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THE UNIFORM CONTINUITY OF THE MODULUS OF ROTATION AUTOMORPHIC FUNCTIONS

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Let $D=\{z\colon |z|<1\}$ and let $f(z)$ be a function meromorphic in D. We say that $f(z)$ is a *rotation automorphic function* in D if there exists a Fuchsian group Γ acting on D such that for each $T\in\Gamma$ there exists a rotation S_T of the Riemann sphere W such that $f(T(z))=S_T(f(z))$ for each $z\in D$. We will use F_0 to denote the fundamental region for the Fuchsian group Γ. If Γ contains more than the identity element, there are many possible choices for a fundamental region, and we will fix F_0 to be a connected hyperbolically convex set which satisfies the conditions for a fundamental region. Let $d(z_1, z_2)$ denote the hyperbolic distance between the points z_1 and z_2 in D, and let $\chi(w_1, w_2)$ denote the chordal distance, that is, the usual distance in real 3-space, between the points w_1 and w_2 in W, where we identify points on W with points on the extended complex plane in the usual way. If G is a subset of D, we say that a function $f(z)$ defined on D is *uniformly continuous hyperbolically* on G if for each $\varepsilon>0$ there exists a $\delta>0$ such that $\chi(f(z_1), f(z_2))<\varepsilon$ whenever z_1 and z_2 are points in G such that $d(z_1, z_2)<\delta$. We note that the definition does not require $f(z)$ to be a meromorphic function, and below we will use the idea of "uniformly continuous hyperbolically" for functions which are not meromorphic. In addition, we let $\bar{G}$ denote the closure of G.

If $f(z)$ is a meromorphic function in D, we say that $f(z)$ is a *normal function* if $\sup\{(1-|z|^2)f^{\#}(z)\colon z\in D\}<\infty$, where $f^{\#}(z)=|f'(z)|/(1+|f(z)|^2)$.

A concept similar to that of "uniformly continuous hyperbolically" was considered by Hayman [4]. The definition for a normal function as given above is due to Lehto and Virtanen [5].

In [2, Theorem 1], we obtained the following result.

Theorem. *If $f(z)$ is a meromorphic rotation automorphic function in D such that $f(z)$ is uniformly continuous hyperbolically on $\bar{F}_0\cap D$, then $f(z)$ is a normal function.*

In this theorem, the condition on $f(z)$ in the fundamental region F_0 is sufficiently strong that no restrictions on the group Γ are needed. For our results below, we will require some restrictions on the group Γ. Our first result deals with a finitely generated group Γ.

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Theorem 1. *Let $f(z)$ be a meromorphic rotation automorphic function such that $h(z)=|f(z)|$ is uniformly continuous hyperbolically in $\bar{F}_0 \cap D$. If Γ is a finitely generated Fuchsian group, then $f(z)$ is a normal function.*

In addition, we can obtain a similar result by changing the nature of the restriction on the group Γ. We say that the fundamental region F_0 is *thick* if for each sufficiently small $r>0$ there exists a number $r'>0$ such that for each sequence $\{z_n\}$ of points in $\bar{F}_0$ there exists a sequence of points $\{z_n'\}$ in F_0 such that, for each positive integer n both $d(z_n, z_n')<r$ and the set $U(z_n', r')=\{z \in D\colon d(z, z_n')<r'\}$ is a subset of F_0. The concept of "thick" was introduced in [1] with a slight difference in the statement of the definition. (In [1], the sequence $\{z_n\}$ was required to be in F_0, not its closure. It is a simple exercise to show that the concept as given here is equivalent to that in [1].)

Our second result is the following.

Theorem 2. *Let $f(z)$ be a meromorphic rotation automorphic function such that $h(z)=|f(z)|$ is uniformly continuous hyperbolically in $\bar{F}_0 \cap D$. If F_0 is thick, then $f(z)$ is a normal function.*

In view of Theorems 1 and 2, it is reasonable to ask the following general question. *If $f(z)$ is a rotation automorphic function such that $h(z)=|f(z)|$ is uniformly continuous hyperbolically on $\bar{F}_0 \cap D$, is $f(z)$ a normal function*? Although we suspect that the answer to this question is negative, we do not have an example to show this. Theorems 1 and 2 show that such an example must involve a fundamental region F_0 with a reasonably complicated structure.

We prove Theorems 1 and 2 below.

Proof of Theorem 1. Let $f(z)$ be a rotation automorphic function relative to a finitely generated Fuchsian group Γ such that $h(z)=|f(z)|$ is uniformly continuous hyperbolically on $\bar{F}_0 \cap D$, and suppose that $f(z)$ is not a normal function. By a theorem of Lohwater and Pommerenke [6, Theorem 1, page 3], there exist a sequence of points $\{z_n\}$ in D and a sequence $\{p_n\}$ of positive real numbers such that $p_n/(1-|z_n|) \to 0$ and the sequence of functions $\{g_n(t)=f(z_n+p_n t)\}$ converges uniformly on each compact subset of the complex plane to a function $g(t)$ meromorphic and non-constant on the complex plane. Since Γ is a Fuchsian group, for each positive integer n there exists $T_n \in \Gamma$ such that $T_n(z_n) \in \bar{F}_0 \cap D$. The family $\{S_{T_n}(g_n(t))\}$ is a normal family because the family $\{g_n(t)\}$ is a normal family. Further, since $f(z)$ is rotation automorphic relative to Γ, we have that $S_{T_n}(g_n(t))=f(T_n(z_n+p_n t))$, which means that, by taking subsequences, if necessary, we may assume that the sequence $\{f(T_n(z_n+p_n t))\}$ converges uniformly on each compact subset of the complex plane to a non-constant meromorphic function $g_*(t)$. Let $z_n'=T_n(z_n)$ and let $\Phi_n(t)=T_n(z_n+p_n t)-z_n'$. Since $p_n/(1-|z_n|) \to 0$, we have that $d(z_n', z_n'+\Phi_n(t)) \to 0$ for each fixed complex number t. It is no loss of generality to assume that the sequence

$\{z_n'\}$ converges to a point on the boundary of D, since f is continuous on D. Also, since Γ is finitely generated, there are only two possibilities: (1) $\{z_n'\}$ converges to a point on the closure of a free boundary arc of $\bar{F}_0 \cap \{z: |z|=1\}$, or (2) $\{z_n'\}$ converges to a parabolic vertex P of $\bar{F}_0$.

Assume Case (1). Then there exists a number q_0 (depending on the sequence $\{z_n'\}$) and two real numbers α_1 and α_2, with $0 \leqq \alpha_1 < \alpha_2 \leqq 2\pi$ and $\alpha_2 - \alpha_1 \geqq \pi/2$ and such that the set $D_n = \{z: 0 < d(z, z_n') < q_0,\ \alpha_1 < \arg(z - z_n') < \alpha_2\}$ is a subset of F_0 for all sufficiently large n. Since T_n preserves both angles and hyperbolic distances, for n sufficiently large the set $T_n^{-1}(D_n)$ contains a hyperbolic sector at z_n with an opening containing an angle at least $\alpha_2 - \alpha_1$. By choosing a subsequence, if necessary, we have that if t is a complex number taken from an appropriate fixed sector of the complex plane formed by an angle of at least $(\alpha_2 - \alpha_1)/2$ at the origin, then $z_n + p_n t \in T_n^{-1}(D_n)$ and thus $z_n' + \Phi_n(t) \in D_n$. Since $h(z) = |f(z)|$ is uniformly continuous hyperbolically in $\bar{F}_0 \cap D$, and since $d(z_n', z_n' + \Phi_n(t)) \to 0$, it follows that $|g_*(t)| = |g_*(0)|$. But t can be chosen arbitrarily from a fixed sector on the plane, so we have that the function $g_*(t)$ must be a constant function, which contradicts our previous assumptions about $g_*(t)$. Thus, the assumptions we have made are untenable in Case (1). This means that Case (1) cannot occur.

Now assume that Case (2) occurs. The point P is the fixed point of a parabolic element $T_P \in \Gamma$, where T_P sends the region F_0 onto an adjacent copy of F_0. Let S_{T_P} denote the rotation of the Riemann sphere which satisfies the condition $f(T_P(z)) = S_{T_P}(f(z))$. Let A and B denote the two points of the Riemann sphere fixed by the rotation S_{T_P}. We consider three subcases: (2a) $A=0$ and $B=\infty$ (this includes the case when $S_{T_P} =$ identity); (2b) $0 < |A| < |B| < \infty$; and Case (2c) $|A| = |B| = 1$. The nature of the fixed points of a rotation of the Riemann sphere gives these three cases as the only possibilities.

Assume that Case (2a) occurs. Fix t with $|t| > 0$, and let $\zeta_n = z_n' + \Phi_n(t) = T_n(z_n + p_n t)$. For each positive integer n sufficiently large, there exists an integer k_n such that $(T_P)^{k_n}(\zeta_n) \in \bar{F}_0$. But since $d(z_n', \zeta_n) \to 0$, it follows from the nature of the action of parabolic elements of Fuchsian groups that $d(z_n', (T_P)^{k_n}(\zeta_n)) \to 0$ also. Thus, since $h(z) = |f(z)|$ is uniformly continuous hyperbolically on $\bar{F}_0 \cap D$, we have that $\chi(|f(z_n')|, |f((T_P)^{k_n}(\zeta_n))|) \to 0$. But since the fixed points of S_{T_P} are 0 and ∞, it follows that S_{T_P} is a rotation of the complex plane and so $|f(T_P(\zeta_n))| = |S_{T_P}(f(\zeta_n))| = |f(\zeta_n)|$, which means that $|f((T_P)^{k_n}(\zeta_n))| = |f(\zeta_n)|$. But this implies that $|g_*(t)| = |g_*(0)|$. But since t is an arbitrary non-zero complex number, we must conclude that $g_*(t)$ is a constant function, in violation of our previous assumptions. Thus, we have that Case (2a) cannot occur.

Now assume that Case (2b) occurs. For each $\delta > 0$ let $A(\delta) = \{w: \chi(w, A) < \delta\}$ and let $B(\delta) = \{w: \chi(w, B) < \delta\}$. Since $|A| < |B|$, there exists a number $\delta_0 > 0$ such that $\sup\{|w|: w \in A(\delta_0)\} < \inf\{|w|: w \in B(\delta_0)\}$. Let t_1 and t_2 be two complex numbers such that $g_*(t_1) \in A(\delta_0)$ and $g_*(t_2) \in B(\delta_0)$. Then $d(z_n' + \Phi_n(t_j), z_n') \to 0$

for $j=1, 2$. Also, for $j=1, 2$, there exist integers $k_n^{(j)}$ such that $T_P^{k_n^{(j)}}(z_n'+\Phi_n(t_j))\in\bar{F}_0$. As in Case (2a), we have $d(z_n', T_P^{k_n^{(j)}}(z_n'+\Phi_n(t_j)))\to 0$ for $j=1, 2$, which means, since $h(z)=|f(z)|$ is uniformly continuous hyperbolically on $\bar{F}_0\cap D$, that both $|g_*(0)|\in\{|w|: w\in A(\delta_0)\}$ and $|g_*(0)|\in\{|w|: w\in B(\delta_0)\}$. But this is impossible by the way in which δ_0 was chosen. Thus, Case (2b) cannot occur.

Now assume that Case (2c) occurs. The boundary of F_0 consists of a finite number of arcs of circles, two of which meet at the point P. We designate one of these two arcs as the left boundary and the other as the right boundary, and we denote these two boundaries by LB and RB, respectively. For convenience, we will choose these arcs so that $T_P(LB)=RB$. Let $C_{LB}=C_{LB}(f, P)$, the cluster set of f at P relative to the arc LB, and let $C_{RB}=C_{RB}(f, P)$, the cluster set of f at P relative to the arc RB. Finally, let $C_{F_0}(f, P)$ denote the cluster set of f at P relative to the set F_0. We wish to show that our accumulated assumptions imply that $C_{LB}=C_{RB}=C_{F_0}(f, P)$, and that each of these sets is a singleton set, say $\{\alpha\}$, where either $\alpha=A$ or $\alpha=B$.

If $w\in C_{LB}$ then there exists a sequence $\{\omega_n\}$ of points in LB such that $f(\omega_n)\to w$. But T_P sends each point ω_n into RB, and also $d(\omega_n, T_P(\omega_n))\to 0$. Since $h(z)=|f(z)|$ is uniformly continuous hyperbolically in $\bar{F}_0\cap D$, we have that $h(T_P(\omega_n))=|S_{T_P}(f(\omega_n))|\to|w|$, and thus $|S_{T_P}(w)|=|w|$. Now suppose that C_{LB} is not contained in any great circle through the points A and B. Let w_1 and w_2 be points in C_{LB}, where w_1, w_2, A, and B are all different points and w_2 does not lie on the great circle C_1 determined by A, B, and w_1. Since the rotation S_{T_P} fixes the two points A and B on the unit circle, and $|S_{T_P}(w_1)|=|w_1|$, we have that S_{T_P} sends the circle C_1 onto another circle C_1^*, where C_1^* contains both the points A and B and C_1^* is the reflection of C_1 across the great circle determined by A, B, and ∞. Similarly, if we denote by C_2 the great circle determined by A, B, and w_2, then $S_{T_P}(C_2)=C_2^*$, where C_2^* is the reflection of C_2 across the great circle determined by A, B, and ∞. However, S_{T_P} is a rotation of W, which means that C_1 and C_2 should be rotated by the same angle. But since C_1 and C_2 are different circles, one of the pairs C_1 and C_1^* or C_2 and C_2^* occurs between the other, which means that the angle between C_1 and C_1^* cannot be the same as the angle between C_2 and C_2^*. Thus, we have shown that C_{LB} (and hence C_{RB}) lies on a great circle through A and B.

Suppose that C_{LB} and C_{RB} each consist of more than one point. If C_{LB} is a subset of the unit circle $\{w: |w|=1\}$, then C_{RB} is a subset of this same circle, which means that $S_{T_P}(w)=e^{i\lambda}/w$ for some choice of λ, and $C_{F_0}\cup S_{T_P}(C_{F_0})$ contains the outer angular cluster set of f at P. Since we have assumed that $(1-|z_n'|^2)\cdot f^{\#}(z_n')\to\infty$ and that $z_n'\to P$ radially, we must have that the outer angular cluster set of f at P is total. But this means that C_{F_0} must contain at least one of the points 0 or ∞.

Now suppose that C_{LB} contains points w such that $|w|\neq 1$. Without loss of generality, we may assume that C_{LB} contains a point w with $|w|<1$. Let $w_1\in C_{LB}$

be such that $|w_1|=\min\{|w|: w\in C_{LB}\}$. Since we have shown that $|S_{T_P}(w)|=|w|$ for $w\in C_{LB}$, we conclude that $|w_1|>0$, for we are dealing with a case where 0 is not a fixed point of S_{T_P}. Let $\{\beta_n\}$ be a sequence of points in LB converging to P such that $f(\beta_n)\to w_1$, and let γ_n denote the circle through β_n which is internally tangent to the unit circle at P. Then $T_P(\beta_n)\in\gamma_n$ and $d(\beta_n, T_P(\beta_n))\to 0$. Then $f(\gamma_n)$ is a curve on the Riemann sphere containing both $f(\beta_n)$ and $S_{T_P}(f(\beta_n))$. If t_n is a point of $\gamma_n\cap F_0$, then since $h(z)=|f(z)|$ is uniformly continuous hyperbolically on the closure of F_0, we must have that $|f(t_n)|\to|w_1|$. But by the continuity of f we can choose the point t_n so that $f(t_n)$ lies on the great circle through the points 0, ∞, and $(w_1+S_{T_P}(w_1))/2$. Since the limit points of the sequence $\{f(t_n)\}$ lie in the set C_{F_0}, it follows that C_{F_0} contains a point in the component of the complement of $C_{LB}\cup C_{RB}$ which contains the origin. A basic result of cluster set theory says that the boundary of the set $C_{F_0}(f, P)$ is contained in the union of the two sets C_{LB} and C_{RB} [3, Theorem 5.2.1, page 91]. Thus, we must have that $0\in C_{F_0}$. From this reasoning, it follows that under the assumptions we have made, we must have either $0\in C_{F_0}$ or $\infty\in C_{F_0}$. Suppose, for definiteness, that $0\in C_{F_0}(f, P)$. Then there exists a sequence $\{\tau_n\}$ in F_0 such that $f(\tau_n)\to 0$. Letting γ_n again be an arc of the circle through P and τ_n which is internally tangent to the unit circle at P, we can repeat the same reasoning as in the previous paragraph to conclude that $0\in C_{LB}$ and $0\in C_{RB}$. But this violates what we have shown above. Thus, the only possibility remaining is that C_{LB} and C_{RB} coincide as a singleton set, which must be a fixed point of S_{T_P}. Now, it follows from the argument just completed that $C_{F_0}(f, P)=C_{LB}=C_{RB}$. For definiteness, let A be the point which is the single element of the sets C_{LB}, C_{RB}, and $C_{F_0}(f, P)$.

Now let $U_n=U(z_n', b)$ be the hyperbolic disc with center at z_n' and hyperbolic radius b, where $b>0$ is a fixed number. Since $C_{F_0}(f, P)=\{A\}$, for a given number $\delta>0$ there exists a number $\beta>0$ such that $\chi(f(z), A)<\delta$ whenever $z\in\bar{F}_0\cap D$ and $|z-P|<\beta$. Let $D_\beta=\{z: |z-(1-(\beta/2))P|<\beta/2\}$. For $z\in D_\beta$ there exists an integer $n(z)$ such that $T_P^{n(z)}(z)\in D_\beta\cap\bar{F}_0$, which means that $\chi(f(z), A)=\chi(S_{T_P}^{n(z)}(f(z)), A)<\beta$. But, for n sufficiently large, the disk U_n is a subset of D_β. Hence, for each complex number t, we have $\chi(g_*(t), A)<\delta$, which, for $\delta<1/2$, contradicts the assumption that $g_*(t)$ is a non-constant meromorphic function. Thus, Case (2c) is also impossible.

We have now shown that the assumption that $f(z)$ is not a normal function is inconsistent with the other assumptions on $f(z)$ and Γ, and so Theorem 1 is proved.

Proof of Theorem 2. Assume that $f(z)$ is a rotation automorphic function such that $h(z)=|f(z)|$ is uniformly continuous hyperbolically in $\bar{F}_0\cap D$, where F_0 is thick, and assume that $f(z)$ is not a normal function. Again, by the result of Lohwater and Pommerenke cited above, there exist a sequence of points $\{z_n\}$ in D and a sequence of positive real numbers $\{p_n\}$ such that $p_n/(1-|z_n|)\to 0$ and the sequence of functions $\{f_n(t)=f(z_n+p_n t)\}$ converges uniformly on each compact subset of the complex plane to a non-constant meromorphic function $g(t)$.

As in the proof of Theorem 1, there exists a sequence of elements $\{T_n\}$ in Γ such that for each n, $z'_n=T_n(z_n)\in\bar{F}_0$ and, if $\Phi_n(t)=T_n(z_n+p_n t)-z'_n$, then the sequence of functions $\{f(z'_n+\Phi_n(t))\}$ converges uniformly on each compact subset of the plane to a non-constant meromorphic function $g_*(t)$, where we replace $\{z'_n+\Phi_n(t)\}$ by a subsequence, if necessary. Also, we have that $d(z'_n, z'_n+\Phi_n(t))\to 0$ for each complex number t. As in the proof of Theorem 1, it is no loss of generality to assume that the sequence $\{z'_n\}$ converges to a point of the unit circle $\{z\colon |z|=1\}$.

Let $r>0$ be fixed. Since F_0 is thick, there exists a real number $r'>0$ and a sequence $\{\zeta_n\}$ in F_0 such that $d(z'_n, \zeta_n)<r$ and the hyperbolic disk $U_n(\zeta_n, r')=\{z\in D\colon d(z, \zeta_n)<r'\}$ is a subset of F_0. Let Δ_n denote the hyperbolic triangle formed by taking z'_n as one vertex and letting the side opposite z'_n be a hyperbolic chord L_n of $U(\zeta_n, r')$ with hyperbolic length r', and such that Δ_n is isosceles with the altitude on the side L_n as large as possible. Since r' is independent of the sequence $\{z'_n\}$, there exists a number $\alpha>0$, independent of n, such that for each n the size of the angle of Δ_n at the vertex z'_n is at least α. (Here, Δ_n is a hyperbolic triangle with hyperbolic altitude at most $r+r'$ and hyperbolic base r', where r and r' are fixed. By applying a linear transformation of D which sends the point z'_n to the origin, it is possible to calculate a specific value of α in terms of r and r', but it is enough for our purposes to know that α is a fixed positive number.) Since F_0 is hyperbolically convex, Δ_n°, the interior of Δ_n, is a subset of $F_0\cap D$. Let $\Delta'_n=T_n^{-1}(\Delta_n^\circ)$. Then Δ'_n is an open hyperbolic triangular region with a vertex angle at least α at the vertex z_n, and the hyperbolic distance from z_n to the opposite side of the triangular region is at least r'. Let $B_n=\{t\colon z_n+p_n t\in\Delta'_n\}$. By simple geometric considerations, each of the regions Δ'_n contains a Euclidean triangle X_n, where X_n has a vertex at z_n with vertex angle $\alpha/2$ and the hyperbolic distance from z_n to the opposite side of X_n is bounded away from zero. Since $p_n/(1-|z_n|)\to 0$, it follows that B_n contains a triangle π_n with vertex at the origin and such that, if h_n is the Euclidean altitude to the side of the triangle π_n opposite the origin then $h_n\to\infty$. Thus, there exists a sector S at the origin with opening $\alpha/2$ such that, if $t\in S$ then $t\in B_n$ for infinitely many n.

Recall that we are assuming that the sequence $\{f(z'_n+\Phi_n(t))\}$ converges uniformly on each compact subset of the complex plane to the non-constant function $g_*(t)$. But if $t\in S$ then $z'_n+\Phi_n(t)\in\Delta_n^\circ$. However, the conditions that $d(z'_n, z'_n+\Phi_n(t))\to 0$ and $h(z)=|f(z)|$ is uniformly continuous hyperbolically in $\bar{F}_0\cap D$ mean that $|g_*(t)|=|g_*(0)|$ whenever $t\in S$. But this implies that the function $g_*(t)$ is a constant function, in violation of our previous assumptions. It follows that f is a normal function, and the proof of Theorem 2 is complete.

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