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WEAKLY QUASISYMMETRIC EMBEDDINGS OF R INTO C

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1. Introduction

In this note we classify all continuous functions $f\colon \mathbf{R}\to\mathbf{C}$ which satisfy

$$|f(x+y)-f(x)| = |f(x-y)-f(x)| \tag{1.1}$$

for all x and y in $\mathbf{R}$.

Theorem 1. *If $f\colon \mathbf{R}\to\mathbf{C}$ is continuous and satisfies* (1.1) *then exactly one of the following holds:*

(i) $f(x)=B$ *for some complex constant B and all x in* $\mathbf{R}$,

(ii) $f(x)=Ax+B$ *for some complex constants $A\neq 0$, B, and all x in* $\mathbf{R}$,

(iii) $f(x)=Ae(\theta x)+B$ *for some complex constants $A\neq 0$, B, a real constant $\theta\neq 0$ and all x in* $\mathbf{R}$. *(We write $e(t)=e^{2\pi i t}$.)*

A useful class of mappings which satisfy (1.1) is the class of weakly 1-quasisymmetric embeddings of $\mathbf{R}$ into $\mathbf{C}$. The notion of a weakly quasisymmetric embedding was introduced by Tukia and Väisälä [TV]. In general if $(\mathscr{X}_1, d_1)$ and $(\mathscr{X}_2, d_2)$ are metric spaces, an embedding $f\colon \mathscr{X}_1\to\mathscr{X}_2$ is weakly H-quasisymmetric if $H\geqq 1$ is a constant such that

$$d_2\{f(x),f(y)\} \leqq Hd_2\{f(x),f(z)\}$$

whenever

$$d_1(x,y) \leqq d_1(x,z).$$

While an injective mapping $f\colon \mathbf{R}\to\mathbf{C}$ which satisfies (1.1) need not be weakly 1-quasisymmetric, it is clear that any weakly 1-quasisymmetric embedding must satisfy (1.1). Furthermore, if $f\colon \mathbf{R}\to\mathbf{C}$ is a weakly 1-quasisymmetric embedding then f is unbounded [TV, (2.1) and (2.16)]. It will be convenient to combine these remarks with Theorem 1.

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Corollary 2. *If* f: $\mathbf{R}\to\mathbf{C}$ *is a weakly* 1*-quasisymmetric embedding then* $f(x)=Ax+B$ *for some complex constants* $A\neq 0$, B, *and all* x *in* $\mathbf{R}$.

The statement of Corollary 2 is used in [M] to establish the existence of uniformly quasiconformal groups acting on $\mathbf{R}^n$, $n\geqq 3$, which have small dilatation. We remark that weakly 1-quasisymmetric embeddings of $\mathbf{R}$ into $\mathbf{R}^n$, $n\geqq 3$, need not be affine. This can be seen by considering an embedding

$$f(x) = (x, \cos x, \sin x)$$

whose image is a helix.

2. Mapping Z into C

Throughout this section we assume that f: $\mathbf{Z}\to\mathbf{C}$ satisfies the identity

$$|f(n+l)-f(n)| = |f(n-l)-f(n)| \tag{2.1}$$

for all integers l and n. This is equivalent to the requirement that for each fixed integer $l\geqq 1$ the function

$$n \to |f(n+l)-f(n)| \tag{2.2}$$

is periodic with period l. In particular,

$$n \to |f(n+1)-f(n)|$$

is constant. If $|f(1)-f(0)|=0$ then f is constant on $\mathbf{Z}$, a case we no longer need to consider. Therefore we assume that $|f(1)-f(0)|\neq 0$ and then we may also assume that $|f(1)-f(0)|=1$.

For each integer n we define

$$\delta(n) = f(n+1)-f(n),$$

$$\mu(n) = \delta(n+1)\overline{\delta(n)},$$

$$\alpha(n) = \Re\{\mu(n)\},$$

and

$$\varepsilon(n) = \operatorname{sgn}(\Im\{\mu(n)\}).$$

From our previous assumptions we have

$$|\delta(n)| = |\mu(n)| = 1$$

and

$$\mu(n) = \alpha(n)+i\varepsilon(n)\{1-\alpha(n)^2\}^{1/2} \tag{2.3}$$

for all n.

Lemma 3. *The function* $n\to\alpha(n)$ *has period* 2.

Proof. By hypothesis the function

$$n \to |f(n+2)-f(n)|^2$$

has period 2. We also have

$$\begin{aligned}|f(n+2)-f(n)|^2 &= |\delta(n+1)+\delta(n)|^2\\ &= |\delta(n+1)|^2+2\Re\{\delta(n+1)\overline{\delta(n)}\}+|\delta(n)|^2\\ &= 2+2\alpha(n),\end{aligned}$$

which proves the lemma.

Using Lemma 3 and the obvious fact that $-1\leqq\alpha(n)\leqq 1$ for all n, we consider three separate cases.

Case 1: $\alpha(0)^2=\alpha(1)^2=1$,

Case 2: $\alpha(0)^2<1$ and $\alpha(1)^2<1$,

Case 3: $\alpha(0)^2<1$ and $\alpha(1)^2=1$ or $\alpha(0)^2=1$ and $\alpha(1)^2<1$.

In Case 1 we have $\mu(n)=\alpha(n)$ and therefore μ has period 2. It remains to show that μ is also periodic in the other cases.

Lemma 4. *In Case* 2 *the functions* $n\to\varepsilon(n)$ *and* $n\to\mu(n)$ *both have period* 6.

Proof. By hypothesis the function

$$n \to |f(n+3)-f(n)|^2$$

has period 3. Now, we have

$$\begin{aligned}|f(n+3)-f(n)|^2 &= |\delta(n+2)+\delta(n+1)+\delta(n)|^2\\ &= |\delta(n+2)\overline{\delta(n+1)}\delta(n+1)\overline{\delta(n)}+\delta(n+1)\overline{\delta(n)}+1|^2\\ &= |1+\mu(n)+\mu(n)\mu(n+1)|^2\\ &= 3+2\Re\{\mu(n)+\mu(n+1)+\mu(n)\mu(n+1)\}\\ &= 3+2\{\alpha(n)+\alpha(n+1)+\alpha(n)\alpha(n+1)\\ &\quad -\varepsilon(n)\varepsilon(n+1)(1-\alpha(n)^2)^{1/2}(1-\alpha(n+1)^2)^{1/2}\}.\end{aligned}$$

By Lemma 3 the function

$$\alpha(n)+\alpha(n+1)+\alpha(n)\alpha(n+1)$$

is constant and, in Case 2,

$$(1-\alpha(n)^2)^{1/2}(1-\alpha(n+1)^2)^{1/2}$$

is a nonzero constant. Hence we find that

$$n \to \varepsilon(n)\varepsilon(n+1)$$

has period 3. It follows that

$$\{\varepsilon(n)\varepsilon(n+1)\}\{\varepsilon(n+1)\varepsilon(n+2)\}\{\varepsilon(n+2)\varepsilon(n+3)\} = \varepsilon(n)\varepsilon(n+3)$$

is a constant function of n. Of course this constant is $+1$ or -1, so

$$\begin{aligned}1 &= (\varepsilon(n)\varepsilon(n+3))^2\\ &= \varepsilon(n)\varepsilon(n+3)\varepsilon(n+3)\varepsilon(n+6)\\ &= \varepsilon(n)\varepsilon(n+6).\end{aligned}$$

This shows that $n \to \varepsilon(n)$ has period 6. Finally, (2.3) and Lemma 3 imply that $n \to \mu(n)$ has period 6.

Lemma 5. *In* Case 3 *the functions* $n \to \varepsilon(n)$ *and* $n \to \mu(n)$ *both have period* 8.

Proof. We assume that $\alpha(0)^2 < 1$ and $\alpha(1)^2 = 1$. Then $\varepsilon(n) = 0$ for odd integers n, so

$$\varepsilon(2m+1) = \varepsilon(2m+1+8)$$

is trivial. Thus we must show that

$$\varepsilon(2m) = \varepsilon(2m+8)$$

for all integers m.

By hypothesis the function

$$m \to |f(2m+4) - f(2m)|^2 \tag{2.4}$$

has period 2. As in the proof of Lemma 4, we expand the right-hand side of (2.4) into terms involving α and μ. We find that

$$\begin{aligned}|f(2m+4)-f(2m)|^2 &= 4+2\{\alpha(2m)+\alpha(2m+1)+\alpha(2m+2)\}\\ &\quad +2\Re\{\mu(2m)\mu(2m+1)+\mu(2m+1)\mu(2m+2)\}\\ &\quad +2\Re\{\mu(2m)\mu(2m+1)\mu(2m+2)\}.\end{aligned}$$

Of course

$$m \to \alpha(2m) \quad \text{and} \quad m \to \alpha(2m+1)$$

are constant. Since we are in Case 3,

$$m \to \Re\{\mu(2m)\mu(2m+1)\} = \alpha(2m)\alpha(2m+1)$$

and

$$m \to \Re\{\mu(2m+1)\mu(2m+2)\} = \alpha(2m+1)\alpha(2m+2)$$

are also constant. It follows that

$$m \to \Re\{\mu(2m)\mu(2m+1)\mu(2m+2)\}$$

must have period 2. But in Case 3,

$$\begin{aligned}&\Re\{\mu(2m)\mu(2m+1)\mu(2m+2)\}\\ &= \alpha(2m)\alpha(2m+1)\alpha(2m+2) - \varepsilon(2m)\varepsilon(2m+2)\alpha(2m+1)(1-\alpha(2m)^2).\end{aligned}$$

From our previous remarks and the fact that

$$m \to (1-\alpha(2m)^2)$$

is a nonzero constant, we conclude that

$$m \to \varepsilon(2m)\varepsilon(2m+2)$$

has period 2. Thus for every integer m,

$$\begin{aligned} 1 &= (\{\varepsilon(2m)\varepsilon(2m+2)\}\{\varepsilon(2m+2)\varepsilon(2m+4)\})^2 \\ &= \{\varepsilon(2m)\varepsilon(2m+4)\}\{\varepsilon(2m+4)\varepsilon(2m+8)\} \\ &= \varepsilon(2m)\varepsilon(2m+8), \end{aligned}$$

and so $2m \to \varepsilon(2m)$ has period 8. The corresponding result for μ follows from (2.3) and Lemma 3.

Next we suppose that f: $\mathbf{Z} \to \mathbf{C}$ satisfies (2.2), $|f(1)-f(0)|=1$, and that the corresponding function μ has period p, where $p \geqq 1$.

Theorem 6. *Let q be an integer. Then the function*

$$n \to f(q+pn)$$

satisfies exactly one of the following conditions:

(i) *$f(q+pn)=B$ for some complex constant B and all $n\in\mathbf{Z}$,*

(ii) *$f(q+pn)=An+B$ for complex constants $A\neq 0$, B, and all $n\in\mathbf{Z}$,*

(iii) *$f(q+pn)=Ae(\theta n)+B$ for complex constants $A\neq 0$, B, a real constant θ with $0<\theta<1$, and all $n\in\mathbf{Z}$.*

Proof. Since μ has period p it follows that

$$\gamma = \prod_{j=0}^{p-1} \mu(m+j)$$

is a constant function of m and of course $|\gamma|=1$. Therefore

$$\begin{aligned} \delta(m+p) &= \left\{\prod_{j=0}^{p-1} \mu(m+j)\right\}\delta(m) \\ &= \gamma\delta(m) \end{aligned}$$

and, replacing m by $m-p$,

$$\delta(m-p) = \gamma^{-1}\delta(m).$$

Thus we have

$$\delta(m+pn) = \gamma^n \delta(m)$$

for all integers m and n.

If $n \geqq 1$ we have

$$\begin{aligned} f(q+pn)-f(q) &= \sum_{j=0}^{pn-1} \delta(q+j) \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{p-1} \delta(q+pk+l) \\ &= \sum_{k=0}^{n-1} \sum_{l=0}^{p-1} \gamma^k \delta(q+l) \\ &= \{f(q+p)-f(q)\} \sum_{k=0}^{n-1} \gamma^k. \end{aligned}$$

If $f(q+p)-f(q)=0$ then $n\to f(q+pn)$ clearly satisfies condition (i). If

$$f(q+p)-f(q)\neq 0 \quad\text{and}\quad \gamma=1$$

then

$$f(q+pn)=f(q)+\{f(q+p)-f(q)\}n$$

and an identical formula holds for $n\leqq 0$. Thus f satisfies condition (ii). If

$$f(q+p)-f(q)\neq 0 \quad\text{and}\quad \gamma\neq 1$$

we may write $\gamma=e(\theta)$, $0<\theta<1$, and

$$f(q+pn)=f(q)+\{f(q+p)-f(q)\}\left\{\frac{e(\theta n)-1}{e(\theta)-1}\right\}. \tag{2.5}$$

This formula also extends easily to $n\leqq 0$. Of course (2.5) shows that f satisfies condition (iii).

We note that the conclusion of Theorem 6 continues to hold as stated if we drop the assumption that $|f(1)-f(0)|=1$ and set $p=24$. Also, we have only used the fact that (2.2) has period l for $l=1, 2, 3$ and 4.

3. Proof of Theorem 1

Let f: $\mathbf{R}\to\mathbf{C}$ be continuous and satisfy (1.1). Clearly we may assume that f is not constant on $\mathbf{R}$. For each $\alpha>0$ the function $n\to f((24)^{-1}\alpha n)$ maps $\mathbf{Z}$ into $\mathbf{C}$ and satisfies (2.1). By Theorem 6 and the remarks following that theorem, the function $n\to f(\alpha n)$ must be one of the three types described in Theorem 6. It will be convenient to formalize these observations as follows. For each $\alpha>0$ we define f_α: $\mathbf{Z}\to\mathbf{C}$ by $f_\alpha(n)=f(\alpha n)$. Then for each $\alpha>0$ the function f_α satisfies exactly one of the conditions:

(i′) $f_\alpha(n)=B(\alpha)$ for some complex number $B(\alpha)$ and all $n\in\mathbf{Z}$,

(ii′) $f_\alpha(n)=A(\alpha)n+B(\alpha)$ for some complex numbers $A(\alpha)\neq 0$, $B(\alpha)$, and all $n\in\mathbf{Z}$,

(iii′) $f_\alpha(n)=A(\alpha)e(\theta(\alpha)n)+B(\alpha)$ for some complex numbers $A(\alpha)\neq 0$, $B(\alpha)$, some real number $\theta(\alpha)$ with $0<\theta(\alpha)<1$, and all $n\in\mathbf{Z}$.

Since f is not constant there are distinct real numbers x_1 and x_2 such that $f(x_1)\neq f(x_2)$. Then by the continuity of f there exists $\eta>0$ such that $f(y_1)\neq f(y_2)$ whenever $|x_1-y_1|<\eta$ and $|x_2-y_2|<\eta$. Next we fix a choice of α in the interval $0<\alpha<\frac{1}{2}\eta$. It follows that

$$\left|x_1-\alpha\left[\frac{x_1}{\alpha}\right]\right|<\eta$$

and

$$\left|x_2-\alpha\left[\frac{x_2}{\alpha}\right]\right|<\eta,$$

where $[\xi]$ is the integer part of the real number ξ. Therefore

$$f_\alpha\left(\left[\frac{x_1}{\alpha}\right]\right) \neq f_\alpha\left(\left[\frac{x_2}{\alpha}\right]\right),$$

which shows that for our choice of α the function f_α must have the form (ii′) or (iii′).

Let $\beta=2^{-m}\alpha$ where $m\geqq 1$ is an integer. Since $0<\beta<\frac{1}{2}\eta$ we see that f_β must also have the form (ii′) or (iii′). In fact, we claim that f_α and f_β are either both of the form (ii′) or both of the form (iii′). To see this we note that

$$f_\alpha(n) = f_\beta(2^m n), \tag{3.1}$$

for all $n\in\mathbf{Z}$. If f_α has the form (ii′) then f_α is unbounded, hence f_β is unbounded and therefore f_β must also have the form (ii′). Conversely, if f_β has the form (ii′) then f_β is unbounded on the subsequence $\{2^m n\colon n\in\mathbf{Z}\}$. Thus f_α is unbounded and has the form (ii′).

Now suppose that f_α and f_β both have the form (ii′). Then we may write (3.1) as

$$A(\alpha)n+B(\alpha) = A(2^{-m}\alpha)2^m n+B(2^{-m}\alpha)$$

for all $n\in\mathbf{Z}$. Setting $n=0$ and $n=1$ it follows that

$$B(\alpha) = B(2^{-m}\alpha)$$

and

$$A(\alpha) = A(2^{-m}\alpha)2^m.$$

Since $A(\alpha)\neq 0$ and $A(2^{-m}\alpha)\neq 0$ we find that

$$f_\beta(n) = A(\alpha)2^{-m}n+B(\alpha)$$

or

$$f(\alpha 2^{-m}n) = A(\alpha)2^{-m}n+B(\alpha) \tag{3.2}$$

for all $n\in\mathbf{Z}$. Let

$$\mathcal{D} = \{2^{-m}n\colon m\in\mathbf{Z},\ n\in\mathbf{Z} \text{ and } m\geqq 1\}, \tag{3.3}$$

so that $\mathcal{D}$ is dense in **R**. We have

$$f(\alpha x) = A(\alpha)x+B(\alpha) \tag{3.4}$$

for all x in $\mathcal{D}$ by (3.2) and therefore (3.4) holds for all x in **R**. This shows that f has the form (ii) in the statement of Theorem 1.

Finally, we suppose that f_α and f_β both have the form (iii′). Then

$$A(\alpha)e\big(\theta(\alpha)n\big)+B(\alpha) = A(2^{-m}\alpha)e\big(\theta(2^{-m}\alpha)2^m n\big)+B(2^{-m}\alpha) \tag{3.5}$$

for all $n\in\mathbf{Z}$. Of course $0<\theta(\alpha)<1$ and therefore $\theta(2^{-m}\alpha)2^m$ is not an integer. By computing the mean value

$$\lim_{N\to\infty} N^{-1}\sum_{n=1}^{N} f_\alpha(n)$$

we see that

$$B(\alpha) = B(2^{-m}\alpha).$$

Then we set $n=0$ in (3.5) and find that

$$A(\alpha) = A(2^{-m}\alpha).$$

Let $g\colon \mathbf{R}\to\mathbf{C}$ be the continuous function

$$g(x) = A(\alpha)^{-1}\{f(\alpha x)-B(\alpha)\}, \tag{3.6}$$

and for each integer $m\geqq 1$ let

$$\mathscr{D}_m = \{2^{-m}n\colon\ n\in\mathbf{Z}\}.$$

Then $\mathscr{D}_1\subseteqq\mathscr{D}_2\subseteqq\dots$, each $(\mathscr{D}_m, +)$ is a subgroup of $(\mathbf{R}, +)$, $\mathscr{D}=\bigcup_{m=1}^{\infty}\mathscr{D}_m$, and $(\mathscr{D}, +)$ is a dense subgroup. Since

$$\begin{aligned} g(2^{-m}n) &= A(\alpha)^{-1}\{f_\beta(n)-B(\alpha)\} \\ &= e\big(\theta(2^{-m}\alpha)n\big), \end{aligned}$$

we conclude that g restricted to $\mathscr{D}$ is a homomorphism into the circle group $\mathbf{T}=\{\zeta\in\mathbf{C}\colon\ |\zeta|=1\}$. Since g is continuous on $\mathbf{R}$ it follows that $g\colon \mathbf{R}\to\mathbf{T}$ is a homomorphism, that is, g is a group character. Hence

$$g(x) = e(\theta\alpha x) \tag{3.7}$$

for some real $\theta\neq 0$ and all real x. Now (3.6) shows that f has the form (iii) in the statement of the Theorem. (That g must have the shape (3.7) is proved, for example, in [R, p. 12].)

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