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ON THE MAPPING WITH COMPLEX DILATATION $ke^{i\theta}$

EDGAR REICH

Abstract. By considering the explicit form of the quasiconformal mapping $F_k(z)$ of the complex plane with complex dilatation $ke^{i\theta}$ it is possible to determine a sharp version for a variational lemma for extremal quasiconformal mappings of the unit disk. Additionally, we are led to an explicit elementary solution of a geometric extremal problem involving mappings of conic sections, which, in the case of ellipses, is equivalent to Teichmüller's shift theorem, and, in the case of parabolas, reduces to a known example of non-unique extremality.

1. The mapping $F_k(z)$

Let k be a constant, $-1<k<1$. Suppose $f(z)$ is a qc (quasiconformal) mapping of the region Ω of the complex plane $\mathbf{C}$ satisfying the Beltrami equation

$$\varkappa_f(z) = \frac{f_{\bar z}}{f_z} = kz/|z| = ke^{i\theta} \quad (z = re^{i\theta}). \tag{1.1}$$

The complex dilatation $\varkappa_f(z)$ of (1.1) is of "Teichmüller" form,

$$\varkappa_f(z) = k\overline{\varphi(z)}/|\varphi(z)|,$$

with

$$\varphi(z) = \frac{1}{z}. \tag{1.2}$$

If $\Phi(z)$ is a local determination of the integral

$$\Phi(z) = \tfrac{1}{2}\int \sqrt{\varphi(z)}\,dz = \sqrt{z},$$

then

$$\zeta = g(z) = \Phi(z) + k\overline{\Phi(z)} = \sqrt{z} + k\sqrt{\bar z}$$

is a local solution of (1.1). The general qc solution, $w=f(z)$, of (1.1) in a particular region Ω therefore has the form $\Lambda(g(z))$, where $\Lambda(\zeta)$ is analytic when $\zeta\in\operatorname{range} g$, and Λ is chosen so that $\Lambda(g(z))$ is single-valued homeomorphic in Ω. In particular,

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choosing

$$A(\zeta) = \frac{\zeta^2}{1-k^2},$$

and denoting $A(g(z))$ by $F_k(z)$, we obtain

$$w = F_k(z) = \frac{1}{1-k^2}(z+2k|z|+k^2\bar{z}). \tag{1.3}$$

It is easily verified that $F_k(z)$ is one-to-one and maps **C** onto itself. Therefore, up to a linear transformation of the w plane, $F_k(z)$ is the unique qc map of **C** with complex dilatation (1.1). For the real and imaginary parts, $w=u+iv$, of the mapping (1.3) we have

$$u = \frac{1}{1-k^2}\left[(1+k^2)x+2k\sqrt{x^2+y^2}\right], \tag{1.4}$$

and

$$v = y. \tag{1.5}$$

We note that $F_b \circ F_a = F_c$, with $c=(a+b)/(1+ab)$. In particular,

$$F_k^{-1}(w) = F_{-k}(w) = \frac{1}{1-k^2}(w-2k|w|+k^2\bar{w}). \tag{1.6}$$

2. Extremal qc mappings of the disk and the class $\mathcal{N}$

Let Ω be a proper subregion of the plane, and let $Q=Q_\Omega$ denote the class of qc mappings of Ω. For $f\in Q$, let

$$k_f = \operatorname{ess\,sup}\{|\varkappa_f(z)|: z\in\Omega\},$$

and, for $g\in Q$, let

$$Q[g] = \{f\in Q: f|_{\partial\Omega} = g|_{\partial\Omega}\}.$$

The basic extremal problem is to determine

$$k^*[g] = \inf\{k_f: f\in Q[g]\} \tag{2.1}$$

and corresponding extremal mapping(s) $f^*\in Q[g]$.

Ahlfors' [1] class $\mathcal{N}(\Omega)$ which plays a basic role in the variational approach [2, 3] to the problem (2.1) is defined as follows. First, let $\mathcal{B}(\Omega)$ denote the class of functions $\varphi(z)$ holomorphic in Ω and belonging to $\mathcal{L}^1(\Omega)$:

$$\|\varphi\| = \iint_\Omega \varphi(z)\,dx\,dy < \infty.$$

Then,

$$\mathcal{N}(\Omega) = \left\{\nu\in\mathcal{L}^\infty(\Omega): \iint_\Omega \nu(z)\varphi(z)\,dx\,dy = 0 \text{ for all } \varphi\in\mathcal{B}(\Omega)\right\}.$$

The fundamental idea that is exploited in the variational approach is that if $f \in Q_\Omega$, and $\varkappa_f \in \mathcal{N}(\Omega)$, then $f(z)$ is close to the identity on $\partial\Omega$. Our purpose here is to derive a best possible quantitative version of this fact when Ω is *simply connected.* To this end, let

$$\chi(t) = \sup\{k^*[f]\colon f \in Q_\Omega, \varkappa_f \in \mathcal{N}(\Omega),\ k_f \leqq t\}, \quad 0 \leqq t < 1. \tag{2.2}$$

We observe that if $z \leftrightarrow \tilde{z}$ is a conformal map between Ω and $\tilde{\Omega}$, then the rules

$$\varphi(z)\,dz^2 = \tilde{\varphi}(\tilde{z})\,d\tilde{z}^2, \quad \nu(z)\frac{\overline{dz}}{dz} = \tilde{\nu}(\tilde{z})\frac{\overline{d\tilde{z}}}{d\tilde{z}},$$

establish a correspondence between $\mathcal{N}(\Omega)$ and $\mathcal{N}(\tilde{\Omega})$, which leaves $\chi(t)$ unchanged. Thus $\chi(t)$ depends only on the value of t, not on the choice of Ω. Our result is as follows.[1]

Theorem 2.1. *Suppose Ω is simply connected. Then*

$$\chi(t) = t^2 + O(t^3), \quad \textit{as} \quad t \to 0.$$

Proof. In view of the observation regarding invariance, we can assume that Ω is the unit disk $U = \{|z| < 1\}$.

An upper bound for $\chi(t)$ is obtained using the following known inequality [4]:

$$\frac{k^*[f]}{1-k^*[f]} \leqq \frac{k_f^2}{1-k_f^2} + I, \tag{2.3}$$

where

$$I = \sup\left\{\left|\iint_U \varkappa_f(z)\varphi(z)\frac{dx\,dy}{1-|\varkappa_f(z)|^2}\right|\colon \varphi \in \mathcal{B}(U),\ \|\varphi\| \leqq 1\right\}.$$

Since, $\varkappa_f \in \mathcal{N}(U)$,

$$\iint_U \varkappa_f \varphi \frac{dx\,dy}{1-|\varkappa_f|^2} = \iint_U \varkappa_f |\varkappa_f|^2 \varphi \frac{dx\,dy}{1-|\varkappa_f|^2}.$$

Therefore, $I \leqq k_f^3/(1-k_f^2)$. Hence, by (2.3),

$$k^*[f] \leqq (k_f^2 + k_f^3)/(1+k_f^3) \leqq k_f^2 + k_f^3.$$

We therefore conclude [5] that

$$\chi(t) \leqq t^2 + t^3. \tag{2.4}$$

Consider, now, the mapping

$$F(z) = F_t(z),$$

where $F_k(z)$ as given by (1.2), is restricted to $\{|z| \leqq 1\}$. On ∂U, $F(z)$ agrees with the affine mapping

$$f_0(z) = \frac{1}{1-t^2}(z + 2t + t^2\bar{z}),$$

[1] A very closely related result is obtained by Lehto in [3].

which has complex dilatation

$$\varkappa_{f_0}(z) = t^2.$$

Since the affine mapping of a region of finite area is extremal in its class, it follows that

$$k^*[F] = k_{f_0} = t^2.$$

On the other hand, since

$$\int_0^{2\pi} e^{i\theta}\varphi(re^{i\theta})\,d\theta = 0, \quad 0 \leqq r < 1,$$

for every function $\varphi(z)$ holomorphic in U, we have $\varkappa_{f_0} \in \mathcal{N}(U)$. Consequently,

$$\chi(t) \geqq t^2. \tag{2.5}$$

3. Mapping of conic sections

By a conic section we shall understand a region bounded by a circle, ellipse, parabola, or by a single branch of a hyperbola, or an angular region bounded by two rays.

Theorem 3.1. *$w = F_k(z)$ maps every conic section with focus at the origin onto a conic section with focus at the origin. If the conic section Ω is a disk, ellipse, or parabola, then $F_k(z)$ is the unique extremal qc mapping among the class of mappings $g(z)$ that satisfy*

$$g|_{\partial\Omega} = F_k|_{\partial\Omega}, \quad g(0) = 0. \tag{3.1}$$

Proof. Let $\Gamma = \partial\Omega$. We can assume that the major axis of the conic section lies on the x axis. Γ satisfies an equation

$$|z| = \lambda x + p, \quad (z = x + iy \in \Gamma), \tag{3.2}$$

where λ is real and $p \geqq 0$. If $-1 < \lambda < 1$, then Γ is an ellipse with eccentricity $|\lambda|$ and foci at the points $z=0$ and $z = 2p\lambda/(1-\lambda^2)$; if $\lambda = \pm 1$, and $p > 0$, then Γ is a parabola; if $\lambda^2 > 1$, then Γ is a branch of a hyperbola or the union of two rays. Substituting (3.2) into (1.3), we find that $F_k(z)$ is the restriction to Γ of the following affine mapping

$$w = A(z) = \frac{1}{1-k^2}[(1+k\lambda)z + (k+\lambda)k\bar{z} + 2kp]. \tag{3.3}$$

Solving (3.3) for z in terms of w,

$$z = A^{-1}(w) = \frac{1}{1+2k\lambda+k^2}[(1+k\lambda)w - (k+\lambda)k\bar{w} - 2kp]. \tag{3.4}$$

When we substitute (3.4) into (1.6), we obtain

$$|w| = \frac{(1+k^2)\lambda + 2k}{1+k^2+2k\lambda}u + \frac{1-k^2}{1+k^2+2k\lambda}p, \quad (w = u + iv \in F_k(\Gamma)), \tag{3.5}$$

from which we see that $F_k(\Omega)$ is a conic section and $w=0$ a focus. Since $A(z)$ is a homeomorphism, it is clear that if Γ is a circle or an ellipse, then its image is also a circle or an ellipse, etc.

As regards the assertion of the theorem pertaining to the extremality of $F_k(z)$, subject to the side conditions (3.1), one needs to make use of the fact, (1.2), that $F_k(z)$ is a Teichmüller mapping of $\Omega\setminus\{0\}$, with $\varphi(z)=1/z$. Since $1/z$ is holomorphic in $\Omega\setminus\{0\}$, unique extremality of $F_k(z)$, given the boundary values on $\partial[\Omega\setminus\{0\}]$, follows from classical facts [8], providing

$$\iint_{\Omega\setminus\{0\}} \left|\frac{1}{z}\right| dx\, dy < \infty. \tag{3.6}$$

Condition (3.6) is satisfied when Ω is bounded, i.e., a circle or ellipse. For the case when Ω is a parabolic region, the integral (3.6) is infinite, and not even extremality alone, with or without uniqueness, is *a-priori* clear. However, the fact that a qc mapping of a parabolic region with complex dilatation (1.1) is uniquely extremal for its boundary values on the parabola and the focus was established by the author previously [7], using a more refined analysis.

We note that the affine mapping $A(z)$ of (3.3) has

$$\varkappa_A(z) = \frac{A_{\bar z}}{A_z} = \frac{(k+\lambda)k}{1+k\lambda}.$$

In particular, $\lambda=\pm 1$, $0<k<1$ implies $k_A=k$. This reflects the previously known fact [6, 7] that $F_k(z)$ *and* $A(z)$ are *both* extremal mappings of parabolic regions with focus at $z=0$, *given the boundary values of* $A(z)$ *on the parabola alone.*

4. Shift mappings

Theorem 4.1. *Let E be the Jordan region bounded by an ellipse of eccentricity τ, whose foci are at z_1 and z_2. There is a uniquely extremal qc map h of E onto itself, keeping all points of ∂E fixed, and mapping z_1 onto z_2. The extremal dilation k^* has the value $k^*=k_h=\tau$. If $z_1=0$, and z_2 is on the negative real axis, then*

$$h(z) = \frac{1}{1-\tau^2}(z+2\tau|z|+\tau^2\bar z)+z_2. \tag{4.1}$$

Proof. If the foci are at $z=2p\tau/(1-\tau^2)$ and at $z=0$, the equation of ∂E is

$$|z| = -\tau x+p, \quad (z = x+iy\in\partial E),$$

with $p>0$. Consider now the image of E under the mapping $w=F_k(z)$, where we set $k=\tau$. By (3.5), with $\lambda=-\tau$, $k=\tau$, the equation of $F_\tau(\partial E)$ is

$$|w| = \tau u+p, \quad (w = u+iv\in F_\tau(\partial E)).$$

$F_\tau(E)$ is an ellipse with foci at $w=0$ and at $w=2p\tau/(1-\tau^2)$. From (3.3), it is, in fact, clear that $F_\tau(\partial E)$ is obtained from ∂E by a *pointwise* translation to the right by a distance $2p\tau/(1-\tau^2)$. Thus, $h(z)$, as defined by (4.1), has the property that it keeps all points of ∂E fixed. The unique extremality property is a consequence of Theorem 3.1.

The above considerations are useful in clarifying Teichmüller's construction of an extremal qc mapping of the disk onto itself which keeps all points of the circumference fixed and shifts the origin to a specified point. Suppose $z=G(\zeta)$ is a conformal map of a region $\mathscr{R}$ onto the elliptical region E of Theorem 4.1, with the foci located at $z=0$, $z=z_2<0$. Say, $G(\zeta_0)=0$. Then the mapping $T(\zeta)$, defined by

$$T(\zeta) = G^{-1} \circ h \circ G(\zeta)$$

maps $\mathscr{R}$ onto itself, keeping all boundary prime ends fixed, while shifting ζ_0 to the point $G^{-1}(z_2)$. Since

$$\varkappa_h(z) = \tau z/|z|, \quad z \in E,$$

it follows that $T(\zeta)$ is a Teichmüller mapping with complex dilatation

$$\varkappa_T(\zeta) = \frac{\overline{G'(\zeta)}}{G'(\zeta)} \varkappa_h(G(\zeta)) = \tau\overline{\varphi(\zeta)}/|\varphi(\zeta)|, \tag{4.2}$$

where

$$\varphi(\zeta) = [G'(\zeta)]^2/G(\zeta). \tag{4.3}$$

The mapping $T(\zeta)$ is uniquely extremal within the class of mappings which are the identity on $\partial\mathscr{R}$ and which shift ζ_0 to $T(\zeta_0)$, since $\varphi(\zeta)$ is holomorphic in $\mathscr{R}\setminus\{\zeta_0\}$, and

$$\iint_{\mathscr{R}\setminus\{\zeta_0\}} |\varphi(\zeta)|\, d\sigma_\zeta = \iint_{E\setminus\{0\}} \frac{d\sigma_\zeta}{|z|} < \infty.$$

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University of Minnesota
School of Mathematics
Minneapolis, MN 55455
USA

ETH-Zentrum
Forschungsinstitut für Mathematik
CH-8092 Zürich
Switzerland

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