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WANDERING DOMAINS FOR MAPS OF THE PUNCTURED PLANE

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1. Introduction and results

The iteration theory of Fatou [8, 9] and Julia [12] applies to analytic maps $f: D \to D$ where the domain D belongs to $\hat{\mathbf{C}}$, and introduces the sets $N(f)=\{z; z \in D, (f^n)$ is a normal family in some neighbourhood of $z\}$ and $J(f)=D \setminus N(f)$. To avoid trivial cases it is supposed that f is not a Moebius transformation. The theory studies the way in which $J(f)$ divides the components of $N(f)$. To obtain interesting results it is necessary to assume that the complement of D consists of at most two points, since otherwise $J(f)$ is empty. We may assume that the complement of D is $\emptyset$, $\{\infty\}$, or $\{0, \infty\}$ and with this normalisation there are essentially the following cases

I. $D = \hat{\mathbf{C}}$, f rational,

II. $D = \mathbf{C}$, f entire,

III. $D = \mathbf{C}^* = \{z; 0 < |z| < \infty\}$.

In the third case there are four types of function f, depending on the behaviour of f at the isolated (potentially) singular points $0, \infty$,

(a) $f=kz^n$, $k \neq 0$, $n \in \mathbf{Z}$, $n \neq 0, \pm 1$
(N. B. we are excluding Moebius transformations),

(b) $f(z)=z^n \exp(g(z))$, g non-constant entire, $n \in \mathbf{N}$,

(c) $f(z)=z^{-n} \exp(g(z))$, g non-constant entire, $n \in \mathbf{N}$,

(d) $f(z)=z^m \exp\{g(z)+h(1/z)\}$, g, h non-constant entire functions, $m \in \mathbf{Z}$.

Here we have made the normalisation that if f has exactly one essential singularity it is ∞. Note that (a), (b) may be regarded as belonging to cases I, II, and that for any $k \geqq 2$, and f of type (c) we have f^k of type (d).

The cases I, II have been discussed by Fatou [8, 9], Julia [12] and many other authors, case III by Rådström [14] and Bhattacharyya [6].

In all cases the set $J(f)$ is closed, non-empty and even perfect in D, with the invariance property $f(J(f))=f^{-1}(J(f))=J(f)$, (sometimes called "complete invariance") and the further one that $J(f^p)=J(f)$ for $p \in \mathbf{N}$. $N(f)$ may be empty, as is the case for $f(z)=\exp z$.

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If the components of $N(f)$ are denoted by N_j, then for each N_j there is an N_k such that $f(N_j)\subset N_k$. N_1 is a wandering component if $f^m(N_1)\cap f^n(N_1)\neq\emptyset$, $m, n\in\mathbf{N}$, implies that $m=n$. Sullivan [15] has shown that in case I there are no wandering components, while Baker [3, 4, 5], Eremenko and Lyubich [7], Herman [11] and Sullivan [15] have shown that wandering components may occur in case II. These wandering components may be either simply- or multiply-connected. In particular we have the following results.

Theorem A. [4] *If f is transcendental entire and U is a multiply-connected component of $N(f)$, then U is a wandering component. Further, $f^n\to\infty$ in U $(n\to\infty)$ and, for large n, $f^n(U)$ contains a closed curve γ_n, whose distance from 0 is arbitrarily large and whose winding number about 0 is non-zero. Moreover, every component of $N(f)$ is bounded.*

Theorem B. [5] *For any ϱ such that $0\leqq\varrho\leqq\infty$ there is an entire function f of order ϱ, which has multiply-connected wandering components of $N(f)$. In the case $\varrho=0$ the connectivity of the wandering component may be infinite.*

If U is one of the wandering components described in Theorem B, it is clear that all $f^n(U)$, $(n>n_0)$, are multiply-connected and that each is bounded away from 0 and ∞.

It is natural to ask whether wandering components can occur for functions of class III. It turns out that the situation described in Theorems A and B cannot occur but that simply-connected wandering domains are possible.

Theorem 1. *If f is a (non-Moebius) analytic map of $\mathbf{C}^*$ to itself, then the components of $N(f)$ are simply or doubly-connected. There is at most one doubly-connected component, except in the simple case III(a).*

In case II, Theorem A states that any multiply-connected component of $N(f)$ is bounded. See also [2]. By contrast we have

Theorem 2. *For $0<\alpha<1/2$ and $f(z)=\exp\{\alpha(z-z^{-1})\}$, $N(f)$ consists of a single multiply-connected component with 0 and ∞ on its boundary.*

Theorem 3. *There is a function f of class III for which $N(f)$ has a doubly-connected component in which f is analytically-conjugate to a rotation $z\to e^{i\alpha\pi}z$, α irrational.*

This is the case of a Herman ring, which cannot occur for functions of class II.

Theorem 4. *There is a function of class III(b) which has a wandering component.*

We remark that if U is a wandering component for f of class III, then $f^n(U)$ is simply-connected for $n\geqq n_0$. This will simplify attempts to prove that particular classes of such functions do in fact possess no wandering components, using the methods of [15], [4] or [7]. Some classes are already known from [4] and [7], e.g. $\exp(p(z))$, where p is a polynomial.

2. Preliminary lemmas

Suppose throughout that f is one of the functions of class III(a), (b), (c) or (d).

Lemma 2.1. [6] *Let G be a component of $N(f)$ such that some sequence f^{n_k}, where n_k is a strictly-increasing sequence of natural numbers, has a non-constant limit function φ in G. Then for some n_k we have $f^{n_k}(G)$ and $\varphi(G)$ contained in a component G_1 of $N(f)$, which is mapped univalently onto itself by some iterate f^p. Further the identity is a limit function of some sequence f^{m_k} in G_1.*

Lemma 2.2. [6] *If $\alpha \in J(f)$ and Δ is a neighbourhood of α and K is any compact subset of $\mathbf{C}^*$, then there is a natural number n_0 such that for $n > n_0$ we have $f^n(\Delta) \supset K$.*

Lemma 2.3. [6] *The fixed points of iterates of f are dense in $J(f)$.*

The following results were proved in the appropriate form for functions of class II in [1]. The proofs need almost no modification for class III.

Lemma 2.4. *Suppose that n_k is an increasing sequence of natural numbers such that certain branches $z = G_{n_k}(w)$ of the inverse functions of $w = f^{n_k}(z)$ are all defined and regular in the domain G. Then (G_{n_k}) is a normal family in G.*

Lemma 2.5. *Let the set of singularities other than $0, \infty$ of f^{-1} be S, and let E be the set of points of the form $f^n(s)$, $s \in S$, $n \geqq 0$. Then a point belongs to E precisely if it is a singularity other than $0, \infty$ of some inverse function f^{-n} of an iterate of f.*

We may recollect that the singularities of f^{-1} are either algebraic branch points or are asymptotic values approached by $f(z)$ as $z \to 0$ or ∞ along a suitable path.

Lemma 2.6. *Let E be the set defined in Lemma 2.5 and let E'' denote the derived set of E, together with any points which are of the form $f^n(s)$, $s \in S$, for an infinity of values of n. Then any constant limit of a sequence f^{n_k} in a component of $N(f)$ belongs to $L = E \cup E'' \cup \{0, \infty\} = \bar{E} \cup \{0, \infty\}$.*

Lemma 2.7. *If the set L defined in Lemma 2.6 has an empty interior and a connected complement, then no sequence (f^{n_k}) has a non-constant limit function in any component of $N(f)$.*

3. Proof of Theorem 1

Lemma 3.1. *Suppose that f is of class III and that G is a component of $N(f)$. Then if γ is a simple closed curve in G, either* (i) *γ separates* 0, ∞ *or* (ii) *the complement of γ has a compact component which belongs to G.*

Proof. If (i) does not hold let Δ denote the compact component of $C^*\setminus\gamma$ and suppose that $\Delta\cap J(f)\neq\emptyset$. For an arbitrarily small positive ε, $f^n(\Delta)$, $n>n_0(\varepsilon)$, covers $\mathbf{C}^*$ except for an ε-neighbourhood of 0 and ∞. Since $\partial(f^n(\Delta))\subset f^n(\partial\Delta)=f^n(\gamma)$ it follows that $f^n(\gamma)$ meets the ε-neighbourhood of both 0 and ∞ for $n>n_0(\varepsilon)$. Thus if we pick out a subsequence f^{n_k} which is locally uniformly convergent to, say, φ in G, the function φ is non-constant and (cf. Lemma 2.1) $\varphi(\gamma)$ is a compact subset of some component of $N(f)\subset\mathbf{C}^*$. This contradicts the fact that $d(f^n(\gamma), \infty)\to 0$ as $n\to\infty$.

Corollary. G has at most two boundary components, so that the first part of Theorem 1 is proved.

Lemma 3.2. *Suppose that γ_1, γ_2 are disjoint Jordan curves in $N(f)$, f of class III, which separate* 0, ∞. *Then the region Δ bounded by γ_1, γ_2 contains no points of $J(f)$ except in the case when f has the form III* (a).

Proof. Suppose that $\Delta\cap J(f)\neq\emptyset$. Then for arbitrary positive ε, $f^n(\Delta)$ covers $\mathbf{C}^*$ except for an ε-neighbourhood of 0 and ∞ for $n>n_0(\varepsilon)$. Now if some f^{n_k} has a non-constant limit function φ in the component G_1 of $N(f)$ which contains γ_1, it follows from Lemma 2.1 that for large n, $f^n(\gamma_1)$ is close to the compact set $\gamma_1'\cup f(\gamma_1')\cup\ldots\cup f^{p-1}(\gamma_1')\subset\mathbf{C}^*$, $\gamma_1'=\varphi(\gamma_1)$ where p is the (smallest) positive integer such that f^p maps to itself the component of $N(f)$ which contains $\varphi(G_1)$. By the covering property of $f^n(\Delta)$ it follows that for large n, $f^n(\gamma_2)$ contains points near both 0 and ∞, so that f^n has no constant limit functions in the component G_2 of $N(f)$ which contains γ_2. But then $f^n(\gamma_2)$ must also approximate a certain compact subset of $\mathbf{C}^*$, as is the case for $f^n(\gamma_1)$. This again contradicts the covering property of $f^n(\Delta)$, as $n\to\infty$.

Thus for any sufficiently large n we have either $|f^n|<\varepsilon$ on γ_1, $|f^n|>\varepsilon^{-1}$ on γ_2 or $|f^n|>\varepsilon^{-1}$ on γ_1 and $|f^n|<\varepsilon$ on γ_1.

Thus the set $f^n(\gamma_1)$ or $f^n(\gamma_2)$ contains a simple closed curve $\gamma(\varepsilon)$ in $|z|>1/\varepsilon$, which separates 0, ∞. Further, on $\gamma(\varepsilon)$ we have either $|f|<\varepsilon$ or $|1/f|<\varepsilon$. Applying this for $\varepsilon=1/n\to 0$ and noting that for m much larger than n the curves $\gamma(1/m)$, $\gamma(1/n)$ are disjoint, we see that either f or $1/f$ is bounded in a neighbourhood of ∞ and hence f is analytic or has a pole at ∞. A similar argument applies at 0. Thus f has the form given by III(a).

Combining Lemma 3.2 with the corollary to Lemma 3.1 completes the proof of Theorem 1.

4. Proof of Theorem 2

The function given by

$$f(z) = \exp\{\alpha(z - z^{-1})\},$$

where α is a constant such that $0<2\alpha<1$, is of type III(d). The unit circumference γ is mapped so that $z=e^{i\vartheta}$ gives $f(z)=e^{i\varphi}$, where $\varphi=2\alpha\sin\vartheta$, so that $|\varphi|\leqq 2\alpha|\vartheta|$, and $f^n(z)=e^{i\vartheta_n}$, where $|\vartheta_n|\leqq(2\alpha)^n\,\vartheta\to 0$. Thus γ belongs to the domain of attraction of the attractive fixed point 1 for which $f(1)=1$, $f'(1)=2\alpha$.

The only singularities of f^{-1} are 2 algebraic branch points, over $e^{2\alpha i}$ and $e^{-2\alpha i}$ respectively, and transcendental singularities over 0, ∞. Denote by G that component of $N(f)$ which contains 1, and hence an annulus A: $1-\delta<|z|<1+\delta$ for some $\delta>0$. Now we can reach all branches of $f^{-1}(1)$ by continuation from the value $f^{-1}(1)=1$ along paths in A. By the invariance properties of $N(f)$ it follows that all branches of $f^{-1}(1)$ belong to G. A similar argument shows that for any $z\in G$, all values of $f^{-1}(z)$ belong to G, by considering continuation of f^{-1} from 1 to z along a path in G.

Thus we have shown that G is completely invariant. G must therefore extend to the essential singularities 0 and ∞. It remains to show that there are no other components of $N(f)$.

Now in the notation of Lemma 2.6 $\bar{E}=\{f^n(e^{\pm 2\alpha i}),\ n=0, 1, \ldots,\}\cup\{1\}$, which is a compact subset of G, and $L=\bar{E}\cup\{0, \infty\}$. If H is a component of $N(f)$ other than G, then the only possible limit functions of any subsequence of f^n in H are 0 and ∞. Thus $(f^n+f^{-n})\to\infty$ in H as $n\to\infty$. Since

$$(f^n)'(\tau) = f^n\left\{\Pi_{\nu=1}^{n-1}\alpha\left(f^\nu+\frac{1}{f^\nu}\right)\right\}\alpha(1+z^{-2})$$

we see that if there is a sequence of n-values such that $f^n\to\infty$ in H, then for such n-values we also have $(f^n)'\to\infty$ in H.

Put $g_n=\alpha(f^n-f^{-n})$. Then for large n in the given sequence we have g_n' large in H and so, by Bloch's theorem, $g_n(H)$ contains a disc Δ of diameter at least 2π. Then $f^{n+1}(H)\supset\exp(\Delta)$ contains a circle γ' of the form $|z|=\text{constant}$. By Lemma 3.2 γ' belongs to G and by the complete invariance of G, $H=G$ against assumption.

Thus if $H\neq G$ the only remaining possibility is that $f^n\to 0$ in H as $n\to\infty$. Put $h_n=1/(f^n)$, so that

$$(h^n)'(z) = -(f^n)^{-1}\left[\Pi_{\nu=1}^{n-1}\left\{\alpha\left(f^\nu+\frac{1}{f^\nu}\right)\right\}\right]\alpha(1+z^{-2}).$$

Then $|(h^n)'|\to\infty$ in H and if $k_n=\alpha(h_n^{-1}-h_n)$, then $|k_n'|=|\alpha(1+(f^n)^2)h_n'|\to\infty$, so that $k_n(H)$ contains a disc Δ of diameter at least 2π. As above we then find that $f^{n+1}(H)=\exp(k_n(H))$ contains a circumference $|z|=\text{constant}$, and hence $H=G$.

Remark. The map $z_1=f(z)$ is semiconjugated by $z=e^{it}$, $z_1=e^{it_1}$ to $t_1=2\alpha\sin t=g(t)$, say. Our results give independent confirmation of the fact that $N(g)$ is a single simply-connected region in which $g^n\to 0$, a situation discussed e.g. by R. L. Devaney (unpublished).

5. Proof of Theorem 3

Modify the preceding example by setting

$$g(z) = e^{2\pi i\beta}z\exp\{\alpha(z-z^{-1})\},$$

where β is a real constant. For $0<2\alpha<1$ the function g gives an orientation preserving homeomorphism of the unit circumference γ to itself. Indeed, putting $e^{i\vartheta}=e^{2\pi ix}$ on γ we may represent $g|_\gamma$ by $x\to G(x)=x+\beta+(\alpha/\pi)\sin(2\pi x)\pmod 1$, which is monotone increasing and satisfies $G(x+1)=G(x)+1$. We recall the definition of the rotation number ϱ of g

$$\varrho = \lim_{n\to\infty}\frac{G^n(x)}{n}\quad (\mathrm{mod}\ 1),$$

which is independent of x. ϱ varies continuously with g and G, in particular with β, and so β may be chosen in such a way that ϱ is an irrational number which satisfies a diophantine condition D.C.: — There exist $b\geqq 0$, $c>0$ such that for every $p/q\in\mathbf{Q}$ we have $|\varrho-(p/q)|\geqq cq^{-(2+\beta)}$. Such ϱ are of measure 1 in [0, 1].

As was proved by Yoccoz [16], extending earlier work of Herman [10], g is then real analytically conjugate on γ to the rotation $z\to e^{2\pi i\varrho}z$. The conjugacy is then in fact complex analytically valid in a neighbourhood Δ of γ, and Δ belongs to a component of $N(g)$ of the type whose existence was asserted in Theorem 3. There are necessarily points of $J(g)$ near the essential singularities 0 and ∞ of g, so that the component is certainly multiply- and hence doubly-connected. This example is very closely related to the example $z\to z+(a/2\pi)\sin(2\pi z)+b$ for case II, which was described in a slightly different context by Herman [11].

6. Proof of Theorem 4

Here we use a method of construction of wandering domains first introduced by A. Eremenko and M. Lyubich [7]. It is based on an approximation theorem.

If F denotes a closed subset of $\mathbf{C}$ and $C_a(F)$ the functions which are continuous on F and analytic in $\mathring{F}$, we say that F is a Carleman set (for $\mathbf{C}$) if for any positive continuous functions ε on F and for any g in $C_a(F)$ there is an entire f such that $|g(z)-f(z)|<\varepsilon(z)$, $z\in F$. By Arakelyan's theorem $\hat{\mathbf{C}}\setminus F$ must be connected and locally connected at ∞. A. H. Nersesjan [13] showed that if we add the following we have a sufficient condition for F to be Carleman: for each compact K the union $W(K)$ of those components of $\mathring{F}$ which meet K is relatively compact (in $\mathbf{C}$).

It will follow that the set $F=B\cup\bigcup_{m=10}^{\infty}\{A_m\cup L_m\}$, is a Carleman set.

Now denote

$$L_m = \{z;\ \mathrm{Re}\, z = 4m\},\quad m \geqq 10,$$

$$A_m = \{z;\ |z-4m-2| \leqq 1\},\quad m \geqq 10,$$

$$B = \{z;\ |z+6| \leqq 1\},$$

and let δ, δ'_m be positive numbers so small that $|w-\pi i-\log 6|<\delta$ implies $|e^w+6|<1/2$, and $|w-\log(4m+2)|<\delta_m$ implies $|e^w-(4m+2)|<1/2$. Using the approximation lemma we find an entire function f such that

$$|f(z)-\pi i-\log 6| < \delta, \qquad z\in L_m,\quad m \geqq 10,$$

$$|f(z)-\pi i-\log 6| < \delta, \qquad z\in B,$$

and

$$|f(z)-\log(4m+6)| < \delta_{m+1}, \quad z\in A_m.$$

It follows that $g=e^f$, which is a function of class III(b), satisfies $g(A_m)\subset A_{m+1}$, so that $g^n\to\infty$ in each A_m, $m\geqq 10$. On the other hand g maps B into the smaller disc $|z+6|<1/2$, so that B contains an attractive fixed point ξ such that $g^n\to\xi$ in B, and $B\subset N(g)$.

Finally g maps L_m, $m\geqq 10$, into B so that $L_m\subset N(g)$ and further L_m belongs to a component of $N(g)$ different from the component G_m to which A_m belongs. Thus each G_m is a wandering component, mapping to G_{m+1} under $z\to g(z)$.

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