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THE GEOMETRY OF UNIFORM, QUASICIRCLE, AND CIRCLE DOMAINS

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Abstract. We introduce a class of collared domains. We solve the Kreisnormierungsproblem for collared quasicircle domains in $\mathbf{R}^2$. Then we give geometric conditions which ensure that a quasicircle domain be collared and in fact be uniform. Finally, we state our results in terms of hyperbolic geometry.

1. Introduction

Throughout this paper D denotes a domain in euclidean n-space $\mathbf{R}^n$. We are primarily interested in the plane $\mathbf{R}^2$, which we identify with $\mathbf{C}$, so except where explicitly noted otherwise we always have $n=2$.

We call D *uniform* if there is a constant c such that each pair of points z_1, z_2 in D can be joined by an arc α in D satisfying

(1) $$l(\alpha) \leqq c|z_1 - z_2|$$

and

(2) $$\min_{j=1,2} l(a_j) \leqq c\, d(z, \partial D) \quad \text{for all} \quad z \in \alpha.$$

Here $l(\alpha)$ is the euclidean length of α, $d(\cdot, \cdot)$ is euclidean distance, and α_1, α_2 are the components of $\alpha - \{z\}$. Condition (1) says that $l(\alpha)$ is comparable to $|z_1 - z_2|$; condition (2) says that away from z_1, z_2 α stays away from ∂D. Martio and Sarvas first introduced uniform domains [MS] and since then they have appeared quite naturally throughout function theory. Gehring has a recent survey article [G3] which lists many of the properties of uniform domains and has an extensive bibliography.

There are numerous alternative descriptions of uniform domains, however, with all of these it is often a non-trivial task to verify that a domain is uniform. In Theorem

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1.2 we give sufficient conditions for a domain to be uniform. These are geometric conditions in terms of the domain's boundary components and often can easily be verified. As a byproduct of the proof of Theorem 1.2 we solve a special case of the Kreisnormierungsproblem; see Theorem 2.6.

As the following result indicates, the complementary regions determined by a uniform domain in the plane must satisfy certain geometric conditions. It is not hard to see that the boundary components of a uniform domain must be linearly locally connected and hence are quasicircles [MS, Corollary 2.33], [G1, Theorem 4.1]. We call D a *K-quasicircle* domain if each of its boundary components is either a point or a K-quasicircle.

1.1. Theorem. *Suppose D is uniform. Then D is a K-quasicircle domain, $K=K(c)$, and there exists a constant $a=a(c)$ such that*

$$\min_{j=1,2} \operatorname{dia}(B_j) \leqq a\, d(B_1, B_2) \tag{3}$$

for any two components B_1, B_2 of $\overline{\mathbf{C}}-D$.

Condition (3) says that any two complementary regions determined by a uniform domain cannot be too close together unless one of them is small [G2, Theorem 2 and Lemma 3]; this condition can be derived directly from conditions (1) and (2).

Thus a simply connected proper subdomain of the plane is uniform if and only if it is a quasidisk, and in general a uniform domain is a quasicircle domain. In fact, a quasicircle domain is quasiconformally equivalent to a uniform domain if and only if it itself is uniform [GM, Corollary 3.17]. So whether or not a plane domain is uniform depends on two criteria: First, its boundary components must have the right shape; they must be quasicircles or points. Second, its boundary components must be in the right position with respect to each other; the internal moduli of the domain must satisfy appropriate conditions.

It is easily seen that condition (3) is not strong enough to guarantee that D be uniform, e.g., consider the quasicircle domain $\mathbf{C}-\mathbf{Z}$. However, we do have the following result.

1.2. Theorem. *Let B_0 be the component of $\overline{\mathbf{C}}-D$ which contains ∞. Suppose D is a K-quasicircle domain and there exist constants a, b such that*

$$\operatorname{dia}(B_1) \leqq a\, d(B_1, B_0) \tag{4}$$

and

$$\min_{j=1,2} d(B_j, B_0) \leqq b\, d(B_1, B_2) \tag{5}$$

for any two components B_1, B_2 of $\overline{\mathbf{C}}-D-B_0$. Then D is uniform with $c=c(a, b, K)$.

Condition (4) is a weak-version of (3) and thus is necessary for D to be uniform. Condition (5) says that $B_0 \neq \{\infty\}$ and that any two bounded complementary regions determined by D cannot be too close together unless they are close to B_0 (and hence

by (4) small). It is easy to see that there are uniform D which do not satisfy (5); e.g. $D=\{z\in\mathbf{C}\colon \operatorname{Re}(z)>0\}-\{2^n, 2^n+1\colon n=1, 2, \ldots\}$.

We prove Theorem 1.2 in Section 3. It turns out that we need only consider the special case of *circle* domains which have points and circles for boundary components. The general case follows from a result on quasiconformal mappings, Theorem 2.6, together with the quasi-invariance of the hypotheses.

In Section 2 we state and prove Theorem 2.6. In particular, this theorem gives sufficient conditions for the existence of a conformal mapping of a quasicircle domain onto a circle domain.

In Section 4 we conclude by stating our results in terms of hyperbolic geometry.

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2. Extension of quasiconformal mappings

Our ultimate result here, Theorem 2.6, is that certain quasicircle domains can be mapped conformally onto circle domains via quasiconformal self-homeomorphisms of $\overline{\mathbf{C}}$.

2.1. *Notation.* We use the following standard notation. Let Γ be a nonempty family of curves. We call ϱ *admissible* for Γ, denoted $\varrho\in\operatorname{adm}(\Gamma)$, if ϱ is non-negative and Borel measurable and if $\int_\gamma \varrho(z)|dz|\geqq 1$ for each locally rectifiable $\gamma\in\Gamma$. The *modulus* of Γ is $M(\Gamma)=\inf_\varrho \iint \varrho(z)^2\,dx\,dy$, where the infimum is taken over all $\varrho\in\operatorname{adm}(\Gamma)$.

A *ring* is a doubly connected domain; thus every ring has exactly two boundary components. The *modulus* of a ring R is defined by

$$\operatorname{mod}(R) = 2\pi M(\Gamma_1) = 2\pi/M(\Gamma_2),$$

where Γ_1, Γ_2 are the families of curves in R which respectively separate, join the boundaries of R. The *extremal ring* of Grötzsch is $\{z\in\mathbf{C}\colon |z|<1\}-[0, r]$ and its modulus is denoted by $\mu(r)$, $0<r<1$. See [LV].

Two lemmas, each of independent interest, are required for the proof of Theorem 2.6. For each of these lemmas we use the following notation: R is a ring with boundary components C_0 and C, C is a Jordan curve, and D and D_0 are respectively the components of $\overline{\mathbf{C}}-C$ and $\overline{\mathbf{C}}-C_0$ containing R.

2.2. Lemma. *Let α and β be disjoint arcs in C and let Γ_R, Γ_D be the families of curves in R, D respectively which join α and β. Then*

$$M(\Gamma_R) \leqq M(\Gamma_D) \leqq \varphi(\operatorname{mod} R)\,M(\Gamma_R),$$

where $\varphi(t)=[1+\exp(\pi^2/2t)]/2$.

Proof. As $\Gamma_R \subset \Gamma_D$, $M(\Gamma_R) \leqq M(\Gamma_D)$ and we need only prove the second inequality.

Map D conformally onto a rectangle D_1 so that α and β are sent onto the vertical sides $[-a-i, -a+i]$ and $[a-i, a+i]$. Let R_1, C_1, Γ_{R_1}, Γ_{D_1} be the images of R, C_0, Γ_R, Γ_D respectively. Note that $M(\Gamma_D)=M(\Gamma_{D_1})=1/a$.

Let $[iy_1, iy_2] \subset [-i, i]$ be the projection of C_1 onto the imaginary axis. For each $y \in (-1,1)-[y_1 y_2]$, $\gamma_y=\{x+iy: -a<x<a\} \in \Gamma_{R_1}$, so for each $\varrho \in \mathrm{adm}(\Gamma_{R_1})$

$$1 \leqq \left(\int_{\gamma_y} \varrho(z)\,|dz|\right)^2 = \left(\int_{-a}^{a} \varrho(x+iy)\,dx\right)^2 \leqq 2a \int_{-a}^{a} \varrho(x+iy)^2\,dx,$$

whence

$$\iint \varrho(z)^2\,dx\,dy \geqq \left(\int_{-1}^{y_1} + \int_{y_2}^{1}\right) dy/(2a) = [2-(y_2-y_1)]/(2a).$$

Taking the infimum over all $\varrho \in \mathrm{adm}\,(\Gamma_{R_1})$ yields $M(\Gamma_{R_1}) \geqq [1-(y_2-y_1)/2]\,M(\Gamma_{D_1})$, whence

$$M(\Gamma_D) \leqq cM(\Gamma_R),\ c = 1/[1-(y_2-y_1)/2].$$

Let R_2 be the Steiner symmetrization of R_1 with respect to the imaginary axis [H], let R_3 be the ring with boundary components $C_3=[iy_1, iy_2]$ and $\{x \pm i: x \in \bar{\mathbf{R}}\}$, and let $R_4=\psi \circ \varphi(R_3)$ where

$$\varphi(z) = \tanh\left(\frac{\pi z}{4}\right),\quad \psi(w) = \frac{w-iv_1}{1+iv_1 w},\quad iv_j = \varphi(iy_j) = i\tan\left(\frac{\pi y_j}{4}\right)\quad (j=1,2).$$

Then

$$m = \mathrm{mod}\,(R) = \mathrm{mod}\,(R_1) \leqq \mathrm{mod}\,(R_2) \leqq \mathrm{mod}\,(R_3) = \mathrm{mod}\,(R_4) = \mu(t),$$

where $it=\psi(iv_2)$. Next,

$$\mu^{-1}(m) \geqq t = \frac{\tan s_2 - \tan s_1}{1-\tan s_1 \tan s_2} \geqq \sin(s_2-s_1),$$

where $s_j=(\pi/4)y_j$ $(j=1,2)$. Thus

$$y = (y_2-y_1)/2 \leqq \sin(\pi y/2) \leqq \mu^{-1}(m),$$

so [LV, (2.9), p. 61]

$$\log\frac{1+y}{1-y} \leqq \mu\left(\frac{1-y}{1+y}\right) = \frac{\pi^2}{2\mu(y)} \leqq \frac{\pi^2}{2m}.$$

Finally,

$$c = 1/(1-y) = \left(1+\frac{1+y}{1-y}\right)\Big/2 \leqq [1+\exp(\pi^2/2m)]/2.$$

2.3. Lemma. *Suppose that C is a K-quasicircle. Let f be a K_0-quasiconformal mapping of R, and suppose that the boundary component C' of $R'=f(R)$ which corresponds to C under f is a K'-quasicircle. Then f extends to a quasiconformal mapping of D_0 with maximal dilatation bounded above by a constant which depends only on K_0, K, K' and $M(R)=2\pi/\mathrm{mod}\,(R)$.*

Proof. Let $g, g'\colon \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ be K, K'-quasiconformal mappings of C, C' onto the real axis with $g(R)$, $g'(R')$ lying in the upper half plane. To get the desired extension of f it suffices to extend $g' \circ f \circ g^{-1}$ to the lower half plane. Thus we may assume that R and R' are rings in the upper half plane $D = D'$ each having the real axis as a boundary component and that $f\colon R \to R'$ is K_0-quasiconformal with $f\colon \partial D \to \partial D'$. To extend f to the lower half plane we need only show [LV, Theorem II.6.1] that there exists a constant $a > 0$, which depends only on K_0 and $M(R)$, such that for all $x \in \mathbf{R}$ and for all $t > 0$ we have

$$1/a \leqq M(\Gamma_{D'}) \leqq a, \tag{6}$$

where $\Gamma_{D'}$ is the family of curves in D' joining the intervals $[f(x-t), f(x)]$, $[f(x+t), f(\infty)]$.

So fix $x \in \mathbf{R}$ and $t > 0$. Let α and β be the intervals $[f(x-t), f(x)]$ and $[f(x+t), f(\infty)]$. Then let $\Gamma_{R'}$ and $\Gamma_{D'}$ be the families of curves in R' and D' respectively which join α, β and let Γ_R and Γ_D be the corresponding families of curves in R and D. By Lemma 2.2 $1 = M(\Gamma_D) \leqq \varphi(\mathrm{mod}\, R)\, M(\Gamma_R)$, and thus, letting $a = K_0 \varphi(2\pi/M(R))$, we get

$$1/a \leqq (1/K_0) M(\Gamma_R) \leqq M(\Gamma_{R'}) \leqq M(\Gamma_{D'}),$$

which gives the first inequality in (6). The second inequality follows by symmetry.

2.4. Corollary. *If the boundary components of a ring are quasicircles, then the ring is a quasiannulus, i.e., the image of an annulus under a quasiconformal self-homeomorphism of* $\overline{\mathbf{C}}$.

Proof. Let R be a ring with boundary components the K-quasicircles C_0 and C. Let D be the component of $\overline{\mathbf{C}} - C$ containing R. Choose a conformal mapping f of R onto an annulus A. Using Lemma 2.3 we extend f to a K'-quasiconformal mapping g of D where $K' = K'(K, M(R))$. We extend g by reflection to a K^2K'-quasiconformal mapping h of $\overline{\mathbf{C}}$. Then $R = h^{-1}(A)$ is a K^2K'-quasiannulus.

2.5. *Collars.* A component B of $\overline{\mathbf{C}} - D$ is said to have a *collar* of modulus m in D if there exists a ring in D with modulus at least m and having ∂B as a boundary component. Let B_0 be a fixed component of $\overline{\mathbf{C}} - D$. We say that D is *m-collared* relative to B_0 if each component B of $\overline{\mathbf{C}} - D - B_0$ has a collar of modulus m in D such that the collars are pairwise disjoint.

2.6. Theorem. *Let D be a K-quasicircle domain. Let B_0 be a component of $\overline{\mathbf{C}} - D$ with ∂B_0 a quasicircle. Suppose that D is m-collared relative to B_0 with $m > 0$. Then there exists a K'-quasiconformal self-homeomorphism of $\overline{\mathbf{C}}$ which maps D conformally onto a circle domain, where $K' = K'(K, m)$.*

Proof. We assume that B_0 is the lower half-plane $\overline{\mathbf{C}} - U$ and that $i \in D$. Let $B_1, B_2, \ldots$ denote the other components of $\overline{\mathbf{C}} - D$. For $n = 1, 2, \ldots$ let $D_n =$

$U-\bigcup_{j=1}^{n} B_j$ and choose a conformal mapping $f_n\colon D_n \to D_n' = U-\bigcup_{j=1}^{n} B_j^{(n)}$ of D_n onto a circle domain D_n'. Here each $B_j^{(n)}$ is a closed disk or a point and ∂U, ∂B_j, i correspond to ∂U, $\partial B_j^{(n)}$, i. See Goluzin's text [G, Theorem 2, p. 237] or the original work of Koebe for the existence of the mappings f_n.

Next, for $j=1, 2, \ldots, n$ let $R_j^{(n)}=f_n(R_j)$, where R_j is the collar associated with B_j. By Lemma 2.3 there exists a K'-quasiconformal extension $F_n^{(j)}\colon R_j \cup B_j \to R_j^{(n)} \cup B_j^{(n)}$ of $f_n|R_j$, $K'=K'(K, m)$. Then

$$F_n = \begin{cases} F_n^{(j)} & \text{in} \quad R_j \cup B_j \\ f_n & \text{elsewhere} \end{cases}$$

defines a K'-quasiconformal self-homeomorphism of U with $F_n|D_n=f_n$.

Thus we have a sequence of K'-quasiconformal mappings $F_n\colon U \to U$ with $F_n|D$ conformal, $F_n(i)=i$, $F_n\colon \partial U \to \partial U$. By normality [LV, Theorem II.5.1] there exists a subsequence, which we continue to call $\{F_n\}$, which converges uniformly on compact subsets of U to some function F. Since $F(i)=i$, F must be a K'-quasiconformal self-homeomorphism of U [LV, Theorem II.5.5].

We extend F to $\overline{\mathbf{C}}$ by reflection. It remains to show that $F(D)$ is a circle domain. Write

$$F(D) = D' = U-\bigcup_{j=1}^{\infty} B_j', \; B_j' = F(B_j).$$

Fix a B_j' and assume it is not a point. Fix $w_1, w_2, w_3, w_4 \in \partial B_j'$. For $k=1, 2, 3, 4$ choose $z_k \in \partial B_j$ so that $w_k=F(z_k)$ and let $w_k^{(n)}=F_n(z_k)$. Since $\partial B_j^{(n)}$ is a circle, it follows that the cross ratio

$$[w_1, w_2, w_3, w_4] = \lim_{n\to\infty} [w_1^{(n)}, w_2^{(n)}, w_3^{(n)}, w_4^{(n)}]$$

must be real. Thus $\partial B_j'$ is a circle, and D' is a circle domain.

3. Proof of Theorem 1.2

The simplest geometric situation to consider is when D is a circle domain. We prove Theorem 1.2 for such domains and then appeal to Theorem 2.6 for the general case. The success of this strategy depends on showing that Theorem 2.6 applies and that the hypotheses of Theorem 1.2 are quasi-invariant with respect to quasiconformal self-homeomorphisms of $\mathbf{C}$.

The following result is useful [GO, Lemma 3].

3.1. Lemma. *There exists a constant $\beta=\beta(n, K)$ such that for any K-quasiconformal $f\colon \mathbf{R}^n \to \mathbf{R}^n$*

$$\frac{|f(x_1)-f(x_0)|}{|f(x_1)-f(x_2)|}+1 \leqq \beta\left(\frac{|x_1-x_0|}{|x_1-x_2|}+1\right)^{1/\alpha}, \quad \alpha = K^{1/(1-n)}$$

for all $x_0, x_1, x_2 \in \mathbf{R}^n$.

The next two results show that the hypotheses of Theorem 1.2 are quasi-invariant with respect to quasiconformal self-homeomorphisms of $\mathbf{R}^n$.

3.2. Lemma. *Let* $B_0, B_1 \subset \mathbf{R}^n$ *be closed. Suppose condition* (4) *holds. Then there exists a constant* $b=b(a, n, K)$ *such that*

$$\operatorname{dia}(fB_1) \leqq b\, d(fB_1, fB_0)$$

for any K-quasiconformal $f: \mathbf{R}^n \to \mathbf{R}^n$.

Proof. Choose points $y_1=f(x_1)$, $y_2=f(x_2)\in f(B_1)$ and $y_0=f(x_0)\in f(B_0)$ so that $|y_1-y_0|=d(fB_1, fB_0)$, $2|y_1-y_2|\geqq d(fB_1)$. Then by Lemma 3.1

$$\frac{\operatorname{dia}(fB_1)}{d(fB_1, fB_2)} \leqq 2\frac{|y_1-y_2|}{|y_1-y_0|} \leqq 2\left[\beta\left(\frac{|x_1-x_2|}{|x_1-x_0|}+1\right)^{1/\alpha}-1\right] \leqq 2[\beta(a+1)^{1/\alpha}-1].$$

3.3. Lemma. *Let* $B_0, B_1, B_2 \subset \mathbf{R}^n$ *be closed. Suppose conditions* (4) *and* (5) *hold. Then there exists a constant* $c=c(a, b, n, K)$ *such that*

$$\min_{j=1,2} d(fB_j, fB_0) \leqq c\, d(fB_1, fB_2)$$

for any K-quasiconformal $f: \mathbf{R}^n \to \mathbf{R}^n$.

Proof. For $j=1, 2$ choose points $y_j=f(x_j)\in f(B_j)$ so that $d(fB_1, fB_2)=|y_1-y_2|$. Assume that $d(B_1, B_0)\leqq d(B_2, B_0)$. Choose $x_0\in B_0$ with $d(B_1, B_0)=d(B_1, x_0)$. Then using (4) and (5) we obtain

$$|x_1-x_0| \leqq \operatorname{dia}(B_1)+d(B_1, B_0) \leqq (a+1)d(B_1, B_0) \leqq b(a+1)d(B_1, B_2).$$

Thus by Lemma 3.1

$$\frac{\min_{j=1,2} d(fB_j, fB_0)}{d(fB_1, fB_2)} \leqq \frac{|y_1-f(x_0)|}{|y_1-y_2|} \leqq \beta\left(\frac{|x_1-x_0|}{|x_1-x_2|}+1\right)^{1/\alpha}-1 \leqq \beta\big(b(a+1)+1\big)^{1/\alpha}-1.$$

The corollary of the next lemma ensures that Theorem 2.6 can be applied; the lemma is also needed in the proof of Theorem 1.2 for a circle domain.

3.4. Lemma. *Let* $B_0, B_1, B_2 \subset \mathbf{R}^n$ *be closed. Suppose conditions* (4) *and* (5) *hold. Then*

$$\max_{j=1,2} d(B_j, B_0) \leqq (1+ab+b)\, d(B_1, B_2).$$

Proof. Suppose that $d(B_1, B_0)\leqq d(B_2, B_0)$. Then by (4), (5) and the triangle inequality

$$d(B_2, B_0) \leqq d(B_1, B_0)+\operatorname{dia}(B_1)+d(B_1, B_2)$$

$$\leqq (1+a)d(B_1, B_0)+d(B_1, B_2) \leqq \big((1+a)b+1\big)\, d(B_1, B_2).$$

3.5. Corollary. *Assume the hypotheses of Theorem* 1.2. *Then* D *is m-collared relative to* B_0, *where* $m=m(a, b)>0$.

Proof. To each component B of $\overline{\mathbf{C}}-D-B_0$ we associate the ring

$$R = \{z\in\overline{\mathbf{C}}-B\colon d(z, B) < \varepsilon d(B, B_0)\},$$

where $2\varepsilon=1/(1+ab+b)$. By Lemma 3.4 these rings are contained in D and are pairwise disjoint. We claim that $\operatorname{mod}(R)\geqq 2/(1+a/\varepsilon)^2$.

Let $r=\operatorname{dia}(B)$, $s=\varepsilon d(B, B_0)$. Note that $r/s\leqq a/\varepsilon$. Fix $z_0\in B$. Define

$$\varrho(z) = \begin{cases} 1/s & \text{in } B(z_0, r+s) \\ 0 & \text{elsewhere.} \end{cases}$$

Then $\varrho\in\operatorname{adm}(\Gamma)$ where Γ is the family of curves in R joining its boundary components. Thus $M(\Gamma)\leqq\iint\varrho^2=\pi(1+r/s)^2$, whence $\operatorname{mod}(R)=2\pi/M(\Gamma)\geqq 2/(1+a/\varepsilon)^2$.

We require the following two results about annuli whose proofs are left for the reader. We remark that the necessary curves can be obtained by using straight line segments together with arcs of circles.

3.6. Lemma. *An annulus A is uniform with $c=O(1), O((1-e^{-m})^{-1})$ as $m=\operatorname{mod}(A)\to\infty, 0$ respectively.*

3.7. Lemma. *Let A be an annulus with inner, outer boundary circles C_1, C_2 respectively. Then points $z_1\in A$, $z_2\in C_2$ can be joined by an arc σ in $A\cup C_2$ such that for every $z\in\sigma$*

$$l(\sigma(z_1, z)) \leqq \min\{c_0|z_1-z|,\ b_0 d(z, C_1)\},$$

where $c_0=2(1+\pi)$, $b_0=O(1), O(e^m)$ as $m=\operatorname{mod}(A)\to\infty, 0$ respectively.

Here, and below in 3.8, we use the notation $\gamma(z, w)$ to denote the subarc between z and w of an arc γ.

3.8. *Proof of Theorem 1.2.* By appealing to Theorem 2.6 and the quasi-invariance of our hypothesis with respect to quasiconformal self-homeomorphisms of $\mathbf{C}$, we may assume that $B_0=\overline{\mathbf{C}}-U$ and that $D=U-\bigcup_{j=1}^{\infty}B_j$. Here each B_j is a closed euclidean disk or a point in U and either U is the open upper half-plane if $\infty\in\partial B_0$ or U is the open unit disk if ∞ is an interior point of B_0.

Set $r_j=\operatorname{dia}(B_j)$, $d_j=d(B_j, B_0)$ and let $A_j=\{z\in\mathbf{C}-B_j\colon d(z, B_j)<\varepsilon d_j\}$ where $4\varepsilon=1/(1+ab+b)$. Then as in the proof of Corollary 3.5 we find that we have disjoint annuli A_j in D with $\partial A_j=\partial B_j\cup C_j$ and $\operatorname{mod}(A_j)\geqq m=m(a, b)>0$. Also, we see that if $z\in\overline{A}_j$, then $d(z, \partial D)=d(z, B_j)$. Moreover, if $z\in D$ and $d(z, \partial D)=d(z, B_0)$, then $z\notin\overline{A}_j$.

Fix $z_1, z_2\in D$. We must exhibit an arc α in D which joins z_1, z_2 and satisfies (1), (2). By Lemma 3.6 we may assume that z_1, z_2 do not both lie in some annulus A_j. By relabeling, if necessary, we may assume that $d(z_k, \partial D-B_0)=d(z_k, B_k)$ for $k=1, 2$. Let β be the subarc of the circle through z_1, z_2 which is orthogonal to ∂U.

If $z_k\in A_k$, choose $w_k\in\beta\cap C_k$ and let σ_k be the curve given by Lemma 3.7 which joins z_k and w_k in $A_k\cup C_k$; otherwise set $w_k=z_k$ and $\sigma_k=\{z_k\}$. Next, let J denote

the set of all j for which β meets C_j twice, and for each $j\in J$: let ζ_{1j}, ζ_{2j} be the points of $\beta\cap C_j$ with ζ_{kj} the point first encountered as β is traversed from z_k, let $\beta_j=\beta(\zeta_{1j}, \zeta_{2j})$, and let γ_j be the component of $C_j-\{\zeta_{1j}, \zeta_{2j}\}$ with minimal euclidean length. Finally, let

$$\alpha = \sigma_1\cup\left[(\beta(w_1, w_2)-\bigcup_J \beta_j)\cup\bigcup_J \gamma_j\right]\cup\sigma_2.$$

We show that α satisfies (1) and (2).

By Lemma 3.7 for $k=1, 2$

$$l(\sigma_k) \leqq c_0|z_k-w_k| \leqq c_0 l(\beta(z_k, w_k)),$$

and by geometry for all $j\in J$

$$l(\gamma_j) \leqq \pi/2\ |\zeta_{1j}-\zeta_{2j}| \leqq \pi/2\ l(\beta_j).$$

Since $c_0\geqq\pi/2$, we conclude that

$$l(\alpha) \leqq c_0 l(\beta) \leqq c_1|z_1-z_2|,$$

where $c_1=c_0\pi/2$. Thus condition (1) is satisfied.

We note that the same reasoning shows that

$$l(\alpha(z, w)) \leqq c_0 l(\beta(z, w)) \quad \text{if} \quad z, w\in\alpha\cap\beta. \tag{7}$$

Fix $z\in\alpha$. First, suppose $d(z, \partial D)=d(z, \partial U)$. Then $z\in\beta$ and using (7)

$$\min_{j=1,2} l(\alpha_j) \leqq c_1 d(z, \partial D),$$

so in this case (2) is satisfied. Next, suppose $d(z, \partial D)=d(z, B_k)$ for some $k=1, 2, \ldots$. We consider the possibilities $z\in\bar{A}_k$, $z\notin\bar{A}_k$ separately.

Suppose $z\in\bar{A}_k$. Then $k\in\{1, 2\}\cup J$. If $k\in\{1, 2\}$, then $\alpha_k=\sigma_k(z_k, z)$ and by Lemma 3.7

$$l(\alpha_k) \leqq c_2 d(z, B_k) = c_2 d(z, \partial D),$$

where $c_2=c_2(a, b)$. So (2) holds. On the other hand, suppose $k\in J-\{1, 2\}$. Then $z\in\gamma_k$ and $d(z, \partial D)=d(C_k, B_k)=\varepsilon d_k$. Now for any $\zeta\in C_k$

$$d(\zeta, B_0) \leqq \varepsilon d_k+r_k+d_k \leqq (1+a+\varepsilon)d_k,$$

whence by (7)

$$\min_{j=1,2} l(\alpha(z_j, \zeta_{jk})) \leqq c_3 d(z, \partial D),$$

where $c_3=c_1(1+a+\varepsilon)/\varepsilon=c_3(a, b)$. Since $l(\gamma_k)\leqq\pi r_k/2\leqq(\pi a/2\varepsilon)d(z, \partial D)$, (2) holds.

Finally, suppose $z\notin\bar{A}_k$. Then $z\in\beta$ and $d(z, \partial D)=d(z, B_k)>\varepsilon d_k$. If

$$d(z, B_0) \leqq 2h, \quad h = r_k+d_k,$$

then by (7)

$$\min_{j=1,2} l(\alpha_j) \leqq c_0 \min_{j=1,2} l(\beta(z_j, z)) \leqq 2c_1 h \leqq 2c_1(1+a)/\varepsilon\ d(z, \partial D),$$

while if

$$nh \leqq d(z, B_0) \leqq (n+1)h \quad \text{for some} \quad n \geqq 2,$$

then since $d(z, \partial D)=d(z, B_k)\geqq d(z, B_0)-h\geqq(n-1)h$, we conclude that

$$\min_{j=1,2} l(\alpha_j) \leqq c_1 d(z, B_0) \leqq c_1(n+1)h \leqq 3c_1 d(z, \partial D).$$

Thus in all cases (2) holds, so D is uniform.

3.9. Corollary. *Let $D'=D-E$ where D is uniform. Suppose there exists a positive constant d such that*

$$\min_{j=1,2} d(\zeta_j, \partial D) \leqq d|\zeta_1-\zeta_2| \quad \textit{for all distinct points} \quad \zeta_1, \zeta_2 \in E.$$

Then D' is uniform with $c'=c'(c, d)$.

Proof. Fix $z_1, z_2 \in D$. By [GO, Theorem 5] there exists a K-quasidisk $D_0 \subset D$ with $z_1, z_2 \in \overline{D}_0$, $K=K(c)$. Since $d(\zeta, \partial D_0)\leqq d(\zeta, \partial D)$ for $\zeta \in D_0$, it is easy to see that conditions (4) and (5) hold for the quasicircle domain $D_0'=D_0-E$. Hence there exists an arc α in $D_0' \subset D'$ joining z_1, z_2 with $l(\alpha)\leqq c'|z_1-z_2|$, and $\min_{j=1,2} l(\alpha_j)\leqq c' d(z, \partial D_0')\leqq c' d(z, \partial D')$ for all $z \in \alpha$, where $c'=c'(c, d)$.

4. Hyperbolic geometry

The euclidean conditions given in Theorems 1.1, 1.2 and in Corollary 3.9 can be stated in terms of the hyperbolic, or Poincaré, metric. The *hyperbolic distance* between two points z_1, z_2 in a domain $D \subset \overline{\mathbf{C}}$ with more than two boundary points is given by

$$h_D(z_1, z_2) = h_B(\pi^{-1}(z_1), \pi^{-1}(z_2))$$

where π: $B \to D$ is a holomorphic projection, $B=\{w \in \mathbf{C}: |w|<1\}$ and $h_B(w_1, w_2)=\log(|1-w_1\overline{w}_2|+|w_1-\overline{w}_2|)/(|1-w_1\overline{w}_2|-|w_1-\overline{w}_2|)$. If D has at most two boundary points, then $h_D \equiv 0$.

4.1. Theorem. *Suppose D is uniform. Then there exists a constant $a=a(c)$ such that*

$$h-\operatorname{dia}(B_1) \leqq a \tag{8}$$

for any two components B_1, B_2 of $\overline{\mathbf{C}}-D$, where $h=h_{\overline{\mathbf{C}}-B_2}$.

Proof. Conditions (3) and (8) are equivalent.

4.2. Theorem. *Let B_0 be any nondegenerate component of $\overline{\mathbf{C}}-D$. Suppose D is a K-quasicircle domain and there exist constants a, b such that*

$$h-\operatorname{dia}(B_1) \leqq a < +\infty \tag{9}$$

and

$$h(B_1, B_2) \geqq b > 0 \tag{10}$$

for any two components B_1, B_2 of $\overline{\mathbf{C}}-D-B_0$, where $h=h_{\overline{\mathbf{C}}-B_0}$. Then D is uniform with $c=c(a, b, K)$.

Proof. Conditions (4), (5) and (9), (10) are equivalent.

4.3. Corollary. *Let* $D'=D-E$ *where* D *is uniform. Suppose there exists a constant* d *such that*

$$h_D(\zeta_1, \zeta_2) \geqq d > 0 \quad \textit{for all distinct points} \quad \zeta_1, \zeta_2 \in E.$$

Then D' *is uniform with* $c'=c'(c, d)$.

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