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A CHARACTERIZATION OF THE λ-INVARIANT OF A p-ADIC L-FUNCTION

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1. Let p be a prime, and put $q=p$ if $p>2$ and $q=4$ if $p=2$. Consider the Kubota—Leopoldt p-adic L-function $L_p(s, \theta)$ attached to an even non-principal character θ. Write $\theta=\chi\omega$, where ω is the Teichmüller character mod q and χ is a character with conductor not divisible by pq (all characters are assumed primitive). It is well known that $L_p(s, \chi\omega)$ has a power series expression, say

$$f(X, \chi\omega) = \sum_{j=0}^{\infty} a_j X^j,$$

whose coefficients a_j are integers of the field $\mathbf{Q}_p(\chi)$ generated by the values of χ over the field of p-adic numbers.

By the Ferrero—Washington theorem [2], there exists an index j such that a_j is prime to p. The least such index is called the λ-invariant of $f(X, \chi\omega)$ and denoted by $\lambda=\lambda_\chi$. Observe that λ is characterized by the statement

$$\lambda \geqq h \Leftrightarrow a_0 \equiv a_1 \equiv \ldots \equiv a_{h-1} \equiv 0 \pmod{\mathfrak{p}}, \tag{1}$$

where h denotes a positive integer and $\mathfrak{p}$ is the maximal ideal of the integer ring of $\mathbf{Q}_p(\chi)$. Another characterization of the λ-invariant is given in the present note; see the theorem in Section 2. This has proved useful in some problems concerning λ.

2. To formulate the theorem, we introduce the necessary notation and record some preliminary facts.

Let us write the conductor of the character χ in the form

$$f_\chi = d \quad \text{or} \quad dq, \quad (d, p) = 1.$$

For $n \geqq 0$, denote by Γ_n the multiplicative group of those residue classes $a+dqp^n\mathbf{Z}$ for which $a\equiv 1 \pmod{dq}$. Let $\gamma_n(a)$ denote the image of $a+dqp^n\mathbf{Z}$ under the canonical projection $(\mathbf{Z}/dqp^n\mathbf{Z})^\times \to \Gamma_n$. Since Γ_n is generated by $\gamma_n(1+dq)^{-1}$, we see that the set $I_n=\{a\in\mathbf{Z}\colon 0<a<dqp^n, (a, dp)=1\}$ can be partitioned into the subsets

$$I_{nk} = \{a\in I_n\colon \gamma_n(a) = \gamma_n(1+dq)^{-k}\}, \quad k = 0, \ldots, p^n-1.$$

The power series $f(X, \chi\omega)$ is defined by the congruences

$$f(X, \chi\omega) \equiv \sum_{k=0}^{p^n-1} c_{nk}(1+X)^k \pmod{(1+X)^{p^n}-1},$$

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where the coefficients c_{nk} are integers of $\mathbf{Q}_p(\chi)$ having the following expressions:

$$c_{nk} = -\frac{1}{dqp^n} \sum_{a \in I_{nk}} a\chi(a)$$

(see [5, § 7.2]; cf. also [3] where the definition of $f(X, \chi\omega)$ differs from the present one by a factor 1/2). Note that the coefficients of $f(X, \chi\omega)$ can be given explicitly as follows:

$$a_j \equiv \sum_{k=j}^{p^n-1} \binom{k}{j} c_{nk} \pmod{\mathfrak{p}}, \quad j = 0, \ldots, p^n - 1. \tag{2}$$

Now let b be a positive integer prime to dp. For $n \geqq 0$ and $k, j = 0, \ldots, p^n - 1$, set

$$S_{nk} = -\sum_{a \in I_{nk}} \chi(a) \left[\frac{ba}{dqp^n}\right], \quad T_j^{(n)} = \sum_{k=j}^{p^n-1} \binom{k}{j} S_{nk},$$

where $[z]$ denotes the largest integer $\leqq z$.

Theorem. *Let* $n \geqq 0$ *and* $1 \leqq h \leqq p^n$.

(i) *If* $\lambda \geqq h$, *then* $T_0^{(n)} \equiv T_1^{(n)} \equiv \ldots \equiv T_{h-1}^{(n)} \equiv 0 \pmod{\mathfrak{p}}$.

(ii) *If* $T_0^{(n)} \equiv T_1^{(n)} \equiv \ldots \equiv T_{h-1}^{(n)} \equiv 0$ *and* $\chi(b)b \not\equiv 1 \pmod{\mathfrak{p}}$, *then* $\lambda \geqq h$.

Note that the expression of $T_j^{(n)}$ is integral, contrary to that of a_j. This makes the theorem suitable for the computation of λ. In fact, Ernvall has carried out such computations by using a similar result which may be regarded as a preliminary version of the above theorem (see [1]). We point out that in that version, proved for $p > 2$ only, the definition of $T_j^{(n)}$ is slightly different and the restriction imposed on h is stronger (the present assumption about h being the natural one). The following proof is completely different; its key idea goes back to the author's article [4] concerning the estimation of λ from above. This shows, conversely, that the present theorem also plays an important role in this estimation result.

3. For the proof of the theorem, we keep $n \geqq 0$ fixed. We extend the preceding definition of I_{nk} for all $k \in \mathbf{Z}$ by taking $I_{nk} = I_{nm}$ whenever $k \equiv m \pmod{p^n}$. Choose $t \in \mathbf{Z}$ such that

$$\gamma_n(b) = \gamma_n(1+dq)^{-t}, \quad 0 \leqq t \leqq p^n - 1.$$

The crucial formula of the proof reads

$$S_{nk} = bc_{nk} - \chi(b)^{-1} c_{n,k+t}, \quad k = 0, \ldots, p^n - 1; \tag{3}$$

this will be verified by an argument similar to [4, Lemma 1]. Indeed, let a run through I_{nk} and write

$$ba = dqp^n \left[\frac{ba}{dqp^n}\right] + r_a.$$

Since

$$\gamma_n(r_a) = \gamma_n(ba) = \gamma_n(1+dq)^{-k-t},$$

we find that r_a runs through $I_{n,k+t}$. Hence we may write

$$c_{n,k+t} = -\frac{1}{dqp^n}\sum_{a\in I_{nk}} r_a\chi(r_a),$$

and the right-hand side of (3) becomes

$$-\frac{1}{dqp^n}\sum_{a\in I_{nk}}(ba\chi(a)-r_a\chi(b)^{-1}\chi(r_a)) = -\frac{1}{dqp^n}\sum_{a\in I_{nk}}(ba-r_a)\chi(a).$$

This proves the claim.

It follows from (3) that

$$(4)\qquad T_j^{(n)} = b\sum_{k=j}^{p^n-1}\binom{k}{j}c_{nk}-\chi(b)^{-1}\sum_{k=j}^{p^n-1}\binom{k}{j}c_{n,k+t}\quad (j=0,\ldots,p^n-1).$$

By (2), the first sum on the right-hand side is congruent to a_j (mod $\mathfrak{p}$). As for the second sum, we show that

$$(5)\qquad \sum_{k=j}^{p^n-1}\binom{k}{j}c_{n,k+t}\equiv a_j-\sum_{i=0}^{j-1}d_ia_i \pmod{\mathfrak{p}}$$

($j=0,\ldots,p^n-1$), where the coefficients d_i are rational integers.

We use induction on j. First observe that, as a function of k, c_{nk} is periodic with period p^n. For $j=0$ the left-hand side of (5) equals

$$\sum_{k=0}^{p^n-1}c_{n,k+t} = \sum_{k=0}^{p^n-1}c_{nk}\equiv a_0 \pmod{\mathfrak{p}}.$$

Let $j\geqq 1$. As usual, set $\binom{k}{j}=0$ if $0\leqq k<j$. Making use of the identity

$$\binom{k+t}{j} = \sum_{u=0}^{j}\binom{k}{u}\binom{t}{j-u} = \binom{k}{j}+\sum_{u=0}^{j-1}\binom{k}{u}\binom{t}{j-u},$$

valid for all non-negative k and t, we obtain

$$\sum_{k=0}^{p^n-1}\binom{k}{j}c_{n,k+t} = \sum_{k=0}^{p^n-1}\binom{k+t}{j}c_{n,k+t}-\sum_{u=0}^{j-1}\binom{t}{j-u}\sum_{k=0}^{p^n-1}\binom{k}{u}c_{n,k+t}.$$

Like c_{nk}, also the binomial coefficient $\binom{k}{j}$ modulo p is periodic with period p^n (apply the above identity and note that j and n are positive). Therefore, by the induction hypothesis,

$$\sum_{k=0}^{p^n-1}\binom{k}{j}c_{n,k+t}\equiv a_j-\sum_{u=0}^{j-1}\binom{t}{j-u}\left(a_u-\sum_{i=0}^{u-1}d_{iu}a_i\right) \pmod{\mathfrak{p}}$$

with $d_{iu}\in\mathbf{Z}$. This gives us (5).

From (4) and (5) we conclude that

$$T_j^{(n)}\equiv(b-\chi(b)^{-1})a_j+\chi(b)^{-1}\sum_{i=0}^{j-1}d_ia_i \pmod{\mathfrak{p}}$$

($j=0,\ldots,p^n-1$). By comparing this with (1) we easily infer the theorem.

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