Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 12, 1987, 177—184

ON THE CONTINUATION OF MEROMORPHIC FUNCTIONS

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1. Let W be an open Riemann surface and let V be an arbitrary end of W, i.e., a subregion of W with compact relative boundary $\partial_W V$. Let $MC(V)$ denote the class of meromorphic functions on V which have a limit at every point of the relative Stoïlow ideal boundary β_V of V. Further, let $BV(V)$ denote the class of constants and of meromorphic functions of bounded valence on V, and let $MD^*(V)$ stand for the class of meromorphic functions with a finite spherical Dirichlet integral on V. As in [3] and [4], we are interested in the interrelations of these three classes on certain Riemann surfaces. In Section 2, we discuss the possibility of obtaining $BV(V)\subset MC(V)$ on Riemann surfaces whose boundaries are absolutely disconnected in the sense of Sario [13]. In Section 3, we seek conditions upon β under which $BV(\overline{V})=MC(\overline{V})$. It turns out that AC-removability [4], besides the absolute disconnectivity, is just what is wanted. In Section 4 we will show that $MD^*(\overline{V})=BV(\overline{V})\subset MC(\overline{V})$ provided that W belongs to O_{A^0D} [12]. This result completes a theorem by Matsumoto [9].

In the course of this work we shall make some comments on [7], [8] and [11]. All of them contain invalid argumentation at certain points. We have not been able to restore corresponding results completely.

2. Let W be an open Riemann surface with absolutely disconnected boundary [13, p. 240]. In case W is a plane domain, this means that $\hat{\mathbf{C}}\setminus W$ ($\hat{\mathbf{C}}=\mathbf{C}\cup\{\infty\}$) belongs to N_{SB} [1, p. 105]. An immediate consequence is that all univalent functions on W can be extended to homeomorphisms of the Riemann sphere. More generally, one may expect that all meromorphic functions of bounded valence on W possess a limit at every point of β, the Stoïlow boundary of W, even if W is nonplanar. Actually, Lemma 2 in [11, p. 177] asserts that this is the case for surfaces of finite genus. Unfortunately, the author overlooked serious problems related to the path lifting. We have not managed to restore her assertion. However, we will give a positive result in a special case which covers the canonical mappings treated in [11].

Proposition 1. *Let W be an open Riemann surface, and let V be an end of W such that β_V is absolutely disconnected. Suppose f is a meromorphic function of bounded valence on V such that no $\mathrm{Cl}(f;p)$, the cluster set of f at $p\in\beta_V$, separates the (extended) plane. Then f admits a continuous extension to β_V.*

doi:10.5186/aasfm.1987.1215

Proof. Fix $p_0 \in \beta_V$. We will show that $\mathrm{Cl}(f;p_0)$ reduces to a singleton. Since f has bounded valence, $\mathrm{Cl}(f;\beta_V)=\bigcup_{p\in\beta_V}\mathrm{Cl}(f;p)$ is nowhere dense in $\hat{\mathbf{C}}$; this is readily seen by a simple category argument (see e.g. [3, p. 312]). Hence we may assume, performing an auxiliary linear fractional mapping, that f is bounded in V.

Denote by Γ the family of rectifiable dividing cycles in V which separate p_0 from $\partial_W V$, and let $\lambda(\Gamma)$ denote the extremal length of Γ. Since β_V is absolutely disconnected, $\lambda(\Gamma)=0$ [5, Theorem 5]. Suppose, for the moment, that $\mathrm{Cl}(f;p_0)$ is a proper continuum. Denote by d the diameter of $\mathrm{Cl}(f;p_0)$. It follows from Lemma 2 in [4, p. 44] that $l(f(c))$, the Euclidean length of $f(c)$, exceeds d for every $c\in\Gamma$. The boundedness of $f(V)$ then implies that $\lambda(f(\Gamma))>0$. But this state of affairs contradicts the well-known relation

$$\lambda(f(\Gamma)) \leqq N\lambda(\Gamma),$$

where $N=\max\{v_f(z)=\sum_{f(p)=z} n(p;f) \mid z\in\mathbf{C}\}$ and $n(p;f)$ denotes the multiplicity of f at p. Thus $\mathrm{Cl}(f;p_0)$ is a singleton as was asserted. □

Remark 1. It follows from [4, Theorem 1] that $\mathrm{Cl}(f;\beta_V)$ is totally disconnected. It seems, however, that it need not be of class N_{SB} (see [13, p. 289]).

Remark 2. Let W, V and β_V be as above, and suppose that f defines a proper meromorphic mapping of V onto $f(V)$, i.e., the valence function v_f is finite and constant in $f(V)$. Then f admits a continuous extension to β_V, and $\mathrm{Cl}(f;\beta_V)$ is of class N_{SB}. This result is due to Jurchescu [6, Theorem 1].

Corollary. *Let W be an open Riemann surface of finite genus, let V be an end of W with β_V absolutely disconnected. Suppose f is a meromorphic function of bounded valence on V such that each $\mathrm{Cl}(f;p)$, $p\in\beta_V$, is a line segment. Then f admits a continuous extension to β_V. Accordingly, the conclusion holds for the parallel slit mappings discussed, e.g., in* [10] *and* [11].

As usual, let O_{AD} denote the class of Riemann surfaces which do not carry nonconstant analytic functions with a finite Dirichlet integral (AD-functions). Further, let $O_{AD,n}$, $1\leqq n<\infty$, denote the class of Riemann surfaces tolerating no AD-function f with $\max\{v_f(z)|z\in\mathbf{C}\}\leqq n$. These classes were introduced in [7, p. 381]. The authors' object was to establish the relation

$$(1) \qquad \left(\bigcap_{n=1}^{\infty} O_{AD,n}\right)\setminus O_{AD} \neq \emptyset.$$

Unfortunately, their argument to this end is incorrect. A source of error is the identification of AD-functions with related homeomorphisms onto the Riemannian image. We will indicate how (1) can be obtained *provided* the assertion of Mori, cited at the outset of this section, holds true at least for planar surfaces (cf. [7, p. 381]). Let E be a compact set in $\hat{\mathbf{C}}$ which belongs to $N_{SB}\setminus N_D$ [1, p. 105]. Then $W=\hat{\mathbf{C}}\setminus E$ does not belong to O_{AD}. Further, let f be an AD-function of bounded valence on

W. Then $m(f(W))$, the two-dimensional measure of $f(W)$, is finite. This implies that $\hat{\mathbf{C}}\setminus f(W)$ does not belong to N_{SD} [1, p. 105]. But $N_{SD}=N_{SB}$ [1, p. 116]. Therefore we can find an injective bounded analytic function φ in $f(W)$. Then $g=\varphi\circ f$ is both bounded and of bounded valence in W. By our provision, g admits a continuous extension g^* to $W\cup\beta=\hat{\mathbf{C}}$. Since g^* is bounded in $\hat{\mathbf{C}}$, we infer from [3, Lemma 3] (cf. also [4, Theorem 1]) that $v_g(z)=\infty$ for "most" $z\in g(W)$, i.e., $v_f(z)=\infty$ for "most" $z\in f(W)$. This contradiction completes the argument.

3. Suppose that β, the ideal boundary of W, is admissible, i.e., for each $p\in\beta$ there is an end $V\subset W$ with $p\in\beta_V$ such that $MC(V)$ is non-trivial [4, p. 34]. Then the AC-removability of β [4, p. 40] can be characterized by the condition $MC(\bar{V})\subset BV(\bar{V})$ for each end $V\subset W$ [4, Theorems 1 and 2]. Thus, one may ask what more is needed to insure $MC(\bar{V})=BV(\bar{V})$ for each end $V\subset W$. Theorem 9 in [3, p. 311] suggests that the absolute disconnectivity of β would do the job. This is the case as shown by

Theorem 1. *Let W be an open Riemann surface with admissible ideal boundary β. Suppose β is absolutely disconnected and AC-removable. Then $MC(\bar{V})=BV(\bar{V})=MD^*(\bar{V})$ for each end $V\subset W$. Conversely, if $MC(\bar{V})=BV(\bar{V})$ for each end $V\subset W$, then β is absolutely disconnected and AC-removable.*

Proof. Suppose β is absolutely disconnected and AC-removable. As noted before, this implies $MC(\bar{V})\subset BV(\bar{V})$ for each end $V\subset W$. To establish the reverse inclusion, let V be an end of W and let $f\in BV(\bar{V})$. Fix $p_0\in\beta_V$ and choose a subend V' of V such that $p_0\in\beta_{V'}$ and $MC(\bar{V}')$ contains a nonconstant function f_0. Reducing V' and performing a preliminary linear fractional mapping, we may assume that f_0 is bounded in V'. Let f_0^* stand for the extension of f_0 to $\beta_{V'}$. Since $\beta_{V'}$ is AC-removable, $f_0^*(\beta_{V'})$ is of class N_C' [4, p. 38]. We are going to show that $f_0^*(\beta_{V'})\in N_D$ also.

Since $f_0^*(\beta_{V'})$ is totally disconnected, we may arrange $f_0^*(\beta_{V'})\cap f_0(\partial_W V')=\emptyset$. For $z\in\mathbf{C}\setminus f_0(\partial_W V')$, let $i(z; f_0(\partial_W V'))$ denote the index of z with respect to $f_0(\partial_W V')$ [4, p. 34]. By [4, Lemma 1], $v_{f_0}(z)<i(z; f_0(\partial_W V'))$ for each $z\in f_0^*(\beta_{V'})$. Set $n=\max\{i(z; f_0(\partial_W V'))\,|\,z\in\mathbf{C}\}$ and $E_i=\{z\in f_0^*(\beta_{V'})\,|\,v_{f_0}(z)\leqq i\}$. Of course, $E_{n-1}=f_0^*(\beta_{V'})$. Assume that E_0 is not of class N_D. Since E_0 belongs to N_C', it also fails to be of class N_{SB}. Therefore, we can find a nonweak boundary point, say z_0, of the planar surface $\hat{\mathbf{C}}\setminus E_0$ [13, p. 152 and p. 239]. Pick out a point p in $(f_0^*)^{-1}(z_0)$ and denote by n_0 the local degree of f_0^* at p (see [4, p. 35]). Next choose a Jordan domain $D\subset\mathbf{C}$ such that $\partial D\cap f_0^*(\beta_{V'})=\emptyset$ and $n_0=$the constant valence of $f_0|U$ in $D\setminus E_{n-1}$, where U denotes the component of $f_0^{-1}(D)$ with $p\in\beta_U$ (cf. [3, p. 306]).

By [5, Theorem 5], we can find a nonnegative Borel function ϱ on $D\setminus E_0$ and a positive δ such that

$$\iint_{D\setminus E_0}\varrho^2\,dx\,dy<\infty \quad\text{and}\quad \int_c \varrho\,|dz|\geqq\delta \tag{2}$$

for each rectifiable Jordan curve $c \subset D \setminus E_0$ separating z_0 from ∂D. Denote by $\hat{\varrho}(z)|dz|$ the pullback to U via f_0 of the conformal metric $\varrho(z)|dz|$ on $D \setminus E_0$. Since the valence of $f_0|U$ is bounded by n_0, we have

$$\iint_U \hat{\varrho}^2\, dx\, dy < \infty.$$

By the weakness of p_0, we can find a rectifiable dividing cycle c' in U separating p_0 from $\partial_W U$ such that

$$\int_{c'} \hat{\varrho}\, |dz| < \delta$$

[5, Theorem 5], [13, Theorem IV 2 C]. Then c' is homologous to $\partial_W U'$ for some subend U' of U with $p_0 \in \beta_{U'}$ [13, p. 84]. It follows that $i(z_0; f_0(c')) = n_0$. Clearly $f(c')$ contains a rectifiable Jordan curve $c \subset D \setminus E_0$ which separates z_0 from ∂D. Of course,

$$\int_c \varrho\, |dz| \leq \int_{f(c')} \varrho\, |dz| = \int_{c'} \hat{\varrho}\, |dz| < \delta.$$

This contradiction to (2) proves $E_0 \in N_D$.

Next suppose that, for some i, E_i is of class N_D and fix $z_0 \in E_{i+1} \setminus E_i$. Choose a neighbourhood D of z_0 such that ∂D is a Jordan curve with $\partial D \cap (E_{n-1} \cup f_0(\partial_W V')) = \emptyset$ and $f_0^{-1}(D)$ contains j $(j \leq i+1)$ relatively compact mutually disjoint Jordan regions V_k in V' such that each $z \in D$ has exactly $i+1$ pre-images (with due account of multiplicities) in $\bigcup_{k=1}^{j} V_k$. Then $f_0|V' \setminus \bigcup_{k=1}^{j} V_k$ assumes no value in $E_{i+1} \cap D$. Thus we have the state of affairs treated above. Hence a reproduction of the preceding arguments yields $E_{i+1} \cap D \in N_D$. Further, [12, Theorem VI 1 L] in conjunction with the Lindelöf covering theorem gives $E_{i+1} \in N_D$. It follows that $E_{n-1} = f_0^*(\beta_{V'})$ is of class N_D as was asserted.

Let G denote the component of $\mathbf{C} \setminus f_0(\partial_W V')$ which contains $f_0^*(p_0)$. Let n stand for the constant value of $v_{f_0}(z)$ in $G \setminus f_0^*(\beta_{V'})$. Then f satisfies in $f_0^{-1}(G \setminus f_0^*(\beta_{V'}))$ an identity

$$f^n + \sum_{i=1}^{n} (a_i \circ f_0) f^{n-i} = 0,$$

where $a_1, \ldots, a_n$ are meromorphic functions on $G \setminus f_0^*(\beta_{V'})$. Making use of the relation $f \in BV(\overline{V}')$ and arguing as in [2, pp. 14 and 18], it can be shown that, for every i, a_i admits a meromorphic extension over $f_0^*(\beta_{V'}) \cap G$. Thus we may regard each a_i as defined everywhere in G. Denote by $\tilde{G}$ the Riemann surface of the relation

$$P(z, w) = w^n + \sum_{i=1}^{n} a_i(z) w^{n-i} = 0, \quad z \in G, \tag{3}$$

(note that $\tilde{G}$ is a finite union of connected Riemann surfaces). By an argument involving $\tilde{G}$ and the center and value mappings associated with (3), it can be shown (cf. again the proof of [2, Theorem 1]) that $\mathrm{Cl}(f; (f_0^*)^{-1}(f_0^*(\beta_{V'}) \cap G))$ belongs to N_D. In particular, $\mathrm{Cl}(f; p_0)$ reduces to a singleton, i.e., f admits a continuous extension to p_0. Thus $BV(\overline{V}) \subset MC(\overline{V})$.

The relation $BV(\overline{V})\subset MD^*(\overline{V})$ being trivial, it remains to show that $MD^*(\overline{V})\subset BV(\overline{V})$ for each end $V\subset W$. So let $f\in MD^*(\overline{V})$ be nonconstant. Since the problem is of a local nature, we may assume that $\partial_W V$ is piecewise analytic and $MC(\overline{V})$ contains a bounded nonconstant function f_0. By virtue of $f\in MD^*(\overline{V})$, we may also assume that f omits in $\overline{V}$ a compact set $E\subset \mathbf{C}$ with $m(E)>0$. Then, by an important theorem of Nguyen Xuan Uy [14, Theorem 4.1], we can find a nonconstant analytic function g such that both g and g' are bounded in $\hat{\mathbf{C}}\setminus E$. We will show that $h=g\circ f\in AD(\overline{V})$. To this end, choose $R>0$ such that $E\subset D(0, R)=\{z\in\mathbf{C}\,|\,|z|<R\}$. Set $F_1=f^{-1}(D(0, R))$, $F_2=f^{-1}(\hat{\mathbf{C}}\setminus D(0, R))$ and choose $M>0$ such that $|g'(z)|\leqq M$ for $z\in\hat{\mathbf{C}}\setminus E$. Obviously

$$\iint_{F_1} dh\wedge *d\bar{h} = \iint_{F_1} |g'(f(p))|^2 df\wedge *d\bar{f} \leqq (1+R^2)^2 M^2 \iint_{F_1} \frac{1}{(1+R^2)^2}\, df\wedge *d\bar{f}$$

$$\leqq (1+R^2)^2 M^2 \iint_{F_1} \frac{1}{(1+|f(p)|^2)^2}\, df\wedge *d\bar{f} < \infty.$$

Let φ denote the mapping $z\mapsto 1/z$, $z\in\hat{\mathbf{C}}\setminus D(0, R)$. Then $g|\hat{\mathbf{C}}\setminus D(0, R)=g_1\circ\varphi$ with g_1 analytic in $\overline{D(0, 1/R)}$. Suppose $|g_1'(z)|\leqq M_1$ for $z\in\overline{D(0, 1/R)}$. Then

$$\iint_{F_2} dh\wedge *d\bar{h} = \iint_{F_2} |g_1'(\varphi(f(p)))|^2 d(\varphi\circ f)\wedge *d(\overline{\varphi\circ f})$$

$$\leqq M_1^2 \iint_{F_2} d(\varphi\circ f)\wedge *d(\overline{\varphi\circ f})$$

$$\leqq M_1^2\left(1+\left(\frac{1}{R}\right)^2\right)^2 \iint_{F_2} \frac{1}{(1+|\varphi(f(p))|^2)^2}\, d(\varphi\circ f)\wedge *d(\overline{\varphi\circ f})$$

$$= M_1^2\left(1+\left(\frac{1}{R}\right)^2\right)^2 \iint_{F_2} \frac{1}{(1+|f(p)|^2)^2}\, df\wedge *d\bar{f} < \infty.$$

The assertion follows.

Let G denote a component of $\mathbf{C}\setminus f_0(\partial_W V)$ with $G\cap f_0^*(\beta_V)\neq\emptyset$, and let n denote the constant value of $v_{f_0}(z)$ in $G\setminus f_0^*(\beta_V)$. Then h satisfies in $f_0^{-1}(G\setminus f_0^*(\beta_V))$ an identity

$$h^n + \sum_{i=1}^{n} (a_i\circ f_0) h^{n-i} = 0$$

with a_i analytic in $G\setminus f_0^*(\beta_V)$. As shown before, we have $f_0^*(\beta_V)\in N_D$. Because h is both bounded and Dirichlet bounded, each a_i admits an analytic extension over $f_0^*(\beta_V)\cap G$ (cf. again [2, pp. 14—15 and p. 23]). As before, these extensions permit us to conclude that $\mathrm{Cl}(h;(f_0^*)^{-1}(f_0^*(\beta_V)\cap G))$ belongs to N_D. It follows that $\mathrm{Cl}(h;\beta_V)$ is totally disconnected. Hence by [4, Lemma 2], h has bounded valence. Consequently, the same is true of f, i.e., $f\in BV(\overline{V})$. This completes the first part of the proof.

Next suppose that $MC(\overline{V})=BV(\overline{V})$ for each end $V\subset W$. It is immediate by [4, Theorems 1 and 2] that β is AC-removable. Fix an arbitrary $p_0\in\beta$. It remains to prove that p_0 is weak. So choose an end $V\subset W$ such that $p_0\in\beta_V$ and $MC(\overline{V})$

contains a nonconstant bounded function f_0. Since $f_0^*(\beta_V)$ is of class N'_C [4, p. 40], we may assume that $f_0^*(\beta_V)\cap f_0(\partial_W V)=\emptyset$. In order to prove that $f_0^*(\beta_V)$ is of class N_{SB}, we argue as in the first part of the proof.

Let G denote the component of $\mathbf{C}\setminus f_0(\partial_W V)$ which contains $f_0^*(p_0)$, and let n denote the constant value of $v_{f_0}(z)$ in $G\setminus f_0^*(\beta_V)$. Set $E_i=\{z\in f_0^*(\beta_V)\cap G \mid v_{f_0}(z)\leqq i\}$, $i=0, \ldots, n-1$. Suppose for the moment that E_0 fails to be of class N_{SB}. Then E_0 contains a nonweak boundary point, say z_0, of $\hat{\mathbf{C}}\setminus E_0$. Pick out a point $p\in(f_0^*)^{-1}(z_0)$. By [13, p. 152], we can find a univalent mapping φ of $\hat{\mathbf{C}}\setminus E_0$ into $\hat{\mathbf{C}}$ such that $\mathrm{Cl}(\varphi; z_0)$ is a proper continuum. Given any subend V' of V with $p\in\beta_{V'}$, there is an open neighborhood D of z_0 such that $D\setminus f_0^*(\beta_V)\subset f_0(V')$ [4, Lemma 1]. We conclude that $\mathrm{Cl}(\varphi\circ f_0; p)$ is also a proper continuum. Hence $\varphi\circ f_0$, although a member of $BV(\overline{V})$, does not admit a continuous extension to p. Thus E_0 is of class N_{SB} and, by virtue of $E_0\in N'_C$, even of class N_D.

Now suppose that, for some i, E_i is of class N_D and fix $z_0\in E_{i+1}\setminus E_i$. The argument given in the proof of $BV(\overline{V})\subset MC(\overline{V})$ reduces the situation to that discussed above. It follows that $E_{i+1}\in N_D$. Altogether, $E_{n-1}=f_0^*(\beta_V)\cap G$ is of class N_{SB}. Let U denote the component of $f_0^{-1}(G)$ with $p_0\in\beta_U$. We infer from [13, Theorem X 4 F] or [6, Theorem 1] that β_U is absolutely disconnected. In particular, p_0 is weak. The proof is complete. □

Remark. Recall that the union of two sets of class N_{SB} need not belong to N_{SB} [13, p. 289]. This fact explains the role of the class N_D in the last part of the proof.

The argument used in the proof of the relation $MD^*(\overline{V})\subset BV(\overline{V})$ obviously yields the following removability result, which seems to be new.

Theorem 2. *Let G be a plane domain and let $E\subset G$ be a compact set of class N_D. Then every meromorphic function with a finite spherical Dirichlet integral in $G\setminus E$ can be extended to a meromorphic function in G. In particular, every meromorphic function with a finite spherical Dirichlet integral in $\hat{\mathbf{C}}\setminus E$ is the restriction to $\hat{\mathbf{C}}\setminus E$ of a rational function.*

As mentioned before, $MC(\overline{V})\subset BV(\overline{V})$ for every end $V\subset W$ whenever the ideal boundary of W is AC-removable. This statement can be generalized as follows.

Proposition 2. *Let W and W' be Riemann surfaces with ideal boundaries β and β', respectively. Suppose that every point of β is AC-removable whenever it is admissible. Let f be a nonconstant analytic mapping of W into W' which admits a continuous extension $f^*\colon W\cup\beta\to W'\cup\beta'$. Then f has bounded valence.*

Proof. Let β_1 denote the set $\{p\in\beta \mid f^*(p)\in W'\}$. Clearly β_1 is a relatively open subset of β. For each $p\in\beta_1$, choose an end $V_p\subset W$ such that $p\in\beta_{V_p}$ and $f^*(\overline{V}_p\cup\beta_{V_p})\subset W'$. By assumption and [4, Theorem 2], $f^*(\beta_{V_p})$ is a totally disconnected subset of W'. Thanks to Lindelöf, we can select from the covering $\{V_p\cup\beta_{V_p} \mid p\in\beta_1\}$ of β_1 a countable subcovering $\{V_n\cup\beta_{V_n} \mid n\in N\}$. Thus $f^*(\beta_1)=$

$\bigcup_{n=1}^{\infty} f^*(\beta_{V_n})$ is a closed and totally disconnected subset of W'. It follows that ν_f, the valence function of f, is finite and constant, say m, in $W' \setminus f^*(\beta_1)$. By the lower semicontinuity of ν_f, $\nu_f(q) \leqq m$ for every $q \in W'$. □

Now suppose that W is parabolic. Then every admissible boundary point of W is AC-removable [4, p. 40]. Hence we have immediately the following

Corollary. *Let W and W' be parabolic Riemann surfaces with ideal boundaries β and β', respectively. Let f be a nonconstant analytic mapping of W into W' which admits a continuous extension $f^*\colon W \cup \beta \to W' \cup \beta'$. Then f has bounded valence.*

Remark. This result is contained in [8, Theorem 1]. However, the proof there relies on the incorrect claim that the Stoïlow boundary is a countable set.

4. Next suppose that W belongs to O_{A^0D} [12, p. 17] and V is an end of W. Then, by a result of Matsumoto and Kuroda [12, p. 372], every function $f \in MD^*(\bar{V})$ has the localizable Iversen property [12, p. 365]. By Stoïlow's principle on Inversen's property [12, p. 370], $\mathrm{Cl}(f; \beta_V)$ is either total, i.e., $\mathrm{Cl}(f; \beta_V) = \hat{\mathbf{C}}$, or $\mathrm{Cl}(f; \beta_V)$ is totally disconnected. It was shown by Matsumoto [9, Theorem 3], [12, p. 373] that for $f \in AD(\bar{V})$, only the latter alternative can occur, i.e., every function in $AD(\bar{V})$ admits a continuous extension to β_V. Actually, this is the case with all MD^*-functions as shown by

Theorem 3. *Let W be an open Riemann surface of class O_{A^0D}. Then $BV(\bar{V}) = MD^*(\bar{V}) \subset MC(\bar{V})$ for each end $V \subset W$.*

Proof. Let $V \subset W$ be an end, and suppose that $f \in MD^*(\bar{V})$ is nonconstant. To prove that $f \in BV(\bar{V})$, it clearly suffices to show that for each $p \in \beta_V$ there is a subend V' of V such that $p \in \beta_{V'}$ and $f|\bar{V}'$ has bounded valence. So fix $p \in \beta_V$ and choose a subend V' of V such that $p \in \beta_{V'}$ and f omits in $\bar{V}'$ a compact set $E \subset \mathbf{C}$ with $m(E) > 0$. As in the proof of Theorem 1, we can find a nonconstant analytic function g in $\hat{\mathbf{C}} \setminus E$ such that $h = g \circ (f|\bar{V}') \in AD(\bar{V}')$. By the result of Matsumoto cited previously, $\mathrm{Cl}(h; \beta_{V'})$ is totally disconnected. This implies ([4, Lemma 1] or [12, p. 370]) that h has bounded valence. Thus $f|\bar{V}'$ has bounded valence, too. We conclude that $f \in BV(\bar{V})$. That f also belongs to $MC(\bar{V})$ now follows readily from Stoïlow's principle on Iversen's property. Indeed, provided $\mathrm{Cl}(f; \beta_V)$ is total, a standard argument involving the Baire category (see, e.g., [3, Lemma 3]) gives $f \notin BV(\bar{V})$. Hence $\mathrm{Cl}(f; \beta_V)$ is totally disconnected, whence $f \in MC(\bar{V})$. □

Remark. It is readily verified that $\mathrm{Cl}(f; \beta_V) = f^*(\beta_V)$ is of class N_D: just decompose $f^*(\beta_V)$ by means of the valence function ν_f as in the proof of Theorem **1** and apply [12, Theorem VI 2 C]. It seems that for $f \in BV(\bar{V})$ this result has also been obtained by Qiu Shuxi (see [15, pp. 152—3]). The relation $f^*(\beta_V) \in N_D$ can be used to show that $BV(\bar{V}) = MD^*(\bar{V})$ constitutes a field (see the proofs of [2, Theorem 7] and [3, Theorem 5]).

Let U_S denote the class of open Riemann surfaces whose ideal boundary contains a point of positive harmonic measure [12, p. 385]. Further, let O_{MD^*} denote the class of Riemann surfaces which do not carry nonconstant meromorphic functions with a finite spherical Dirichlet integral. Of course, $O_{MD^*} \subset O_{AD}$. By [13, Theorem X 4 C] we immediately obtain the following corollary, which sharpens another result by Matsumoto [9, Theorem 4], [12, Theorem VI 5 B].

Corollary. *Suppose $W \in U_S \cap O_{A^0D}$ and let K be an arbitrary compact set in W with connected complement. Then $W \setminus K \in O_{MD^*}$.*

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Received 29 April 1986