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ON THE EXTREMALITY AND UNIQUE EXTREMALITY OF AFFINE MAPPINGS IN SPACE, AN ADDENDUM

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The proper use of the number $\exp\sqrt{n/6(n-1)}$ in our above mentioned paper ([1]) depended on the crucial inequality $\psi \geqq 2/3$ *whenever* $\varphi=0$. There were Cases (II) and (III). In Case (II), φ and ψ were the following functions of the two variables α, β: $\psi=F/(n+1)$; $\varphi=\alpha^{k-1}\beta^{m-1}G$, in which $F(\alpha,\beta)=k(\alpha-1)^2+km(\alpha-\beta)^2+m(\beta-1)^2$; $G(\alpha,\beta)=k\beta(\alpha-1)^2+km(\alpha-\beta)^2+m\alpha(\beta-1)^2$. Here k, m, n are positive integers with $k+m=n$ and $n\geqq 2$. The case $\alpha=\beta=1$ is an irrelevant exception to the inequality. Although the case $\alpha=0$ had been treated, we neglected to treat the vanishing of G. It is the purpose of this note to correct this oversight for both this and the companion Case (III).

To begin, and by symmetry, we may assume that $\alpha\leqq\beta$, hence $G=0$ can only occur if $\alpha\leqq 0$. If $\beta<0$, then $F>k+m=n\geqq(2/3)(n+1)$. If $\beta-\alpha>1$, then $F>k+km=k(n+1-k)\geqq n\geqq(2/3)(n+1)$. This could restrict consideration to the triangle $T=\{(a,\beta)\colon 0\leqq\beta\leqq\alpha+1,\ -1\leqq\alpha\leqq 0\}$, but we prefer to consider the square $Q=\{(\alpha,\beta)\colon -1\leqq\alpha\leqq 0, 0\leqq\beta\leqq 1\}$, in which F assumes its absolute minimum at $(\alpha,\beta)=(0,1/(k+1))$, having there the value $(n+1)k/(k+1)$. This in turn is not less than $(2/3)(n+1)$ as soon as $k\geqq 2$. Hence all cases $k\geqq 2$ are done.

In case $k=1$, and on any vertical line (fixed α) which meets Q, G is a quadratic function of β, and since $G(\alpha,0)=(n-1)(\alpha+\alpha^2)\leqq 0<n(\alpha-1)^2=G(\alpha,1)$, it follows that the set $G=0$ meets each vertical section of Q in exactly one point.

We define the pair (B,M) by saying that the level curve $\{F=(2/3)(n+1)\}$ (an ellipse) has its lower meeting with the β-axis when $\beta=B$, having there slope M. The tangent line L: $\beta=M\alpha+B$, which is a lower support line for the sublevel set $K=\{F<(2/3)(n+1)\}$, enters Q from the right at $(0,B)$, and crosses the line $\alpha=-1$ at $\beta=B-M$. As we will soon see, this number is $1-2B\in(0,1]$. Our objective will now be to show that G is nonnegative on L. It will then follow that the locus $G=0$ has no contact with K.

The defining conditions are: B is the smaller root of $\beta^2+(\beta-1)^2=(2n-1)/3(n-1)$, whereas $M=-F_\alpha(0,B)/F_\beta(0,B)=[(n-1)B+1]/(n-1)(2B-1)$. More explicitly, the formula $B=(1/2)(1-\sqrt{(n+1)/3(n-1)})$ shows that B *increases* from 0 to $(1/2)(1-\sqrt{1/3})<0.22$ as n runs from 2 to ∞. In particular, $1-2B\in(0,1]$. The equally crucial and easily checked relation $M=3B-1$ is helpful in simplifying

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the following expressions, where we write $g(\alpha)=G(\alpha, M\alpha+B)=A_0+A_1\alpha+A_2\alpha^2+A_3\alpha^3$, and in which

$$\begin{aligned} A_0 &= B+(n-1)B^2 = nB-(n-2)/6 = 1/3+n[B-(1/6)] \geqq 0, \\ A_1 &= M-2B+(n-1)[2B(M-1)+(B-1)^2] = A_0, \\ A_2 &= B-2M+(n-1)[2M(B-1)+(M-1)^2] = 1+(1/2)n(7-10B), \\ A_3 &= M+(n-1)M^2 = 3nB-(1/2)(n-2) = 1+(1/2)n(-1+6B) = 3A_0 \geqq 0. \end{aligned}$$

Regarding the claim $A_0 \geqq 0$: one sees $B=1/6$ when $n=7$. The case $n=2$ giving $A_0=0$, the remaining cases $n=3, 4, 5, 6$ are checked individually. We now see in addition that $A_2-A_3=4n(1-2B)>0$. We deduce from the variously displayed relations:

$$\begin{aligned} g(\alpha) = A_0+A_1\alpha+A_2\alpha^2+A_3\alpha^3 &= A_0(1+\alpha+3\alpha^2+3\alpha^3)+4n(1-2B)\alpha^2 \\ \geqq A_0(1+\alpha+3\alpha^2+3\alpha^3) &= A_0(1+\alpha)(1+3\alpha^2) \geqq 0, \end{aligned}$$

whenever $-1 \leqq \alpha \leqq 0$, with equality only if $\alpha=0$ and $A_0=0$ $(n=2)$.

Turning to Case (III), the analogue to $(n+1)\psi=F$ may be expressed by $xA\cdot x-2x\cdot v+n$, in which x is the row vector (α, β, γ), v is the row vector (k, m, p), and as before k, m, p, n are positive integers with $k+m+p=n$. Here, A, and for later use, E, are the matrices:

$$A = \begin{bmatrix} k(n+1-k) & -mk & -kp \\ -mk & m(n+1-m) & -mp \\ -kp & -mp & p(n+1-p) \end{bmatrix}, \quad E = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}.$$

We do not, however, adhere to the requirement that k, m, p be integers. Nevertheless, $1 \leqq k \leqq m \leqq p$ is an allowable and useful normalization.

The analogue to the side condition $\varphi=0$ is easily expressed, but in this case there is also a secondary side condition $xE\cdot x=0$, which amounts to $\alpha\beta+\beta\gamma+\alpha\gamma=0$. This all comes about because α, β, γ were originally three *distinct* roots to a certain cubic equation ([1], bottom page 105) which had no linear term. As it turns out, neither the relation $\varphi=0$, nor the assumption that α, β, γ are distinct is required. We simply show $F \geqq (2/3)(n+1)$ whenever $xE\cdot x=0$. The beauty of this side condition is that it is independent of the parameters k, m, p.

The theory of Lagrange multipliers tells us that for any extremal configuration, there is a nontrivial linear dependence of the gradients, which amounts to $\lambda(xA-v)=\mu xE$. If $\lambda=0$, then $xE=0$, hence $x=0$, and $F=n$. In continuing the investigation, we assume $\lambda=1$.

Next assuming $k=m<p$, the first and second coordinates of v are the same. Therefore the same can be said for $x(A-\mu E)$, and we find easily by subtraction that $[k(n+1)+\mu][\alpha-\beta]=0$, hence either $\alpha=\beta$ or $\mu=-k(n+1)$. With $\alpha=\beta$, the side condition reads $\alpha(\alpha+2\gamma)=0$, hence either $\alpha=0$ or $\alpha=-2\gamma$. In the former case $x=(0, 0, \gamma)$ and $F=p(2k+1)\gamma^2-2p\gamma+n$, minimal for $\gamma=1/(2k+1)$, with

value $V_1(k,p)=-p/(2k+1)+n$. In the latter case $x=(-2\gamma, -2\gamma, \gamma)$ and $F=\gamma^2(8k+p+18pk)-2\gamma(p-4k)+n$, minimal for $\gamma=(p-4k)/(8k+p+18pk)$, with value $V_2(k,p)=-(p-4k)^2/(8k+p+18pk)+n$. The final alternative $\mu=-k(n+1)$ leads (with the side condition) back to $x=(0,0,1/(2k+1))$ and $V_1(k,p)$. Since both extremal values $V_1(k,p)$ and $V_2(k,p)$ are less than n, we can in future disregard the first extremal configuration $\lambda=0$. We note that $V_2(k,p)-V_1(k,p)=16k(p-k)(n+1)/(2k+1)(8k+p+18pk)\geqq 0$, so the minimum for the special case $(k,m,p)=(k,k,n-2k)$ is $W_1(k,n)=V_1(k,n-2k)=2k(n+1)/(2k+1)$. We also note $W_1(k,n)\geqq(2/3)(n+1)$.

We next assume $k<m=p$. By algebraic duality with the previous case, we find the minimum to be $W_2(k,n)=V_2((n-k)/2,k)=9k(n+1)(n-k)/(4n-3k+9k(n-k))$. Finally to interpolate between these cases $p=(n-k)/2$ and $p=n-2k$ we observe with x, k, n fixed and m replaced by $n-k-p$, that F is quadratic in p with leading term $-p^2(\beta-\gamma)^2$. Thus for each fixed x, k, n, the minimum of F with respect to p occurs either for $p=n-2k$ or $p=(n-k)/2$. It follows that with k, n fixed, $\min\{F(x)\colon xE\cdot x=0,\ k+m+p=n\}$ is the smaller of the pair $W_1(k,n)$, $W_2(k,n)$, which happens to be the former. Indeed, one finds

$$W_2(k,n)-W_1(k,n) = k(n+1)(n-3k)/(2k+1)(4n-3k+9k(n-k)) \geqq 0.$$

Since, as observed above, $W_1(k,n)\geqq(2/3)(n+1)$, we are about done.

Perhaps one should point out that the case $(k,m,p)=(n/3,n/3,n/3)$, strictly speaking not included in the previous analysis, can nevertheless be treated by continuity from the cases considered. It is not surprising and follows from the last formula that $W_1(n/3,n)=W_2(n/3,n)$. Slightly different about this case, however, are the highly nonunique extremal configurations — a locus comprising a space circle.

Reference

[1] Agard, S., and R. Fehlmann: On the extremality and unique extremality of affine mappings in space. - Ann. Acad. Sci. Fenn. Ser. A I Math. 11, 1986, 87—110.

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