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HOMEOMORPHISMS OF BOUNDED LENGTH DISTORTION

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1. Introduction

1.1. Let D and D' be domains in the plane R^2, and let $f: D \to D'$ be a homeomorphism. We let $l(\alpha)$ denote the length of a path α. If $L \geqq 1$ and if

$$l(\alpha)/L \leqq l(f\alpha) \leqq Ll(\alpha) \tag{1.2}$$

for all paths α in D, we say that f is of *L-bounded length distortion,* abbreviated L-BLD. In a joint article [MV] of O. Martio and the author, we consider more general BLD maps: discrete open maps of domains of R^n into R^n satisfying (1.2). For homeomorphisms and, more generally, for immersions, (1.2) is equivalent to the following condition: Every point in D has a neighborhood U such that $f|U$ is L-bilipschitz, that is,

$$|x-y|/L \leqq |f(x)-f(y)| \leqq L|x-y| \tag{1.3}$$

for all $x, y \in U$. For this reason, the BLD immersions are often called locally bilipschitz maps or local quasi-isometries or just quasi-isometries [Jo], [Ge].

The BLD property can also be defined in terms of upper and lower derivatives. Let $L_f(x)$ and $l_f(x)$ be the upper and lower limits of $|f(x+h)-f(x)|/|h|$ as $h \to 0$. Then a homeomorphism f is L-BLD if and only if $l_f(x) \geqq 1/L$ and $L_f(x) \leqq L$ for all $x \in D$. In particular, if f is differentiable at x, this means

$$|h|/L \leqq |f'(x)h| \leqq L|h| \tag{1.4}$$

for all $h \in R^2$.

Every L-BLD homeomorphism is L^2-quasiconformal, but a quasiconformal map is BLD only if its derivative is a.e. bounded away from 0 and ∞.

The purpose of this paper is to identify the domains $D \subset R^2$ which are BLD homeomorphic to a disk or to a half plane. The corresponding problem for bilipschitz maps was solved in the early eighties by Tukia [Tu_1], [Tu_2], Jerison—Kenig [JK] and Latfullin [La]; see also [Ge]. Their results can be stated as follows: A bounded domain D is bilipschitz homeomorphic to a disk if and only if its boundary ∂D is a rectifiable Jordan curve satisfying the *chord-arc* condition: There is $c \geqq 1$ such that

$$\sigma(x, y) \leqq c|x-y| \tag{1.5}$$

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for all $x, y \in \partial D$; here $\sigma(x, y)$ is the length of the shorter arc of ∂D between x and y. The half plane case is similar; then ∂D is a locally rectifiable Jordan curve through ∞ satisfying (1.5).

We show that the BLD homeomorphic images of the disk and the half plane can be characterized by a somewhat similar condition. However, the euclidean distance $|x-y|$ in (1.5) must be replaced by the *internal distance* $\lambda_D(x, y)$, which is the infimum of the lengths of all arcs joining x and y in D. Moreover, ∂D need not be a Jordan curve. Hence we shall replace ∂D by the prime end boundary $\partial^* D$. Alternatively, the condition can be expressed in terms of the neighborhood system of ∂D in D. An equivalent condition has been considered by Pommerenke [Po_2].

In a forthcoming paper I shall apply the results of the present paper to show that a bounded domain is BLD homeomorphic to a disk if and only if $D \times R^1$ is quasiconformally equivalent to a ball.

1.6. *Notation.* If $x \in R^n$ and $r > 0$, $B(x, r)$ is the open ball with center x and radius r, and $S(x, r)$ is its boundary sphere. We shall write

$$B(r) = B(0, r), \quad B^n = B(1), \quad S(r) = S(0, r), \quad S^{n-1} = S(1).$$

We let $d(A)$ denote the euclidean diameter of a set $A \subset R^n$, and $d(x, A)$ is the distance between A and a point $x \in R^n$.

2. Preliminaries

In this section we introduce the internal chord-arc condition for simply connected domains in R^2. In 2.9 we give a modulus estimate needed in Section 3. Since it may have independent interest, it is formulated for an arbitrary dimension n.

2.1. *Jordan domains.* A domain $D \subset R^2$ is a Jordan domain if its boundary ∂D in the extended plane $\dot{R}^2 = R^2 \cup \{\infty\}$ is a Jordan curve (homeomorphic to a circle). Suppose that D is a Jordan domain and that ∂D is locally rectifiable, that is, every compact arc in $\partial D \setminus \{\infty\}$ is rectifiable. If $a, b \in \partial D \setminus \{\infty\}$, we let $\sigma_D(a, b)$ denote the length of the shorter subarc of ∂D with end points a and b. If $x, y \in \bar{D} \setminus \{\infty\}$, $\lambda_D(x, y)$ will denote the infimum of the lengths of all paths joining x and y in D. We say that D has the *internal chord-arc property* with parameter $c \geqq 1$ if

$$\sigma_D(a, b) \leqq c\lambda_D(a, b) \tag{2.2}$$

for all finite boundary points a, b of D. We abbreviate this by saying that D is c-ICA.

The ordinary chord-arc condition (1.5) clearly implies (2.2). One can show that a Jordan curve through ∞ satisfies (1.5) if and only if both components of its complement have the property (2.2). The domain $D = \{(x, y) \in R^2 \colon x < 0 \text{ or } y > |x|^2\}$ satisfies (2.2) but not (1.5).

2.3. *Prime ends.* We give a brief summary on some known facts on prime ends. In what follows, we assume that D is a simply connected proper subdomain of R^2 which is *finitely connected on the boundary.* This means that every boundary point of D has arbitrarily small neighborhoods U such that $D \cap U$ has only a finite number of components. Equivalently, ∂D is locally connected. Still equivalently, every QC map $f: B^2 \to D$ has a continuous extension $\bar{f}: \bar{B}^2 \to \bar{D}$. See [Nä, 3.2] and [$Po_1$, 9.8].

The prime ends of such a domain D are always of the first kind and can be defined as equivalence classes of tails. By a tail of D we mean a path $\alpha: [a, b) \to D$ such that $\alpha(t) \to z \in \partial D$ as $t \to b$. The point z is written as $h(\alpha)$. A subtail of α is a restriction to a subinterval $[a_1, b)$. If U is a neighborhood of $h(\alpha)$, there is a unique component $W(U, \alpha)$ of $U \cap D$ containing a subtail of α. Two tails α and β are equivalent if $h(\alpha) = h(\beta)$ and if $W(U, \alpha) = W(U, \beta)$ for every neighborhood U of $h(\alpha)$. The equivalence class $\bar{\alpha}$ of a tail α is a boundary element of D, and their collection $\partial^* D$ is the prime end boundary of D. The set $D^* = D \cup \partial^* D$ has a natural topology such that $(D^*, \partial^* D)$ is homeomorphic to $(\bar{B}^2, S^1)$. In fact, every QC homeomorphism $f: B^2 \to D$ has a unique extension to a homeomorphism $f^*: \bar{B}^2 \to D^*$. There is a natural continuous impression map $i: D^* \to \bar{D}$, defined by $i(\bar{\alpha}) = h(\alpha)$ for $\bar{\alpha} \in \partial^* D$ and by $i|D = \mathrm{id}$. If D is locally connected at a boundary point z, $i^{-1}(z)$ consists of a single point, which is often identified with z. In particular, if D is a Jordan domain, we can identify $\partial^* D = \partial D$ and $D^* = \bar{D}$.

Suppose that α is a subarc of $\partial^* D$ .Then $i|\alpha$ is a path in $\dot{R}^2$ and has a well-defined length $l(\alpha) \in (0, \infty]$, called the length of α. If $i(u) = \infty$ for at most one $u \in \partial^* D$, then also written as ∞, and if $l(\alpha) < \infty$ for every compact arc $\alpha \subset \partial^* D \setminus \{\infty\}$, we say that $\partial^* D$ is locally rectifiable. Equivalently, $l(\alpha)$ can be defined as the infimum of all numbers λ such that there is a sequence of arcs $\alpha_j \subset D$ such that (1) $\alpha_j \to \alpha$ in the natural topology of the space of all arcs of D^* and (2) $l(\alpha_j) \to \lambda$.

2.4. *The* ICA *property.* Suppose that D is as in 2.3 and that $\partial^* D$ is locally rectifiable. If u and v are finite points in $\partial^* D$, we let $\sigma_D(u, v)$ denote the length of the shorter arc of $\partial^* D$ between u and v. Furthermore, let $\lambda_D(u, v)$ be the infimum of the lengths of all paths α joining u and v in D. By this we mean that α is an open path which has subpaths representing both u and v. One has always $\lambda_D(u, v) \leqq \sigma_D(u, v)$. If there is a constant $c \geqq 1$ such that

$$\sigma_D(u, v) \leqq c \lambda_D(u, v) \tag{2.5}$$

for all finite $u, v \in \partial^* D$, we say that D is c-ICA.

For Jordan domains D, this definition is equivalent to that given in 2.1. The complement of a ray and a disk with a radial slit are ICA non-Jordan domains.

2.6. Remarks. 1. Pommerenke [Po_2, Theorem 2] considered domains D satisfying the condition

$$\sigma_D(u, v) \leqq c \delta_D(u, v) \tag{2.7}$$

where $\delta_D(u, v)$ is the infimum of the diameters $d(|\alpha|)$ of all paths α joining u and v in D. Since $\delta_D \leqq \lambda_D$, (2.7) implies (2.5). Conversely, (2.5) implies that (2.7) is true with c replaced by a constant $c_1 = c_1(c)$. This follows easily from the results in Section 3. The half plane with an orthogonal slit is $2^{1/2}$-ICA, but satisfies (2.7) only for $c \geqq 2$.

2. It is possible to characterize the ICA property without mentioning prime ends: Let $D \subset R^2$ be simply connected, $D \neq R^2$. Then D is c-ICA if and only if for each pair $a, b \in \partial D \setminus \{\infty\}$ and for each $\varepsilon > 0$ there is $r > 0$ such that if a path α joins points $x \in D \cap B(a, r)$ and $y \in D \cap B(b, r)$ in D, there is a path γ joining x and y in $D \cap (\partial D + \varepsilon B^2)$ with $l(\gamma) \leqq c l(\alpha) + \varepsilon$.

3. One can also show that D is c-ICA if and only if the chord-arc condition

$$\sigma_D(u, v) \leqq c|i(u) - i(v)|$$

is valid for all $u, v \in \partial^* D$ which are the end points of a segmental crosscut of D, that is, the open line segment with end points $i(u)$, $i(v)$ lies in D and represents both u and v.

2.8. *Path families.* In Section 3 we shall consider paths γ joining a boundary point $a \in \partial D$ to a point $b \in \bar{D}$ in D. Such a path defines an element $u \in \partial^* D$ with $i(u) = a$, and we can as well consider γ as a path joining u to b. If Γ is a family of such paths, the modulus $M(\Gamma)$ is always well defined. If γ is a path, we let $|\gamma|$ denote its locus im γ.

2.9. Lemma. *Let $t > 0$ and let $A \subset R^n$ with $d(A) \leqq t$. Let $\lambda > 0$ and let Γ be a family of paths in R^n such that $l(\gamma) \geqq \lambda t$ and $\overline{|\gamma|} \cap A \neq \emptyset$ for all $\gamma \in \Gamma$. Then $M(\Gamma) \leqq \mu_n(\lambda)$, where $\mu_n(\lambda) \to 0$ as $\lambda \to \infty$.*

Proof. We may assume that $t = 1$ and that $A \subset \bar{B}^n$. We may also assume that $\lambda > 1$, since otherwise [Vä$_1$, 7.1] gives $M(\Gamma) \leqq m(B(2))/\lambda^n$. Define $\varrho_1, \varrho_2 \colon R^n \to R^1$ by

$$\varrho_1(x) = \frac{2}{(\ln \lambda)|x|} \quad \text{for} \quad 1 < |x| < \lambda^{1/2}, \quad \varrho_2(x) = 1/\lambda \quad \text{for} \quad |x| < \lambda^{1/2},$$

and $\varrho_j(x) = 0$ for other $x \in R^n$. We show that $\varrho = \max(\varrho_1, \varrho_2)$ belongs to $F(\Gamma)$, that is, the line integral of ϱ along any rectifiable $\gamma \in \Gamma$ is at least one.

If $|\gamma| \subset B(\lambda^{1/2})$, we have

$$\int_\gamma \varrho_2 \, ds \geqq l(\gamma)/\lambda \geqq 1.$$

If $|\gamma| \not\subset B(\lambda^{1/2})$, $\overline{|\gamma|}$ meets the spheres S^{n-1} and $S(\lambda^{1/2})$, and hence

$$\int_\gamma \varrho_1 \, ds \geqq \int_1^{\lambda^{1/2}} \frac{2\,dr}{r \ln \lambda} = 1.$$

Thus $\varrho \in F(\Gamma)$, which implies

$$M(\Gamma) \leqq \int_{R^n} \varrho^n \, dm.$$

Letting Ω and ω denote the volume of B^n and the area of S^{n-1} we have

$$\int_{R^n} \varrho_1^n \, dm \leqq 2^{n-1} \omega (\ln \lambda)^{1-n}, \quad \int_{R^n} \varrho_2^n \, dm = \Omega \lambda^{-n/2},$$

and the lemma follows. □

3. Main results

3.1. In this section we characterize the BLD homeomorphic images of B^2 and H^2. The half plane case is given in 3.4 and the disk case in 3.8. We recall from 1.1 that a homeomorphism $f\colon D \to D'$ is L-BLD if

$$l(\alpha)/L \leqq l(f\alpha) \leqq Ll(\alpha) \tag{3.2}$$

for every path in D or, equivalently, f is locally L-bilipschitz.

3.3. Theorem. *Let $D \subset R^2$ be a convex Jordan domain, and let $f\colon D \to D' \subset R^2$ be an L-*BLD *homeomorphism. Then:*

(1) *f is L-Lipschitz in the euclidean metric.*

(2) *D' is finitely connected on the boundary.*

(3) *f is L-bilipschitz in the metric $\lambda_{D'}$.*

(4) *$\partial^* D'$ is locally rectifiable.*

(5) *f has a unique extension to a homeomorphism $f^*\colon \bar{D} \to D'^*$, which is L-bilipschitz outside ∞ in the metric $\lambda_{D'}$.*

(6) *$f^* | \partial D$ is L-bilipschitz outside ∞ in the metrics σ_D and $\sigma_{D'}$.*

*If, in addition, D has the c-chord-arc property, D' is L^2c-*ICA.

Proof. Observe that since D is convex, λ_D is the euclidean metric. The condition (1) follows at once from convexity. Hence f has a continuous extension $\bar{f}\colon \bar{D} \to \bar{D}'$. Then (2) follows from [Nä, 3.2]. Since D is convex, ∂D is locally rectifiable. The rest of the theorem follows easily from (3.2) and from the considerations in 2.3 and 2.4. □

3.4. Theorem. *A simply connected domain $D \subset R^2$ is* BLD *homeomorphic to the half plane $H^2 = \{(x, y) \in R^2\colon y > 0\}$ if and only if* (1) $D \neq R^2$, (2) *D is finitely connected on the boundary,* (3) *D is* ICA, *and* (4) *D is unbounded.*

Proof. Suppose that $f\colon H^2 \to D$ is an L-BLD homeomorphism. Since the image of the segment $\{0\} \times (0, 1]$ has length at most L, (1) is true. Since H^2 is convex and 1-ICA, (2) and (3) follow from 3.3. Since f^{-1} is locally L-Lipschitz, $\infty = m(H^2) \leqq L^2 m(D)$, which implies (4).

The converse part is considerably harder. We first give an outline of the proof. Suppose that D satisfies the conditions (1)—(4). Choose a conformal map f_1: $H^2 \to D$. It has a homeomorphic extension, still written as f_1: $\bar{H}^2 \to D^*$. We may assume that $f_1(\infty)=\infty$. Choose a homeomorphism g: $R^1 \to \partial^* D \setminus \{\infty\}$ such that $\sigma_D(g(x), g(y))=|x-y|$ for all $x, y \in R^1$ and such that the homeomorphism s: $R^1 \to R^1$ defined by $s(x)=g^{-1}(f_1(x))$ is increasing. Extend s by the Beurling—Ahlfors construction [Ah, p. 69] to a homeomorphism f_2: $\bar{H}^2 \to \bar{H}^2$. Then $f=f_1 f_2^{-1}$: $\bar{H}^2 \to D^*$ is a homeomorphism, and $f|H^2$ will be the desired BLD homeomorphism.

Step 1. We show that s is quasisymmetric (QS). Let $x \in R^1$ and $t>0$. Let Γ be the family of all paths joining the intervals $[x-t, x]$ and $[x+t, \infty]$ in H^2. Then $M(\Gamma)=1$. If γ belongs to the image Γ' of Γ under f_1, γ has end points a, b with $a \in A=f_1[x-t, x]$ and $b \in B=f_1[x+t, \infty]$. The σ-diameter of A is at most its length $s(x)-s(x-t)$, and hence $d(iA) \leqq s(x)-s(x-t)$. Furthermore, $\sigma_D(a, b) \geqq s(x+t)-s(x)$. Since D is c-ICA, this implies $s(x+t)-s(x) \leqq cl(\gamma)$. From 2.8 we obtain the estimate $M(\Gamma') \leqq \mu_2(R)$ with

$$Rc = \frac{s(x+t)-s(x)}{s(x)-s(x-t)}.$$

Since f is conformal, $M(\Gamma)=M(\Gamma')=1$. Since $\mu_2(R) \to 0$ as $R \to \infty$, we obtain an upper bound for Rc. A lower bound is found similarly, changing the roles of $x-t$ and $x+t$. Hence s is $H-$QS with a constant H depending only on c.

Let f_2: $\bar{H}^2 \to \bar{H}^2$ be the Beurling—Ahlfors extension of s. Then $f_2|H^2$ is K-QC and L-bilipschitz in the hyperbolic metric of H^2 [Ah, p. 73] with $L=L(c)$ and $K=L^2$. Then $f=f_1 f_2^{-1}$: $\bar{H}^2 \to D^*$ is a homeomorphism, and $f|R^2=g$; thus

$$\sigma_D(f(x), f(y)) = |x-y| \tag{3.5}$$

for all $x, y \in R^1$.

Step 2. We write $\delta(w)=d(w, \partial D)$ for $w \in D$ and show that there is a constant $M=M(c)$ such that

$$y/M \leqq \delta(f(z)) \leqq My \tag{3.6}$$

for every $z=(x, y) \in H^2$.

Let T be the line through $b=i(f(x))$ and $f(z)$, let R be the component of $T \setminus \{f(z)\}$ not containing b, and let C' be the component of $R \cap D$ with end point $f(z)$. Let Γ be the family of all paths joining the real segment $[x, x+y]$ to $C=f^{-1}C'$ in H^2. Then well known modulus estimates show that $M(\Gamma) \geqq q_0>0$ with a universal constant $q_0>0$, cf. [GV, Lemma 3.3, p. 13]. Assume that $\delta(f(z))=\delta>y$. Since (3.5) implies $\sigma_D(f(x+y), f(x))=y$, the members of $\Gamma'=f\Gamma$ meet the circles $S(b, y)$ and $S(b, \delta)$. Hence $M(\Gamma') \leqq 2\pi/\ln(\delta/y)$. Since $M(\Gamma) \leqq KM(\Gamma')$, we obtain the second inequality of (3.6) with $M=e^{2\pi K/q_0}$.

We turn to the first inequality of (3.6). Fix $z=(x, y) \in H^2$ and set $\delta=\delta(f(z))$. Choose $w_0 \in \partial D$ with $|w_0-f(z)|=\delta$. The segment $[f(z), w_0)$ defines an element

$u_0 \in \partial^* D$ with $i(u_0) = w_0$. Let C_0' be the arc on $\partial^* D$ such that u_0 divides C_0' to two subarcs of length $7c\delta$. Let C_1 be the vertical ray with end point z. Then $C_1' = fC_1$ joins $f(z)$ to ∞ in D. Let J be the subarc of C_1' joining $f(z)$ and a point $w_1 \in S(f(z), \delta)$ in $B(f(z), \delta)$.

Case 1. $|w_1 - w_0| \geqq \delta$. Set $w_2 = (w_0 + f(z))/2$. For every $r \in [\delta/2, 3^{1/2}\delta/2]$ we can choose an arc α_r of $S(w_2, r)$ with end points $a_r \in J$, $b_r \in \partial D$ and with $\alpha_r \setminus \{b_r\} \subset D$. Let Γ' be the family of all these arcs α_r. A standard estimate gives $M(\Gamma) \geqq (\ln 3)/4\pi = q_1$. The arc $\alpha_r' = \alpha_r \setminus \{b_r\}$ defines an element $u_r \in \partial^* D$ with $i(u_r) = b_r$. Moreover, $\alpha_r' \cup [a_r, w_0)$ joins u_r and u_0 in D. Since D is c-ICA, we have

$$\sigma_D(u_r, u_0) \leqq c\big(l(\alpha_r) + |a_r - w_0|\big) \leqq c(2\pi r + |a_r - w_2| + |w_2 - w_0|)$$
$$\leqq 3^{1/2} c\delta(2\pi + 3^{1/2}/2 + 1/2)/2 < 7c\delta.$$

Hence $b_r \in C_0'$. Consequently, the members of $\Gamma = f^{-1}\Gamma'$ join $C_0 = f^{-1}C_0' = [x - 7c\delta, x + 7c\delta]$ and C_1. Thus either $y \leqq 7c\delta$ or $M(\Gamma) \leqq 2\pi/\ln(y/7c\delta)$. Since $M(\Gamma') \leqq KM(\Gamma)$, we obtain

$$y \leqq 7c\delta e^{2\pi K/q_1},$$

which yields the first inequality of (3.6).

Case 2. $|w_1 - w_0| = t < \delta$. We repeat the argument of Case 1 replacing w_2 by $w_3 = (w_0 + w_1)/2$. Since $S(w_3, r)$ meets J and ∂D whenever $t/2 \leqq r \leqq 3^{1/2} t/2$, we obtain the same estimate as in Case 1.

Step 3. We prove that the homeomorphism $f|H^2\colon H^2 \to D$ is BLD. Since $f = f_1 f_2^{-1}$ where $f_1|H^2$ is conformal and $f_2|H^2$ L-bilipschitz in the hyperbolic metric, the diffeomorphism $f|H^2$ is L-bilipschitz in the hyperbolic metrics of H^2 and D. Hence

$$|h|/Ly \leqq \varrho(f(z))|f'(z)h| \leqq L|h|/y$$

for all $z = (x, y) \in H^2$ and $h \in R^2$, where ϱ is the density of the hyperbolic metric in D. It is well known that

$$1/4\delta(w) \leqq \varrho(w) \leqq 1/\delta(w)$$

for all $w \in D$. Together with (3.6), these inequalities show that f is L_1-BLD with $L_1 = 4LM = L_1(c)$. □

3.7. Remark. The proof above shows that the quantitative version of 3.4 is also true: If $f\colon H^2 \to D$ is an L-BLD homeomorphism, D is c-ICA with $c = L^2$. If D is c-ICA and unbounded, there is an L-BLD homeomorphism $f\colon H^2 \to D$ with $L = L(c)$.

3.8. Theorem. *A simply connected domain $D \subset R^n$ is* BLD *homeomorphic to the unit disk B^2 if and only if* (1) *D is finitely connected on the boundary,* (2) *D is* ICA, *and* (3) *D is bounded.*

Proof. Suppose that $f\colon B^2 \to D$ is a BLD homeomorphism. Then f is L-Lipschitz and hence $d(D) \leqq 2L$. The conditions (1) and (2) follow from 3.3. More precisely, since B^2 is π-ICA, D is πL^2-ICA.

The converse part is proved by modifying the proof of 3.4. Suppose that D satisfies (1) and (3) and that D is c-ICA. Then $\partial^* D$ is rectifiable. We normalize the situation by assuming $l(\partial^* D) = 2\pi$. Then there is a lengthpreserving homeomorphism $g\colon S^1 \to \partial^* D$. Let $f_1\colon B^2 \to D$ be a conformal map. It has an extension to a homeomorphism, still written as $f_1\colon \bar{B}^2 \to D^*$. Then $g^{-1}f|S^1 = s$ is a self homeomorphism of S^1, and $l(s\alpha) = l(f_1\alpha)$ for every arc $\alpha \subset S^1$. We may assume that $s|N_3 = \mathrm{id}$ where $N_3 = \{1, e^{2\pi i/3}, e^{4\pi i/3}\}$.

Step 1. We show that $f_1|S^1$ has the following quasisymmetry property: If α and β are adjacent arcs of S^1 with $l(\alpha) = l(\beta)$, then

$$l(f_1\beta) \leqq c_1 l(f_1\alpha) \tag{3.9}$$

with some constant $c_1 = c_1(c)$.

Assume first that $l(\alpha) \leqq \pi/3$. Then we may assume that $\alpha \cup \beta$ does not meet the arc $A = \{e^{i\varphi}\colon 2\pi/3 < \varphi < 4\pi/3\}$. Let a be the end point of A which has the greater distance from $\alpha \cup \beta$. Using the terminology of [LV, I.3.2] we consider the quadrilateral Q consisting of the domain B^2, the three end points of α and β, and the point a. There are two path families Γ_1, Γ_2 associated with Q with moduli $M(\Gamma_1) = 1/M(\Gamma_2)$. The length of a path in either family is at least $d(\alpha) = t$. Hence 2.9 implies $M(\Gamma_j) \leqq \mu_2(1)$ and thus $M(\Gamma_j) \geqq 1/\mu_2(1)$. Let Γ_1 be the family joining α to the opposite side of Q, and suppose that $\gamma \in f_1\Gamma_1 = \Gamma_1'$. The end points of γ divide $\partial^* D$ into two arcs. One of these contains $f_1\beta$ and the other $f_1 A$. Since $sA = A$, we have $l(f_1 A) = 2\pi/3$. Since D is c-ICA, this implies $cl(\gamma) \geqq \min\big(l(f_1\beta), 2\pi/3\big)$. Since $d(if_1\alpha) \leqq l(f_1\alpha)$, 2.9 gives $M(\Gamma_1') \leqq \mu_2(R)$ with

$$R = \frac{\min\big(l(f_1\beta), 2\pi/3\big)}{cl(f_1\alpha)};$$

Since $M(\Gamma_1') = M(\Gamma_1) \geqq 1/\mu_2(1)$ and since $\mu_2(t) \to 0$ as $t \to \infty$, R is bounded by a universal constant c_0. Hence either (3.9) holds with $c_1 = c_0 c$ or $2\pi/3 \leqq c_0 cl(f_1\alpha)$. In the latter case (3.9) holds with $c_1 = 2c_0 c$.

The case $l(\alpha) > \pi/3$ reduces to the case above by dividing α and β to three subarcs, cf. [LV, II.7.1].

Step 2. We want to extend $s\colon S^1 \to S^1$ to a QC homeomorphism $f_2\colon \bar{B}^2 \to \bar{B}^2$. To this end we choose an auxiliary Möbius map h with $hB^2 = H^2$ and $h(1) = \infty$. Then $s_1 = hsh^{-1}|R^1$ is an increasing homeomorphism onto R^1. Moreover, s_1 is (weakly) H-QS with $H = H(c)$. This can be seen for example as follows: Since $l(s\alpha) = l(f_1\alpha)$, (3.9) implies that $s\colon S^1 \to S^1$ is weakly H_1-QS in the arc metric, hence in the euclidean metric, cf. [TV, p. 113]. Since S^1 is of π-bounded turning, s is η-QS with $\eta = \eta_c$ [TV, 2.16]. Hence s is θ-quasimöbius with $\theta = \theta_c$ by [Vä$_2$, 3.2]. Consequently, s_1 is θ-quasimöbius. Since $s_1(\infty)=\infty$, s_1 is θ-QS and hence (weakly) H-QS with $H=\theta(1)$.

Let g: $\bar{H}^2\to\bar{H}^2$ be the Beurling—Ahlfors extension of s_1. It induces a homeomorphism $f_2=h^{-1}gh$: $\bar{B}^2\to\bar{B}^2$. Then $f_2|S^1=s$, and $f_2|B^2$ is K-QC and L-bilipschitz in the hyperbolic metric of B^2 with $L=L(c)$, $K=L^2$.

Step 3. The map $f=f_1f_2^{-1}$: $\bar{B}^2\to D^*$ is the desired map. This follows as in the proof of 3.3 from the inequalities

$$(3.10)\qquad (1-|z|)/M \leqq \delta(f(z)) \leqq M(1-|z|)$$

where $z\in B^2$, $M=M(c)$, $\delta(w)=d(w,\partial D)$. This is proved by a rather obvious modification of the proof of the corresponding inequalities (3.6) of the half plane case. Omitting other details, we describe the construction of the arcs $C_0'=f_1C_0$ and $C_1'=f_1C_1$. We may assume that $7c\delta<1-|z|$. As in the proof of (3.6), C_0' will be a subarc of ∂^*D with $l(C_0')=14c\delta$. This is possible, since $14c\delta<2(1-|z|)\leqq 2<2\pi=l(\partial^*D)$. Then C_1 is chosen to be the line segment with end points z and $-f^{-1}(w_0)$. □

3.11. *The quantitative version of* 3.8. If f: $B^2\to D$ is an L-BLD homeomorphism, D is c-ICA with $c=\pi L^2$. If D is c-ICA and bounded with $l(\partial^*D)=r$, there is an L-BLD homeomorphism f: $B(r)\to D$ with $L=L(c)$.

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