Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 13, 1988, 81–92

ON QUASICONFORMAL RIGIDITY IN SPACE AND PLANE

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1. Introduction

Quasiconformal mappings in space are known to have many rigidity properties. For instance,

1.1. Theorem (Martio–Sarvas, [MS, 3.17]). *Let $D \subset \overline{\mathbf{R}}^n$, $n \geq 3$, be a b-uniform domain. There is a constant $K_0 = K_0(b,n) > 1$ such that every locally K-quasiconformal mapping on D with $K \leq K_0$ is injective.*

1.2. Theorem (Väisälä, [V4, 6.2, 6.12]). *Let $D \subset \mathbf{R}^n$, $n \geq 2$, be a bounded b-uniform domain, and let $f\colon D \to \mathbf{R}^n$ be an s-quasisymmetric embedding. There exists a constant $s_0 = s_0(b,n) > 0$ such that f has a K-quasiconformal extension to $\mathbf{R}^n$, whenever $s \leq s_0$. In addition, $K \to 1$ as $s \to 0$.*

For the undefined notions here and below see Section 2.

In this note we shall study the interrelations of these two aspects of rigidity, the injectivity and extendability of mappings. We begin with a necessary condition for the rigidity of a domain and, in particular, show: If D is a domain in $\mathbf{R}^n$, $n \geq 3$, and if there is a constant $K_0 > 1$ such that every locally K_0-quasiconformal mapping on D is injective, then D must be linearly locally connected. Next, we combine this fact with the above extension theorem of Väisälä and obtain the following result (which has also been announced by Trotsenko in [T1]; for related topics see [T2]).

1.3. Theorem. *Let $D \subset \overline{\mathbf{R}}^n$, $n \geq 3$, be a b-uniform domain. There is a constant $K_1 = K_1(b,n) > 1$ such that every quasiconformal mapping f on D with $K(f) \leq K_1$ has a quasiconformal extension $\tilde{f}\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$. Moreover, $K(\tilde{f}) \to 1$ as $K(f) \to 1$.*

Applying Theorem 1.3, we then get a new and larger class of domains having the injectivity property of Theorem 1.1. We call these domains uniformly collared and show that for them also Theorem 1.3 remains valid.

In the plane the Schwarzian norm

$$(1.4) \qquad \|S_f\|_D = \sup_{z\in D}|S_f(z)|d(z,\partial D)^2, \quad S_f = (f''/f')' - \tfrac{1}{2}(f''/f')^2,$$

doi:10.5186/aasfm.1988.1301

of a locally conformal mapping plays the same role in rigidity as the maximal dilatation $K(f)$ does in space. We shall provide an "explanation" of this analogy and study how far it works. It turns out that looking at the local quasimöbius properties of mappings one can introduce a distortion measure $\kappa_D(f)$ equivalent to $\|S_f\|_D$ in the plane and to $\log K(f)$ in space. Especially, with $\kappa_D(f)$ we obtain formulations of 1.1. and 1.3 valid for quasiconformal mappings in all dimensions n, $n \geq 2$.

2. Preliminaries

2.1. Notation. We shall adopt the fairly standard notation of [V1]. As a rule, D is a domain in $\overline{\mathbf{R}}^n = \mathbf{R}^n \cup \{\infty\}$, $n \geq 2$; $B(x_0, r)$ denotes an open euclidean n-ball; and for a set $A \subset \overline{\mathbf{R}}^n$ we let $C(A) = \overline{\mathbf{R}}^n \setminus A$. The group of the Möbius transformations in $\overline{\mathbf{R}}^n$ will be denoted by $\text{Möb}(\overline{\mathbf{R}}^n)$.

2.2. Quasimöbius and quasisymmetric mappings. Given four distinct points a, b, c, d in $\overline{\mathbf{R}}^n$ we denote their *cross ratio* by

$$|a, b, c, d| = \frac{|a - c||b - d|}{|a - d||b - c|}.$$

If one of these points is ∞, the factors containing that point are omitted. An embedding $f \colon D \to \overline{\mathbf{R}}^n$ is said to be (s-)*quasimöbius*, $s \geq 0$, if for all points $a, b, c, d \in D$ with $|a, b, c, d| \leq 1 + 1/s$ it holds

$$\big|f(a), f(b), f(c), f(d)\big| \leq |a, b, c, d| + s.$$

For s large this definition slightly differs from the original one in [V3].

We similarly arrive at quasisymmetry if, instead of the cross ratio, we consider the ratio $t = |a - x|/|b - x|$ of distinct points $a, b, x \in \mathbf{R}^n$. An embedding $f \colon D \to \mathbf{R}^n, D \subset \mathbf{R}^n$, is called ($s$-)*quasisymmetric*, if there exists a number $s \geq 0$ such that

$$\frac{|f(a) - f(x)|}{|f(b) - f(x)|} \leq \frac{|a - x|}{|b - x|} + s$$

whenever a, b and x lie in D and $|a - x|/|b - x| \leq 1 + 1/s$, cf. [TV] and [V4]. It follows from [V4, 2.3] that if f is quasimöbius or quasisymmetric in the above sense, then $\big|f(a), f(b), f(c), f(d)\big| \leq \theta\big(|a, b, c, d|\big)$ or, respectively, $\big|f(a) - f(b)\big|/\big|f(c) - f(b)\big| \leq \eta\big(|a - b|/|c - b|\big)$ for some homeomorphisms $\theta, \eta \colon [0, \infty) \to [0, \infty)$ and for all distinct points $a, b, c, d \in D$.

In comparison with quasisymmetry the quasimöbius mappings have the advantage (sometimes disadvantage) that they do not single out the point at infinity. Both conditions are global, however, contrary to the notion of quasiconformality; for example, an s-quasimöbius mapping $f \colon D \to D'$ always has an s-quasimöbius extension $f \colon \bar{D} \to \bar{D}'$. Quasisymmetric mappings are always quasimöbius, quasimöbius mappings quasiconformal, and the reverse implications do not hold in general; the maximal dilatation of an s-quasimöbius mapping f satisfies

(2.3) $$K(f) \le (1+s)^{n-1},$$

see [V3, 5.2].

2.4. Uniform and linearly locally connected domains. A domain $D \subset \overline{\mathbf{R}}^n$ is called b-uniform if every pair of points $x, y \in D \setminus \{\infty\}$ can be joined by a b-cigar contained in D. A *b-cigar* is an open set

$$\mathrm{cig}(\gamma, 1/b) = \bigcup_{z \in \gamma} B\big(z, \frac{1}{b} \min\{|x-z|, |z-y|\}\big),$$

where γ is a continuum connecting x to y with

$$\mathrm{diam}(\gamma) \le b|x-y|.$$

A related but larger class consists of linearly locally connected domains, cf. [G2]. A domain $D \subset \overline{\mathbf{R}}^n$ is *c-locally connected* if for each $x_0 \in \mathbf{R}^n$ and $r > 0$

(2.5) points in $D \cap B(x_0, r)$ can be joined in $D \cap B(x_0, cr)$,

(2.6) points in $D \setminus \bar{B}(x_0, r)$ can be joined in $D \setminus \bar{B}(x_0, r/c)$.

Furthermore, D is called *linearly locally connected* if it is c-locally connected for some c. Note that if $T \in \mathrm{Möb}(\overline{\mathbf{R}}^n)$ and D is c-locally connected or b-uniform, then $T(D)$ is c'-locally connected or, respectively, b'-uniform with $c' = c'(c)$ and $b' = b'(b)$, see e.g. [V3].

The following result nicely ties up these notions.

2.7. Lemma ([V3, 5.6 and 4.11]). *Let $D \subset \overline{\mathbf{R}}^n$, $n \ge 2$, be b-uniform and let f be K-quasiconformal on D. If fD is c-locally connected for some $c > 0$, then f is s_1-quasimöbius, where $s_1 = s_1(K, b, c, n)$. If f is s-quasimöbius for some $s > 0$, then fD is b_1-uniform, where $b_1 = b_1(s, b)$.*

3. Quasiconformally rigid domains

We assume throughout this section that $n \ge 3$. For each domain $D \subset \overline{\mathbf{R}}^n$ we let $K(D)$ denote the supremum of the numbers $K \ge 1$ such that every locally K-quasiconformal mapping on D is injective. If $K(D) > 1$, we then say that D is *quasiconformally rigid* or, simply, *rigid*.

The aforementioned result of Martio and Sarvas, Theorem 1.1, tells us that b-uniform domains are rigid with $K(D) \ge K(n, b) > 1$. We also refer to an early result of V.A. Zorič [Z] which implies that $K(\mathbf{R}^n) = \infty$. The definition can also be given by using quasiregular mappings. Indeed, for $n \ge 3$ there is a number $K(n) > 1$ such that every $K(n)$-quasiregular mapping is locally homeomorphic; see [MRV, 4.6].

3.1. Lemma. *Suppose that G is a smooth domain in $\mathbf{R}^n$ separating two points x and y in D and suppose that $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ is quasiconformal. If $f(x) = y$ and $f(z) = z$ for $z \in \bar{G}$, then $K(f) \geq K(D)$.*

Proof. Let $D(x)$ denote the x-component of $D \setminus \bar{G}$. If f is as above, define a mapping g on D by setting $g(z) = f(z)$ for $z \in D(x)$ and $g(z) = z$ for $z \in D \setminus D(x)$. It is obvious that g is locally quasiconformal on D with the dilatation $K(g) \leq K(f)$. On the other hand, g is not injective since $g(x) = y = g(y)$. Hence $K(f) \geq K(g) \geq K(D)$. □

3.2. Remark. If one applies [V2, Theorem 8] in the proof of Lemma 3.1, it is then possible to choose G to be any compact set separating x, y in D.

Lemma 3.1 gives a necessary condition for a domain to be rigid. In fact, we obtain

3.3. Theorem. *Rigid domains are c-locally connected and c depends only on n and $K(D)$.*

Proof. Suppose that D is rigid. If the condition (2.5) does not hold and $x, y \in D \cap B(x_0, r)$ belong to different components of $D \cap B(x_0, cr)$, apply [GP, Lemma 3.1]: There exists a quasiconformal mapping $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ such that $f(x) = y$, $f(z) = z$ for $z \in \overline{\mathbf{R}}^n \setminus B(x_0, cr)$, and

$$\log K(f) \leq 2(n-1)k_B(x, y),$$

where k_B is the quasihyperbolic metric (see [GP]) in the ball $B = B(x_0, cr)$. It follows from the definition of k_B that $k_B(x, y) \leq 2/(c-1)$. Thus, by Lemma 3.1, $\log K(D) \leq \log K(f) \leq 4(n-1)/(c-1)$ and (2.5) holds whenever $c > 1 + 4(n-1) \cdot (\log K(D))^{-1}$.

Since the condition (2.6) can be reduced to (2.5) by an auxiliary Möbius transformation, the above reasoning proves that D is c-locally connected and that, in addition, $c = c(n, K(D))$. □

3.4. Remark. Theorem 3.3 can also be proved by using a method of F.W. Gehring [G3, Lemma 2], to which only obvious modifications are needed. A third proof has been given by G. Martin (unpublished). We illustrate the method of Gehring in an example below in showing that the converse statement for Theorem 3.3 is false.

3.5. Examples. There are linearly locally connected domains which are not rigid.

a) Consider the domain

$$D = \{\,(t_1, t_2, \omega) \in \mathbf{R}^n = \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{n-2}\colon |t_1| < 1\,\}.$$

Clearly D is linearly locally connected. To prove that D is not rigid, apply Lemma 3.1 with $G = \{x \in D\colon t_1^2 + t_2^2 < 1\}$. Since the points $x_j = je_2$ and $y_j = -je_2$, $2 \le j$, lie in different components of $D \setminus \bar{G}$, it suffices to find quasiconformal mappings $g_j\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ such that $g_j(x_j) = y_j$, g_j fixes the points of G and $K(g_j) \to 1$ when $j \to \infty$.

For each $j \in \mathbf{N}$ define the function

$$\phi_j(s) = \max\{0, \pi(\log s)/(\log j)\}$$

and then set, using the polar coordinates in the t_1t_2-plane, $g_j(r,\phi,\omega) = \bigl(r, \phi + \phi_j(r), \omega\bigr)$ and $g_j(\infty) = \infty$. By arguing as in [G3, Lemma 1], we deduce that g_j is an $(1+\pi/\log j)$-quasi-isometry in $\mathbf{R}^n$, hence K_j-quasiconformal with $K_j = (1+\pi/\log j)^{2(n-1)}$, and so g_j does have the required properties.

b) It is equally straightforward to verify that the linearly locally connected wedge domain

$$D = \{(t_1, t_2, \omega) \in \mathbf{R} \times \mathbf{R} \times \mathbf{R}^{n-2}\colon t_1 > 0, \quad 0 < |t_2| < t_1^2\}$$

is not rigid.

4. Extension of quasiconformal mappings

The following simple observation is the link between the injectivity and extendability properties needed in the proof of Theorem 1.3.

4.1. Lemma. *Suppose that $D \subset \overline{\mathbf{R}}^n$, $n \ge 3$, is rigid and that $f\colon D \to D'$ is quasiconformal. If $K(f) < K(D)$, then D' is rigid with*

$$K(D') \ge K(D)/K(f) > 1.$$

Proof. If g is locally quasiconformal in D' and $K(g) < K(D)/K(f)$, then $g \circ f$ is locally quasiconformal in D with $K(g \circ f) \le K(g)K(f) < K(D)$. Thus $g \circ f$, and hence g, is injective. □

4.2. Lemma. *Let $D \subset \overline{\mathbf{R}}^n$, $n \ge 3$, be b-uniform. There is a number $K(n,b) > 1$ such that if f is quasiconformal on D with $K(f) \le K(n,b)$, then $D' = f(D)$ is $\tilde{b}$-uniform and f is s-quasimöbius. Here $\tilde{b} = \tilde{b}(n,b)$ and $s = s(n,b)$.*

Proof. By the Martio–Sarvas injectivity theorem, D is rigid with $K(D) \ge K_0(n,b) > 1$. If f is a quasiconformal mapping on D such that $K(f) \le \frac{1}{2}(K_0+1)$, then $D' = fD$ is rigid by Lemma 4.1 and c-locally connected by Theorem 3.3, $c = c(b,n)$. Hence the conclusion follows from Lemma 2.7. □

Lemma 4.2 gives no information about the behaviour of s when $K(f) \to 1$. However, this can be obtained by a normal family argument, presented in Lemma 4.4, and there 4.2 will be an essential part of the proof. For 4.4 we also need a quasimöbius version of the Carathéodory convergence theorem.

4.3. Lemma. *Let $D_1, D_2, \ldots$ be a sequence of b-uniform domains in $\overline{\mathbf{R}}^n$, $n \geq 2$, let f_j be s-quasimöbius mappings on D_j, and let $a_j \in D_j$. Suppose further that $0, e_1, \infty \in D_j$ and that f_j fixes each of these points. Then we can select subsequences, also denoted by D_j, f_j and a_j, which have the following properties:*

a) There is a b-uniform domain $D \subset \overline{\mathbf{R}}^n$ such that $C(D_j)$ converges to $C(D)$ in the Hausdorff metric;

b) There is an s-quasimöbius mapping f on D such that f_j converges to f uniformly on compact subsets of D;

c) There is a point $a \in \bar{D}$ such that $a_j \to a$ and $f_j(a_j) \to f(a)$ as $j \to \infty$.

Proof. Since the sets $C(D_j)$ are compact in $\overline{\mathbf{R}}^n$, we may assume that they converge to a closed set $F \subset \overline{\mathbf{R}}^n$ in the Hausdorff metric when $j \to \infty$. Here $F \neq \overline{\mathbf{R}}^n$, since each D_j contains a b-cigar connecting 0 to e_1. Similarly, it is easily seen that $D = C(F)$ is a b-uniform domain, cf. [V5, Theorem 3.6].

Moreover, it follows from Lemma 2.7 that $f_j D_j$ are $\tilde{b}$-uniform domains with $\tilde{b} = \tilde{b}(s,b)$. Thus, by further reducing to a subsequence, we also obtain $C(f_j D_j) \to C(\tilde{D})$, as $j \to \infty$, where $\tilde{D}$ is $\tilde{b}$-uniform. Now [G1, Theorems 2 and 3] provide a quasiconformal mapping $f\colon D \to \tilde{D}$ such that f_j converges to f uniformly on compact subsets of D. If $x, y, z, w \in D$ are distinct points , then

$$|f(x), f(y), f(z), f(w)| = \lim_{j\to\infty} |f_j(x), f_j(y), f_j(z), f_j(w)|,$$

and so f is quasimöbius. Hence a) and b) are proved.

For the case c), take subsequences such that $a_j \to a$ and $f_j(a_j) \to a'$ as $j \to \infty$. Here either $a \neq e_1$ or $a \neq 0$; we assume the latter. Connect a_j to 0 by a b-cigar $\mathrm{cig}(\gamma_j, 1/b) \subset D_j$. We may assume that $\gamma_j \to \gamma$ and $f_j(\gamma_j) \to \tilde{\gamma}$, where γ and $\tilde{\gamma}$ are two continua connecting 0 to a and a', respectively. Clearly $\mathrm{cig}(\gamma, 1/b) \subset D$ and since $f_j\big(\mathrm{cig}(\gamma_j, 1/b)\big) \supset \mathrm{cig}\big(f_j(\gamma_j), 1/\tilde{b}\big)$, cf. [V3, 4.11], we also have $\mathrm{cig}(\tilde{\gamma}, 1/\tilde{b}) \subset \tilde{D}$. In addition, $a \in \bar{D}$ and hence $f(a)$ is well defined.

If $x \in \gamma \setminus \{0, a\}$, then $x \in D, f(x) \in \tilde{D}$, and so it follows from b) that $f(\gamma \setminus \{0,a\}) \subset \tilde{\gamma}$. Thus $f(a) \in \tilde{\gamma}$. If $f(a) \notin \{0, a'\}$, by the above $f(a) \in \tilde{D}$, $a \in D$, and b) yields $f(a) = a'$. Therefore, we only need to show that $f(a) \neq 0$. This follows from the quasimöbius properties defined in 2.2. Indeed, $|f_j(e_1), f_j(x), f_j(0), f_j(\infty)| \leq \theta(|x|^{-1})$ and thus $|f(x)| \geq 1/\theta(|x|^{-1})$ whenever $x \in D \setminus \{0\}$. □

4.4. Lemma. *For each $s > 0$ there exists a constant $K_0 = K_0(n, b, s) > 1$ with the following property: If a mapping f is quasiconformal on a b-uniform domain $D \subset \overline{\mathbf{R}}^n$, $n \geq 3$, and if $K(f) \leq K_0$, then f is s-quasimöbius on D.*

Proof. If the claim of the lemma is not true, we can find a number $s_0 > 0$ and sequences of b-uniform domains $D_j \subset \overline{\mathbf{R}}^n$ and K_j-quasiconformal mappings f_j

on D_j, $j \in \mathbf{N}$, such that $K_j \to 1$ as $j \to \infty$ but no f_j is s_0-quasimöbius on D_j. Without loss of generality we may assume that $\infty \in D_j$ and that $f_j(\infty) = \infty$. Moreover, since the mappings f_j are not s_0-quasimöbius, there is by [V3, Theorem 3.8] a number $\tau_0 = \tau_0(s_0) > 0$ such that no f_j is τ_0- quasisymmetric.

Next, we make use of Remark 2.5 in [V4]: There are distinct points $a_j, b_j, x_j \in D_j \setminus \{\infty\}$ such that

$$(4.5) \qquad \left|\frac{a_j - x_j}{b_j - x_j}\right| = t_j \in [\tau_0, 1 + 1/\tau_0] \text{ and } \left|\frac{f_j(a_j) - f_j(x_j)}{f_j(b_j) - f_j(x_j)}\right| = t_j' > t_j + \tau_0.$$

We also normalize the mappings f_j by similarities u_j, v_j such that $u_j(0) = x_j$, $u_j(e_1) = b_j$ and $v_j\big(f_j(x_j)\big) = 0$, $v_j\big(f_j(b_j)\big) = e_1$. If $g_j = v_j \circ f_j \circ u_j$ and $\tilde{D}_j = u_j^{-1} D_j$, then g_j fixes the points $0, e_1, \infty \in \tilde{D}_j$ and (4.5) takes the form $|\tilde{a}_j| = t_j \in [\tau_0, 1 + 1/\tau_0]$ and $\big|g_j(\tilde{a}_j)\big| = t_j' > |\tilde{a}_j| + \tau_0$, where $\tilde{a}_j = u_j^{-1}(a_j)$.

We can now combine the previous lemmas. According to 4.2, for each j large enough, g_j is s_1-quasimöbius with $s_1 = s_1(b, n)$. By 4.3 we can then assume that $g_j \to g$ uniformly on compact subsets of a b-uniform domain D_0. As $K(g_j) \to 1$ and g is non-constant (4.3.b), g is 1-quasiconformal and hence a Möbius transformation. In fact, g is an isometry, since by 4.3.c) it fixes the points $\infty, 0$ and e_1. On the other hand, we have at least for a subsequence that $\tilde{a}_j \to a$, where $\tau_0 \le |a| \le 1 + 1/\tau_0$. Hence, we may use 4.3.c) again and obtain $|a| = \big|g(a)\big| = \lim\big|g_j(\tilde{a}_j)\big| \ge \lim|\tilde{a}_j| + \tau_0 = |a| + \tau_0$, which is a contradiction. □

Proof of Theorem 1.3. Let $D \subset \overline{\mathbf{R}}^n, n \ge 3$, be b-uniform and let f be quasiconformal on D. We can assume that $0 \in C(D)$, that $\infty \in D$, and that $f(\infty) = \infty$. Denote then by D_0 the domain $D \cap B\big(0, 2\,\mathrm{diam}\, C(D)\big)$. Since D_0 is bounded and $\tilde{b}$-uniform, with $\tilde{b}$ depending only on b, we can apply Theorem 1.2 in D_0. Hence, to prove 1.3, it suffices to find for each $s > 0$ a constant $K_0 = K_0(s, n, b) > 1$ such that $f|_{D_0}$ is s-quasisymmetric whenever $K(f) \le K_0$. Because $f(\infty) = \infty$, f is s-quasimöbius only if it is s-quasisymmetric, and thus the claim follows from Lemma 4.4. □

Theorem 1.3 yields plenty of rigid domains which are not uniform. We say that a domain $D \subset \overline{\mathbf{R}}^n$ is (b-)*uniformly collared*, if there is a partition of $C(D)$ into pairwise disjoint compact sets F_i, $i = 0, 1, 2, \ldots$, such that the following two conditions hold:

(4.6.a) $C(F_0)$ is a b- uniform domain.

(4.6.b) For $i \ge 1$ each F_i has a neighbourhood U_i such that $U_i \cap D$ is b- uniform and $U_i \cap U_j = \emptyset$ when $i \ne j$.

We observe that if D is uniform, then it is also uniformly collared since we may choose $F_0 = C(D)$ and $F_i = \emptyset$ for $i \ge 1$. On the other hand, it is easy to exhibit domains which are uniformly collared but not uniform. For example, in the plane the complement of the set $\{0\} \cup \{1/n\}_1^\infty$ is not uniform. Furthermore, we obtain a uniformly collared domain if we take any collection of disjoint balls $B(x_k, r_k)$, $k \in \mathbf{N}$, contained in a uniform domain D_0, choose a number $0 \leq \lambda < 1$, and set $D = D_0 \setminus \bigcup_1^\infty \bar{B}(x_k, \lambda r_k)$; cf. Figure 1.

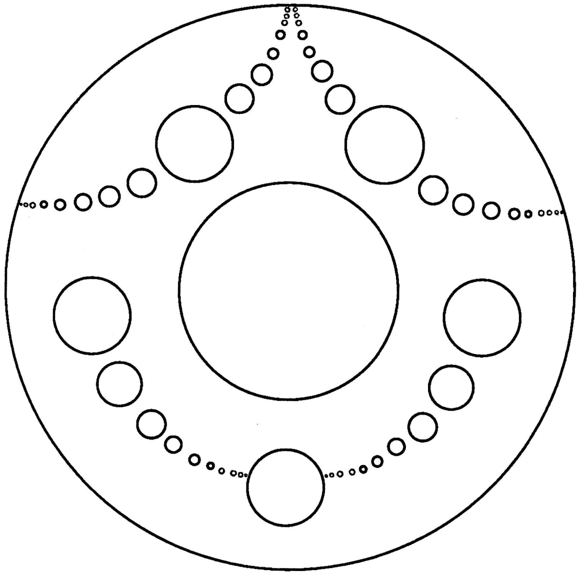

Figure 1.

4.7. Theorem. *Suppose that the domain $D \subset \overline{\mathbf{R}}^n$, $n \geq 3$, is b-uniformly collared and that f is locally quasiconformal on D. Then for every $K > 1$ there is a number $K_0 = K_0(n, b, K) > 1$ such that f is injective and admits a K-quasiconformal extension to $\overline{\mathbf{R}}^n$ whenever $K(f) \leq K_0$.*

Proof. If $C(D) = \bigcup F_i$ and U_i are the neighbourhoods of F_i as above in 4.6.a) and b), it follows from Theorems 1.1 and 1.3 that $f_i = f|_{U_i \cap D}$ has a K_1-quasiconformal extension $\tilde{f}_i\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ as soon as $K(f) \leq K_0 = K_0(n, b, K_1) \leq K_1$. Then the formula

$$\tilde{f}(x) = \begin{cases} f_i(x), & x \in U_i \setminus F_0, \\ f(x), & x \in D \end{cases}$$

defines a locally K_1-quasiconformal mapping on $C(F_0)$. Finally, since $C(F_0)$ is b-uniform, another application of 1.1 and 1.3 implies that, if K_1 is small enough (the smallness depending on n, b and K), then $\tilde{f}$ is injective and extends to a K-quasiconformal mapping of $\overline{\mathbf{R}}^n$. □

4.8. Corollary. *A uniformly collared domain D in $\overline{\mathbf{R}}^n$, $n \geq 3$, is quasiconformally rigid.*

4.9. Remark. There are also rigid domains which are not uniformly collared. To obtain particular examples, we only need to note that if $E \subset \overline{\mathbf{R}}^{n-1}$ is any compact set with empty interior, then $D = \overline{\mathbf{R}}^n \setminus E$ is rigid.

5. Rigidity in plane and space

Suppose that either f is locally conformal on a plane domain D or that f is locally quasiconformal on a domain $D \subset \overline{\mathbf{R}}^n$, $n \geq 3$. Then the natural distortion measures, the Schwarzian norm $\|S_f\|_D$ in the plane, cf. (1.4), and $\log K(f)$ in space behave in a very similar fashion. For example, if D is uniform and $\|S_f\|_D$ small, then f is injective ([MS, Theorem 4.24]) and admits a quasiconformal extension to $\overline{\mathbf{R}}^2$ (see, for instance, Theorem 5.7 below). Conversely, and in analogy with Theorem 3.3, if for some $\lambda > 0$ every locally conformal mapping on D with $\|S_f\|_D \leq \lambda$ is injective, then D is linearly locally connected by [G2]. Furthermore, both $\|S_f\|_D$ and $\log K(f)$ vanish if and only if f is a Möbius transformation.

These facts suggest that there should be a general theory for $\log K(f)$ and $\|S_f\|_D$ which also covers quasiconformal mappings in the plane. A convenient way to study this problem is to introduce the following distortion measure.

5.1. Definition. Suppose that $D \subset \overline{\mathbf{R}}^n$, $n \geq 2$, and $f\colon D \to \overline{\mathbf{R}}^n$ is continuous. We let

$$\kappa_D(f) = \inf\{\, s \geq 0\colon f|_B \text{ is } s\text{-quasimöbius for each Möbius ball } B \subset D\,\}.$$

Here, of course, a Möbius ball is the image of $B(0,1)$ under a Möbius transformation.

Clearly $0 \leq \kappa_D(f) \leq \infty$ and $\kappa_D(f) = 0$ if and only if f is a Möbius transformation. Furthermore, if $T, U \in \text{Möb}(\overline{\mathbf{R}}^n)$, then $\kappa_{U^{-1}D}(T \circ f \circ U) = \kappa_D(f)$.

5.2. Lemma. *Suppose that f is locally quasiconformal on a domain $D \subset \overline{\mathbf{R}}^n$, $n \geq 3$. Then*

$$\log K(f) \leq (n-1)\kappa_D(f). \tag{5.3}$$

Conversely, there is an increasing continuous funtion $\varphi_0\colon [0,\infty] \to [0,\infty]$ which depends only on n and satisfies

$$\kappa_D(f) \leq \varphi_0\bigl(\log K(f)\bigr), \qquad \varphi_0(0) = 0. \tag{5.4}$$

Proof. The estimate (5.3) follows from (2.3) and the converse from Theorem 1.3. Indeed, if $\varphi(\tau)$ denotes the infimum of the numbers $s \geq 0$ such that every locally e^τ-quasiconformal mapping on $B(0,1)$ is s-quasimöbius, then φ is increasing, $\kappa_D(f) \leq \varphi\bigl(\log K(f)\bigr), \varphi(0) = 0$ and $\varphi(\tau) \to 0$ as $\tau \to 0$ by Theorems 1.1 and 1.3. Consequently, we may replace φ by an increasing continuous function $\varphi_0\colon [0,\infty] \to [0,\infty]$ with $\varphi_0(0) = 0$ and $\varphi(\tau) \leq \varphi_0(\tau)$. □

5.5. Lemma. *Suppose that D is a proper subdomain of the finite plane $\mathbf{R}^2$ and that f is locally conformal on D. There exist increasing continuous functions $\varphi_1, \varphi_2\colon [0,\infty] \to [0,\infty]$ which are independent of f and D such that $\varphi_1(0) = \varphi_2(0) = 0$ and*

$$(5.6) \qquad \|S_f\|_D \le \varphi_1(\kappa_D(f)), \qquad \kappa_D(f) \le \varphi_2(\|S_f\|_D).$$

Proof. Fix the point $z_0 \in D$ and let $B = B(z_0, d(z_0, \partial D))$. If $\kappa_D(f) = s < \infty$, then by Lemma 2.7 fB is a simply connected b-uniform domain, i.e., a K_0-quasidisk, $K_0 = K_0(s)$, [MS, 2.33]. Clearly we may assume that $K_0 = 1 + \varphi(s)$ for some continuous increasing function $\varphi\colon [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$. Next, by Lehto's majorization principle, see [L, p. 73], we can estimate

$$\|S_{f|B}\|_B \le 6(K_0^2 - 1)/(K_0^2 + 1) \le 6(K_0 - 1).$$

Thus

$$|S_f(z_0)| d(z_0, \partial D)^2 \le \|S_{f|B}\|_B \le 6\varphi(\kappa_D(f)).$$

Since $z_0 \in D$ was arbitrary, we arrive at the first inequality in 5.6 with $\varphi_1(r) = 6\varphi(r), \varphi_1(\infty) = \infty$.

To obtain the latter inequality, we apply the Ahlfors–Weil extension theorem, cf. [L, II 4.1]. Indeed, if $B \subset D$ is a disk or a halfplane and if $\|S_{f|B}\|_B < 1/2$, then $f|B$ has a K-quasiconformal extension to $\overline{\mathbf{R}}^2$ with $K = (1 + 2\|S_{f|B}\|_B)/(1 - 2\|S_{f|B}\|_B)$. Since $\|S_{f|B}\|_B \le \|S_f\|_D$, $K \to 1$ when $\|S_f\|_D \to 0$, and so [TV, Theorem 2.6] yields the function φ_2. □

5.7. Theorem. *Suppose that D is a b-uniform domain in $\overline{\mathbf{R}}^n$, $n \ge 2$ and that f is locally quasiconformal on D. For every $K > 1$ there is a number $\kappa_0 = \kappa_0(n, b, K) > 0$ such that f is injective and has a K-quasiconformal extension to $\overline{\mathbf{R}}^n$ whenever $\kappa_D(f) \le \kappa_0$.*

Proof. We reduce the proof to the previous arguments and results; a direct proof could, of course, be described along similar lines. In fact, when $n \ge 3$, the theorem is a reformulation of 1.1 and 1.3. When $n = 2$, we assume that $D \subset \overline{\mathbf{R}}^2$; by appealing to the measurable Riemann mapping theorem, see e.g. [L, pp. 68], we write f as a product $f = g \circ \Phi$, where $\Phi\colon \overline{\mathbf{R}}^2 \to \overline{\mathbf{R}}^2$ is quasiconformal, $\Phi(\infty) = \infty$ and g is locally conformal on $D' = \Phi(D)$. Since $K(\Phi) = K(f) \le 1 + \kappa_D(f)$, it suffices to prove that if $\kappa_D(f)$ is small, then g is injective and admits a K-quasiconformal extension to $\overline{\mathbf{R}}^2$ with $K \to 1$ as $\kappa_D(f) \to 0$.

If $\kappa_D(f) \le 1$, Φ is 2-quasiconformal and thus D' is $\tilde{b}$-uniform, $\tilde{b} = \tilde{b}(b)$. By [MS, Theorem 4.24] g is injective whenever $\|S_g\|_{D'} \le \varepsilon_0 = \varepsilon_0(\tilde{b})$. Moreover, imitating the argument that we used in Lemmas 4.1 and 4.2, we see that if $\|S_g\|_{D'}$ is small enough, gD' is first linearly locally connected, then uniform and so g

is s_0-quasimöbius, $s_0 = s_0(\tilde{b})$. Now Lemma 4.3 applies and we deduce as in Lemma 4.4 that g is s-quasimöbius, where $s \to 0$ when $\|S_g\|_{D'} \to 0$, and again the claim follows from Theorem 1.2.

Consequently, it remains to show that $\|S_g\|_{D'} \to 0$ as $\kappa_D(f) \to 0$. To prove this, note that Mori's classical distortion theorem gives an absolute constant $\lambda > 0$ such that for each $z \in D'$ we can find a disk $B \subset D$ with $B(z, \lambda d(z, \partial D')) \subset \Phi(B)$. Thus

$$\begin{aligned} |S_g(z)| d(z, \partial D')^2 &\le \lambda^{-2} \|S_g\|_{\Phi(B)} \le \lambda^{-2} \varphi_1\big(\kappa_{\Phi(B)}(g)\big) \\ &= \lambda^{-2} \varphi_1\big(\kappa_{\Phi(B)}(f|_B \circ \Phi^{-1})\big), \end{aligned}$$

where φ_1 is the function of Lemma 5.5. Moreover, it follows from Theorem 1.2 that $f|_B \circ \Phi|_B^{-1}$ has a K_0-quasiconformal extension to $\overline{\mathbf{R}}^2$ with K_0 depending only on $\kappa_D(f)$ and that $K_0 \to 1$ as $\kappa_D(f) \to 0$. Therefore, by [TV, Theorem 2.6], $\kappa_{\Phi(B)}(f|_B \circ \Phi^{-1})$ approaches zero with $\kappa_D(f)$. □

5.8. Problem. If $D \subset \overline{\mathbf{R}}^2$ is a domain, let $\kappa(D)$ denote the supremum of the numbers $s \ge 0$ such that every locally quasiconformal mapping f on D with $\kappa_D(f) \le s$ is injective. Define similarly $\sigma(D)$ for locally conformal mappings and for $\|S_f\|_D$. Is it true that $\kappa(D) > 0$ if and only if $\sigma(D) > 0$?

In the previous section we saw that Theorem 5.7 remains valid also in the more general uniformly collared domains, when $n \ge 3$. However, in the plane the situation is different.

5.9. Example. Let

$$D = \mathbf{R}^2 \setminus \bigcup_{m,n \in \mathbf{Z}} B\big(m + in, (2\sqrt{2})^{-1}\big).$$

If $\varepsilon > 0$ is given, let $f(z) = \exp(\varepsilon z)$, $z \in D$. Then $\|S_f\|_D = \varepsilon/\sqrt{2} < \varepsilon$, but f is not injective in D, $f(1/2) = f\big(1/2 + (2\pi/\varepsilon)i\big)$. Thus Theorem 5.7 does not hold in all uniformly collared planar domains D.

In conclusion, Theorem 5.7 shows that a general theory with $\kappa_D(f)$ or a joint theory for $\log K(f)$ and $\|S_f\|_D$ is possible in many important cases but, in the light of the above example, the plane still has its peculiarities.

5.10. Remark. One can obtain injectivity and extendability results for uniformly collared planar domains D if additional assumptions are made on the geometric distribution of the components of $C(D)$. As an example, we mention that a domain $D \subset \overline{\mathbf{R}}^2$ is κ_D-rigid ($\kappa(D) > 0$) if it is uniformly collared and if in (4.6) F_0 can be so chosen that for some $\lambda > 0$, $d(x, F_0) \le d(y, F_0)/\lambda$ whenever x, y belong to the same component of

$$D_\lambda = \{\, z \in D \colon d(z, \partial D) < \lambda d(z, F_0) \,\}.$$

This can be proved with a combination of the methods of 5.7 and [GO, Theorem 7]; we omit the details.

Acknowledgements. The authors wish to express their gratitude to F.W. Gehring and J. Väisälä for their valuable remarks related to the topics of this paper.

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Received 11 June 1987