(0, 1)$, the inequality in Corollary 1 reduces to

$$\int_{R^n} (s^+(X, 0)+s^+(X, 1))\, dX < \infty.$$

Hence Theorem C readily follows from Corollary 1.

The author would like to acknowledge that this work was inspired by Yoshida [26—28], and would like to thank Professor Yoshida for showing him the preprints of [26—28].

2. Preliminaries

We shall use the following notation: Let $X_0=(0, \ldots, 0)\in R^n$, $Y_0=(0, \ldots, 0)\in R^m$ and $P_0=(X_0, Y_0)\in R^{n+m}$. Let S^{n-1} be the unit sphere with center at X_0 in R^n and let τ be the surface measure on S^{n-1}. Denote by $B^n(X, r)$ (respectively $B^m(Y, r)$, $B(P, r)$) the n (respectively m, $n+m$)-dimensional open ball with center at X (respectively Y, P) and radius r. We may assume that $D\supset B^m(Y_0, 2)$. Let π: $R^{n+m}\to R^n$ be the projection defined by $\pi((X, Y))=X$ and let $\pi_0(P)=(\pi(P), Y_0)$. We observe that $\Omega(X)=B^n(X, 1)\times D$ is a bounded Lipschitz domain in R^{n+m}. For simplicity we write L for L_D in the sequel.

Unless otherwise specified, A will stand for a positive constant depending only on L, possibly changing from one occurrence to the next, even in the same string. If f and g are positive quantities such that $A^{-1}f\leqq g\leqq Af$, then we write $f\sim g$.

The boundary Harnack principle ([25; Theorem 1]) stated below is a useful tool.

Lemma A. *Suppose that U is a bounded Lipschitz domain in* R^{n+m}, *Q is a point in U, E is a relatively open set on* ∂U, *S is a subdomain of U satisfying* $\partial S\cap\partial U\subset E$. *Then there is a positive constant C, such that whenever u and v are two positive harmonic functions in U vanishing on E and* $u(Q)\leqq v(Q)$, *then* $u(P)\leqq Cv(P)$ *for all* $P\in S$.

Applying Lemma A to $U=\Omega(P)$, $Q=\pi_0(P)$, $E=\partial\Omega(P)\cap\partial L$ and $S==B^n(\pi(P), 1/2)\times D$, we obtain

Lemma 1 ([1; Lemma 1]). *Let* $P\in L$. *Let u and v be positive harmonic functions on* $\Omega(\pi(P))$ *which vanish continuously on* $\partial\Omega(\pi(P))\cap\partial L$. *If* $u(\pi_0(P))\leqq v(\pi_0(P))$, *then* $u(P)\leqq Av(P)$.

We need the following simple Phragmén—Lindelöf principle, which will be improved by Theorem 1.

Lemma 2 ([1; Lemma 2]). *Let L′ be a subdomain of L. If s is subharmonic in*

L', bounded above in L' and nonpositive on $\partial L'$, i.e.,

$$\limsup_{P\to Q} s(P) \leqq 0 \quad \textit{for any} \quad Q \in \partial L',$$

then $s \leqq 0$ in L'.

One of the main difficulties arising in case $m>1$ seems to be caused by the lack of the exact formulas for the Green and Poisson kernels for L (see [6] in case $m=1$ and $D=(0, 1)$). Instead of those formulas we shall use the Riesz—Martin representation ([20] and [19; Chapters 6 and 12]). In the previous paper [1] we determined the Martin compactification of L (see [10] for $m=1$). We shall describe the Martin compactification of L as follows: We denote by $\bar{L}=R^n\times\bar{D}$ the Euclidean closure of L in R^{n+m}. Let $\hat{L}=\bar{L}\cup\{M_\alpha;\ \alpha\in S^{n-1}\}$ be a compact topological space with open base $O_1\cup O_2$, where $O_1=\{U\cap\bar{L};\ U$ is an open set of $R^{n+m}\}$ and $O_2=\{U(\alpha, \varepsilon, R);\ \alpha\in S^{n-1},\ 0<\varepsilon<1 \text{ and } R>0\}$ with $U(\alpha, \varepsilon, R)=\{M_\beta;\ \beta\in S^{n-1},\ \sum_{i=1}^n \alpha_i\beta_i> 1-\varepsilon\}\cup\{(X, Y)\in\bar{L};\ (1-\varepsilon)^{-1}\sum_{i=1}^n x_i\alpha_i>|X|>R\}$. We note that M_α is considered to be an ideal boundary point, and that $P_j=(X_j, Y_j)\in\bar{L}$ converges to M_α if and only if $\lim_{j\to\infty}|X_j|=+\infty$ and $\lim_{j\to\infty} X_j/|X_j|=\alpha$.

Theorem D ([1; Theorem 1]. See also [10], [16; Chapter 8, 4 Appendix]). *The Martin compactification of L is homeomorphic to $\hat{L}$. Every point on $\hat{L}\setminus L$ is a minimal boundary point. More precisely, let G be the Green function for L and let K be the Martin kernel defined by*

$$K(P, Q) = \begin{cases} G(P, Q)/G(P_0, Q) & \text{if } Q\in L, \\ \lim\limits_{M\to Q, M\in L} G(P, M)/G(P_0, M) & \text{if } Q\in\hat{L}\setminus\bar{L}. \end{cases}$$

Then there are a positive constant λ_D^* and a positive continuous function f_D^* on D, vanishing on ∂D and satisfying $f_D^*(Y_0)=1$, such that

$$K(P, M_\alpha) = f_D^*(Y)\exp\left(\lambda_D^* \sum_{i=1}^n \alpha_i x_i\right) \tag{6}$$

for $P=(X, Y)$ and $Q=M_\alpha\in\hat{L}\setminus\bar{L}$.

We write the Laplacian Δ as

$$\Delta = \Delta_X+\Delta_Y = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}+\sum_{j=1}^m \frac{\partial^2}{\partial y_j^2}.$$

Since $\Delta K(\cdot, M_\alpha)=0$, it follows from (6) that $(\Delta_Y+\lambda_D^{*2})f_D^*=0$ on D, so that λ_D^{*2} is an eigenvalue of (3). Furthermore we show

Proposition 1. *The constant λ_D^{*2} is the first positive eigenvalue of* (3). *Hence $\lambda_D=\lambda_D^*$ and $f_D=\text{const}\cdot f_D^*$.*

Proof. Suppose that (3) has a positive eigenvalue $\mu=\lambda^2$ smaller than λ_D^{*2}. Let $f\not\equiv 0$ be an eigenfunction corresponding to λ^2. By a straightforward calculation we see that

$$u(X, Y) = f(Y)e^{\lambda x_1}$$

is harmonic on L. Let $A_0 = \sup_{Y \in D} |f(Y)| < \infty$, $L' = \{X;\ x_1 < 1\} \times D$ and $L'' = \{X;\ x_1 < 0\} \times D$. First we compare u and the bounded positive harmonic function v on L' such that

$$\lim_{P \to Q} v(P) = \begin{cases} A_0 & \text{if} \quad Q \in \{X;\ x_1 = 1\} \times D, \\ 0 & \text{if} \quad Q \in \{X;\ x_1 < 1\} \times \partial D. \end{cases}$$

Lemma 2 applied to $s = |u| - e^{\lambda} v$ leads to

$$|u(P)| \leqq e^{\lambda} v(P) \quad \text{for} \quad P \in L'.$$

Next we compare v and the Martin kernel $K(\cdot, M_\beta)$ with $\beta = (1, 0, \ldots, 0)$. We infer from Lemma 1 that there is a positive constant A_1 such that if $P = (X, Y) \in \{X;\ x_1 = 0\} \times \bar{D}$, then $v(P) \leqq A_1 K(P, M_\beta)$. Hence Lemma 2 applied to $s = v - A_1 K(\cdot, M_\beta)$ yields

$$v(P) \leqq A_1 K(P, M_\beta) \quad \text{on} \quad L''.$$

Therefore we have from (6)

$$|f(Y)| \leqq A_1 e^{\lambda} f_D^*(Y) e^{(\lambda_D^* - \lambda)x_1} \quad \text{if} \quad x_1 < 0.$$

Since $\lambda_D^* > \lambda$, letting $x_1 \to -\infty$, we obtain $f(Y) \equiv 0$ on D, a contradiction.

Remark 1. In case D is a piecewise C^1 domain it is known that if f is a positive eigenfunction for (3), then the eigenvalue corresponding to f is the first positive eigenvalue ([13; p. 458]).

Remark 2. If $m = 1, D = (0, 1)$ and $Y_0 = 1/2$, then $\lambda_D = \pi$ and $f_D(y) = \sin(\pi y)$. If $m \geqq 2$ and $D = B^m(Y_0, r)$, then $\lambda_D = \lambda_0 / r$ and

$$f_D(Y) = 2^{m/2-1} \Gamma\left(\frac{m}{2}\right) \left(\frac{\lambda_0}{r}\right)^{1-m/2} |Y|^{1-m/2} J_{m/2-1}\left(\frac{\lambda_0}{r} |Y|\right),$$

where $J_{m/2-1}$ is the Bessel function of the first kind of order $m/2 - 1$ and λ_0 is the least positive number such that $J_{m/2-1}(\lambda_0) = 0$ (see [23; p. 45] and [6; p. 441]).

Hereafter we let $f_D(Y_0) = 1$. We observe from (6) that

$$\int_{S^{n-1}} K(P, M_\alpha)\, d\tau(\alpha) = f_D(Y) \int_{S^{n-1}} \exp\left(\lambda_D \sum_{i=1}^{n} \alpha_i x_i\right) d\tau(\alpha), \quad P = (X, Y),$$

is a positive harmonic function depending only on $|X|$ and Y. On account of the formulas for Bessel functions in [23; p. 79], we see that the above integral is equal to

$$c_0 f_D(Y) |X|^{1-n/2} I_{n/2-1}(\lambda_D |X|), \tag{7}$$

where c_0 is a positive constant depending only on n and $I_{n/2-1}$ is the Bessel function of the third kind of order $n/2 - 1$. Let $K_{n/2-1}$ be another Bessel function appearing in [23; p. 78], which satisfies

$$K_{n/2-1}(r) \sim r^{-1/2} e^{-r} \quad \text{as} \quad r \to \infty. \tag{8}$$

Since $K_{n/2-1}$ satisfies the same second order differential equation [23; (1) on p. 77] as $I_{n/2-1}$, it follows that

$$f_D(Y)|X|^{1-n/2} K_{n/2-1}(\lambda_D |X|)$$

is positive and harmonic on $\{(X, Y) \in L;\ X \neq X_0\}$.

Let G^D be the Green function for D. We see that if $m=1$ and $D=(a, b)$, then

$$G^D(y, y') = \min\left\{\frac{b-y'}{b-a}(y-a), \frac{y'-a}{b-a}(b-y)\right\} \quad \text{for} \quad y, y' \in D.$$

We observe that $G^D(\cdot, Y')$ is harmonic on $D \setminus \{Y'\}$ and

$$\Delta_Y G^D(\cdot, Y') = \begin{cases} -\delta_{Y'} & \text{if} \quad m = 1, \\ -2\pi\delta_{Y'} & \text{if} \quad m = 2, \\ (2-m)\sigma_m \delta_{Y'} & \text{if} \quad m \geqq 3, \end{cases}$$

where σ_m denotes the surface area of the unit sphere S^{m-1} and $\delta_{Y'}$ denotes the Dirac measure at Y'. We have

Lemma 3. *Let* $G_0(\cdot) = G(\cdot, P_0)$ *and* $G_0^D(\cdot) = G^D(\cdot, Y_0)$. *If* $Y \in D \setminus B^m(Y_0, 1)$, *then*

(i) $G_0((X, Y)) \sim f_D(Y)(1+|X|)^{(1-n)/2} \exp(-\lambda_D |X|)$,

(ii) $G_0^D(Y) \sim f_D(Y)$.

Proof. Applying Lemma 1 to $u = f_D(Y)|X|^{1-n/2} K_{n/2-1}(\lambda_D|X|)$ and $v = G_0$, we obtain that

$$G_0((X, Y)) \sim f_D(Y)|X|^{1-n/2} K_{n/2-1}(\lambda_D |X|) \quad \text{for} \quad |X| = 1.$$

On account of Lemma 2 and (8) we have (i) for $|X| \geqq 1$. From Lemma A with $U = B^n(X_0, 2) \times (D \setminus \overline{B^m(Y_0, 1/2)})$, $u = G_0$ and $v = f_D(Y)|X|^{1-n/2} I_{n/2-1}(\lambda_D |X|)$ we infer (i) for $|X| \leqq 1$ and $Y \in D \setminus B^m(Y_0, 1)$. We regard G_0^D as a positive harmonic function on $\{(X, Y) \in L;\ Y \neq Y_0\}$. Applying Lemma A to the same U as above, $u = G_0^D$ and $v = G_0$, we obtain

$$G_0^D(Y) \sim G_0((X, Y)) \quad \text{for} \quad |X| = 1 \quad \text{and} \quad Y \in D \setminus B^m(Y_0, 1).$$

Hence (i) leads to (ii).

Remark 3. In view of Widman [24; Theorems 2.2 and 2.5], if D is a Liapunov domain in R^m $(m \geqq 2)$, then

$$G_0^D(Y) \sim f_D(Y) \sim \operatorname{dist}(Y, \partial D) \quad \text{for} \quad Y \in D \setminus B^m(Y_0, 1).$$

This relation also holds in case $m=1$.

In [14] Dahlberg studied a relationship between the harmonic measure and the normal derivative of the Green function for a bounded Lipschitz domain. Since G^D and f_D are comparable by Lemma 3 (ii), we can prove the next lemma in a way similar to [14; Lemma 9].

Lemma 4. *Let n_Y be the inward normal at Y with respect to ∂D. For σ-a.e. point Y on ∂D the normal derivative of f_D,*

$$\frac{\partial}{\partial n_Y} f_D(Y) = \lim_{t\downarrow 0} \frac{\partial}{\partial n_Y} f_D(Y+tn_Y)$$

exists and is positive. The normal derivative $\partial f_D/\partial n_Y$ is square integrable with respect to the surface measure σ on ∂D. Furthermore if h is C^2 on a domain including $\bar{D}$, then the following Green's identity holds:

$$\int_D f_D(\Delta_Y+\lambda_D^2)h\,dY = \int_{\partial D} h\frac{\partial f_D}{\partial n_Y}\,d\sigma.$$

Let $\omega(P, E)$ be the harmonic measure at $P\in L$ of $E\subset\partial L$.

Lemma 5. *If E is a Borel measurable set on ∂L, then*

$$\omega(P_0, E)\sim\int_E (1+|X|)^{(1-n)/2}\exp(-\lambda_D|X|)\frac{\partial f_D}{\partial n_Y}(Y)\,dX\,d\sigma(Y).$$

Proof. It is sufficient to prove the lemma in case $E\subset B^n(X_1, 1)\times\partial D$ for some X_1. Let $P_1=(X_1, Y_0)$ and $L'=B^n(X_1, 3)\times D$. By $\omega(\cdot, E, L')$ and G' we denote the harmonic measure of E and the Green function for L'. Applying Lemma A to $U=B^n(X_1, 2)\times(D\setminus\overline{B^m(Y_0, 1/2)})$, $u=G'(\cdot, P_1)$ and $v=G(\cdot, P_1)$, we obtain that $G'((X, Y), P_1)\sim G((X, Y), P_1)$ for $X\in B^n(X_1, 1)$ and $Y\in D\setminus B^m(Y_0, 1)$. By Lemma 3 (i) and a suitable translation we have $G((X, Y), P_1)\sim f_D(Y)$ for $X\in B^m(X_1, 1)$ and $Y\in D\setminus B^m(Y_0, 1)$. Hence we infer from [14; Theorem 3 (b)] that

$$\omega(P_1, E, L')\sim\int_E \frac{\partial f_D}{\partial n_Y}\,dX\,d\sigma(Y).$$

If $|X_1-X_0|\leqq 3$, then the Harnack principle leads to

$$\omega(P_0, E)\sim\omega(P_1, E)\sim\omega(P_1, E, L')\sim\int_E \frac{\partial f_D}{\partial n_Y}\,dX\,d\sigma(Y).$$

In case $|X_1-X_0|>3$, using Lemma 1 for $P\in\partial B^n(X_1, 2)\times D$ and then using Lemma 2, we obtain

$$\omega(P_0, E) \sim G(P_0, P_1)\,\omega(P_1, E, L').$$

Hence by Lemma 3

$$\omega(P_0, E)\sim\int_E (1+|X|)^{(1-n)/2}\exp(-\lambda_D|X|)\frac{\partial f_D}{\partial n_Y}(Y)\,dX\,d\sigma(Y).$$

Remark 4. If D is a Liapunov domain, then $\partial f_D/\partial n_Y \sim 1$ by Remark 3, so that

$$\omega(P_0, E) \sim \int_E (1+|X|)^{(1-n)/2} \exp(-\lambda_D |X|)\, dX\, d\sigma(Y).$$

If $m=1$ and $D=(a, b)$, then

$$\omega(P_0, E) = \omega(P_0, E_a) + \omega(P_0, E_b)$$
$$\sim \int_{\pi(E_a)\cup\pi(E_b)} (1+|X|)^{(1-n)/2} \exp(-\lambda_D |X|)\, dX,$$

where $E_a = \pi(E_a)\times\{a\} = \{(X, y)\in E; y=a\}$ and $E_b = \pi(E_b)\times\{b\} = \{(X, y); y=b\}$.

3. Proofs of Theorems 1 and 2

Brawn [7; Theorem 2, Corollary] (see also [5; Theorem 4]) proved Theorem A by using the Nevanlinna mean $\mathcal{M}(s, r)$ of s defined by

$$\mathcal{M}(s, r) = \int_{S^{n-1}} d\tau(\alpha) \int_0^1 s(r\alpha, y) \sin(\pi y)\, dy$$
$$= \int_0^1 \sin(\pi y)\, dy \int_{S^{n-1}} s(r\alpha, y)\, d\tau(\alpha).$$

In the expressions of $\mathcal{M}(s, r)$, there are two averaging operations,

$$\int_0^1 s \sin(\pi y)\, dy \quad \text{and} \quad \int_{S^{n-1}} s\, d\tau(\alpha).$$

Naturally, the operation $\int_D s f_D(Y)\, dY$ is considered to be a generalization of the first. We shall observe that these operations produce symmetrized subharmonic functions from given subharmonic functions on L (see Lemmas 6 and 7 below). We shall prove Theorem 1 by using this phenomenon. Let us begin with

Lemma 6. *Let s be a nonnegative subharmonic function on L satisfying* (PL). *Then*

$$S(X, Y) = f_D(Y) \int_D s(X, Y') f_D(Y')\, dY'$$

is a subharmonic function on L satisfying (PL).

Proof. On account of (PL) s is bounded on $B^n(X_0, r)\times D$ for every $r>0$. Since f_D is continuous on $\bar{D}$, it follows that S is locally integrable on L. Since s satisfies (PL) and is nonnegative on L,

$$\hat{s}(P) = \begin{cases} s(P) & \text{if } P\in L, \\ 0 & \text{if } P\in R^{n+m}\setminus L, \end{cases}$$

is a subharmonic function on R^{n+m}. On account of [19; Theorem 4.20], there is a nonincreasing sequence of C^2 subharmonic functions s_j on R^{n+m} converging to $\hat{s}$.

In view of (PL) and the construction of the sequence in [19; Theorem 4.20], we may furthermore assume that there are compact subsets K_j of D such that $K_j \uparrow D$ and

$$s_j(P) \leqq 1/j \quad \text{for} \quad P \in B^n(X_0, j+1) \times (R^m \setminus K_j). \tag{9}$$

Let

$$S_j(X, Y) = f_D(Y) \int_D s_j(X, Y') f_D(Y')\, dY'.$$

It follows from the monotone convergence theorem that $S_j \downarrow S$, so that S is upper semicontinuous. Let $P=(X, Y) \in B^n(X_0, j) \times D$. Since $\Delta s_j \geqq 0$, we have from Lemma 4 and (9)

$$\begin{aligned} \Delta S_j(P) &= f_D(Y) \int_D \{\Delta_X s_j(X, Y') - \lambda_D^2 s_j(X, Y')\} f_D(Y')\, dY' \\ &\geqq -f_D(Y) \int_D (\Delta_{Y'} + \lambda_D^2) s_j(X, Y') \cdot f_D(Y')\, dY' \\ &= -f_D(Y) \int_{\partial D} s_j(X, Y') \frac{\partial f_D}{\partial n_{Y'}}(Y')\, d\sigma(Y') \\ &\geqq -\frac{1}{j} f_D(Y) \int_{\partial D} \frac{\partial f_D}{\partial n_{Y'}}(Y')\, d\sigma(Y'). \end{aligned}$$

Since the last term tends to zero as $j \to \infty$, it follows from the dominated convergence theorem that if $\varphi \in C_0^\infty(L)$ and $\varphi \geqq 0$, then

$$\int_L S \Delta\varphi\, dP = \lim_{j\to\infty} \int_L S_j \Delta\varphi\, dP = \lim_{j\to\infty} \int_L \varphi \Delta S_j\, dP \geqq 0.$$

Hence $\Delta S \geqq 0$ on L in the distribution sense, so that S is subharmonic on L.

In the same manner as in Armitage [2; Lemma], we can prove

Lemma 7 (cf. [28; Lemma 1]). *If s is subharmonic on L, then*

$$\int_{S^{n-1}} s(|X|\alpha, Y)\, d\tau(\alpha)$$

is a subharmonic function on L depending only on $|X|$ and Y.

The next lemma is a preliminary version of Theorem 1.

Lemma 8. *Let S be a subharmonic function on L satisfying* (PL). *If*

$$\liminf_{r\to\infty} r^{(n-1)/2} \exp(-\lambda_D r) \sup_{|X|=r, Y\in D} S(X, Y) \leqq 0, \tag{10}$$

then $S \leqq 0$ on L.

Proof. Let $(X_1, Y_1) \in L$ and $\varepsilon > 0$ be given. We find $r > |X_1| + 2$ such that

$$\sup_{|X|=r, Y\in D} S(X, Y) \leqq \varepsilon r^{(1-n)/2} \exp(\lambda_D r).$$

Let h be the bounded harmonic function on $B^n(X_0, r)\times D$ such that

$$h(P)=\begin{cases}\varepsilon r^{1-n/2}I_{n/2-1}(\lambda_D r) & \text{on} \quad \partial B^n(X_0, r)\times D,\\ 0 & \text{on} \quad B^n(X_0, r)\times\partial D.\end{cases}$$

By (1) and the maximum principle we have $S\leqq Ah$ on $B^n(X_0, r)\times D$. Let

$$v(X, Y)=f_D(Y)|X|^{1-n/2}I_{n/2-1}(\lambda_D|X|)$$

and recall that this function is positive and harmonic in L and vanishes on ∂L. Since

$$r^{1-n/2}I_{n/2-1}(\lambda_D r)\leqq A(r-1)^{1-n/2}I_{n/2-1}(\lambda_D(r-1)),$$

it follows that

$$h(X, Y_0)\leqq \varepsilon r^{1-n/2}I_{n/2-1}(\lambda_D r)\leqq A\varepsilon v(X, Y_0) \quad \text{for} \quad |X|=r-1.$$

Hence Lemma 1 leads to

$$h\leqq A\varepsilon v \quad \text{on} \quad \partial B^n(X_0, r-1)\times D.$$

Using the maximum principle, we obtain

$$S\leqq h\leqq A\varepsilon v \quad \text{on} \quad B^n(X_0, r-1)\times D,$$

and in particular $S(X_1, Y_1)\leqq A\varepsilon f_D(Y_1)|X_1|^{1-n/2}I_{n/2-1}(\lambda_D|X_1|)$. Since ε is arbitrary, it follows that $S(X_1, Y_1)\leqq 0$, so that $S\leqq 0$ on L.

Proof of Theorem 1. On account of Lemmas 6 and 7

$$S(X, Y)=f_D(Y)\int_{S^{n-1}}\int_D s^+(|X|\alpha, Y')f_D(Y')\,dY'\,d\tau(\alpha)$$

is a subharmonic function on L satisfying (PL). It follows from (4) that S satisfies (10). Hence Lemma 8 leads to $S\leqq 0$ on L, so that s^+ must identically equal zero. Thus $s\leqq 0$ on L.

Proof of Theorem 2. Let

$$h(P)=\int_{\partial L}s^+(Q)\omega(P, dQ).$$

On account of Lemma 5 and (5), h is positive and harmonic on L. By the aid of [21; 2.24 The Vitali—Carathéodory Theorem] applied to the measure $\omega(P_0, \cdot)$, we find a nonincreasing sequence of nonnegative lower semicontinuous functions v_j on ∂L such that $s^+\leqq v_j$ and

$$h(P_0)\geqq h_j(P_0)-1/j,$$

where

$$h_j(P)=\int_{\partial L}v_j(Q)\omega(P, dQ).$$

We observe that

$$\lim_{j\to\infty} h_j = h \quad \text{on} \quad L. \tag{11}$$

In fact, $h^*=\lim_{j\to\infty} h_j$ is a harmonic function which majorizes h since v_j is non-increasing. We infer from $h^*(P_0)=h(P_0)$ and the maximum principle that $h^*=h$ on L.

Now we claim that h_j majorizes s on L. Let $\varepsilon>0$ and $M\in\partial L$. Since v_j is lower semicontinuous, there is $r>0$ such that

$$v_j(Q) \geqq s^+(M)-\varepsilon \quad \text{for} \quad Q\in B(M, r)\cap\partial L.$$

Hence

$$h_j(P) \geqq (s^+(M)-\varepsilon)\,\omega(P, B(M, r)\cap\partial L) \quad \text{for} \quad P\in L,$$

so that

$$\liminf_{P\to M, P\in L} h_j(P) \geqq s^+(M)-\varepsilon.$$

Therefore

$$\limsup_{P\to M, P\in L} \left(s(P)-h_j(P)\right) \leqq s^+(M)-\liminf_{P\to M, P\in L} h_j(P) \leqq \varepsilon.$$

Since ε and M are arbitrary, $s-h_j$ satisfies (PL). Applying Theorem 1 to $s-h_j$, we obtain $s\leqq h_j$ on L, and $s\leqq h$ on L by (11).

4. Proof of Theorem 3

The Riesz—Martin decomposition ([20] and [19; Chapters 6 and 12]) yields that $s\in\mathscr{A}$ if and only if there are a signed measure ν on $\hat{L}\setminus L$ and a nonnegative measure μ on L such that

$$s(P) = \int_{\hat{L}\setminus L} K(P, Q)\, d\nu(Q) - \int_L G(P, Q)\, d\mu(Q).$$

We first treat the mean of a positive harmonic function, and then treat that of a Green potential.

Lemma 9. *If h is a positive harmonic function on L, then $\mathscr{M}h(Y)$ is harmonic on D or identically $+\infty$ on D.*

Proof. We assume that $\mathscr{M}h(Y')<\infty$ for some $Y'\in D$. Take a compact subset K of D. Then Harnack's inequality [19; Theorem 2.14 and Corollary 2.15] yields that

$$\sup_{Y\in K} h(X, Y) \leqq Ah(X, Y') \quad \text{for all} \quad X\in R^n,$$

where A is a positive constant depending only on Y', K and D. Moreover every first and second order derivative $\mathscr{D}h$ of h satisfies

$$\sup_{Y\in K} |\mathscr{D}h(X, Y)| \leqq A'h(X, Y') \quad \text{for all} \quad X\in R^n,$$

where A' depends only on Y', K and D (see [18; p. 37]). Since $\mathcal{M}h(Y')<\infty$, it follows that

$$\varphi(r)=\sup_{Y\in K}\sum_{i=1}^{n}\int_{\partial B^n(X_0,r)}\left|\frac{\partial}{\partial x_i}h(X,Y)\right|d\tau_r(X)$$

is integrable with respect to r, where τ_r stands for the surface measure on $\partial B^n(X_0,r)$. Hence we can choose $r_j\uparrow\infty$ such that $\varphi(r_j)\to 0$. Using Green's formula, we obtain

$$\begin{aligned}\Delta_Y\mathcal{M}h(Y)&=\int_{R^n}\Delta_Y h(X,Y)\,dX=-\int_{R^n}\Delta_X h(X,Y)\,dX\\&=-\lim_{j\to\infty}\int_{B^n(X_0,r_j)}\Delta_X h(X,Y)\,dX\\&=\lim_{j\to\infty}\int_{\partial B^n(X_0,r_j)}\frac{\partial}{\partial n_X}h(X,Y)\,d\tau_{r_j}(X)=0\end{aligned}$$

for $Y\in K$. Since K is arbitrary, $\mathcal{M}h$ is harmonic on D.

From Lemma 9 we have a relation between G and G^D, which may be of some independent interest.

Proposition 2. *There is a positive constant c_1 depending only on n and m such that*

$$G^D(Y,Y')=c_1\int_{R^n}G((X,Y),(X',Y'))\,dX.$$

Proof. We may assume that $X'=X_0$. Let $Y'\in D$ and put

$$v(Y)=\int_{R^n}G((X,Y),(X_0,Y'))\,dX.$$

We infer from Lemmas 3 and 9 that v is harmonic on $D\setminus\{Y'\}$. It follows from Fatou's lemma that v is lower semicontinuous on D, and from Lemma 3 and Lebesgue's dominated convergence theorem that

$$\lim_{Y\to Y_1}v(Y)=0\quad\text{for}\quad Y_1\in\partial D.$$

The maximum principle yields that $v(Y)\leqq v(Y')\leqq+\infty$ (actually if $m=1$, then $v(Y')<+\infty$ and if $m\geqq 2$, then $v(Y')=+\infty$), so that v is superharmonic on D. Therefore

$$\Delta_Y v=-c(Y',D)\delta_{Y'}$$

in the distribution sense, where $c(Y',D)$ is a positive constant which may depend on Y' and D (see [19; Theorem 5.4]).

What remains is to prove that $c(Y',D)$ does not depend on Y' and D. Take $r>0$ such that $B^m(Y',r)\subset D$. Let $L'=R^n\times B^m(Y',r)$ and let G' be the Green function for L'. Since $G(\cdot,(X_0,Y'))-G'(\cdot,(X_0,Y'))$ is a positive harmonic function on L' decreasing rapidly at the infinity by Lemma 3, it follows from Lemma 9 that

$$v(Y)-\int_{R^n}G'((X,Y),(X_0,Y'))\,dX$$

is harmonic on $B^m(Y', r)$. Hence

$$c(Y', D) = c(Y', B^m(Y', r)) = c(r).$$

We infer from the arbitrariness of r that $c(r)$ is equal to a positive constant depending only on n and m. The proof is complete.

Lemma 10. *If u is a Green potential on L, then $\mathcal{M}u(Y)$ is a Green potential on D or identically $+\infty$ on D.*

Proof. Let

$$u(P) = \int_L G(P, Q)\, d\mu(Q),$$

where μ is a Radon measure on L. We have from Fubini's theorem and Proposition 2 that

$$\mathcal{M}u(Y) = \int_{R^n}\int_L G((X, Y), Q)\, d\mu(Q)\, dX = c_1^{-1}\int_D G^D(Y, Y')\, d\mu_0(Y'),$$

where μ_0 is the measure on D defined by $\mu_0(E) = \mu(R^n \times E)$. If there is a compact set $F \subset D$ such that $\mu_0(F) = \infty$, then $\mathcal{M}u \equiv \infty$ on D. If there is no such compact set, then μ_0 is a Radon measure on D and $\mathcal{M}u$ is a Green potential on D or identically $+\infty$.

Proof of Theorem 3. Since $h - s$ is a nonnegative superharmonic function, it follows from the Riesz—Martin decomposition that

$$h - s = u + p,$$

where u is a nonnegative harmonic function on L and p is a Green potential on L. We infer from the assumption and Lemmas 9 and 10 that

$$\mathcal{M}s(Y) = \mathcal{M}h(Y) - \mathcal{M}u(Y) - \mathcal{M}p(Y)$$

is a subharmonic function on D or identically $-\infty$.

Proof of Corollary 1. The inequality in Corollary 1 implies (5), and hence Theorem 2 shows that

$$h(P) = \int_{\partial L} s^+(Q)\omega(P, dQ)$$

is a nonnegative harmonic majorant of s. We infer from Lemma 5 and Fubini's theorem that

$$\begin{aligned}
&\mathcal{M}h(Y_0) \\
&\sim \int_{R^n}\int_{\partial D}\int_{R^n} s^+(X, Y)(1 + |X - X'|)^{(1-n)/2} \exp(-\lambda_D |X - X'|) \frac{\partial f_D}{\partial n_Y}(Y)\, dX\, d\sigma(Y)\, dX' \\
&\sim \int_{\partial D}\int_{R^n} s^+(X, Y) \frac{\partial f_D}{\partial n_Y}(Y)\, dX\, d\sigma(Y) < \infty.
\end{aligned}$$

Hence $\mathcal{M}h(Y_0) < \infty$, so that the corollary follows from Theorem 3.

Note added in proof: Professor S. J. Gardiner kindly informed the author that in a paper to be published in Bull. London Math. Soc. he obtained results which imply our Theorem 3. His methods are different from ours.

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Received 25 March 1986