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MORI'S THEOREM FOR n-DIMENSIONAL QUASICONFORMAL MAPPINGS

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1. Introduction

In this paper we shall study distortion properties of quasiconformal mappings in two cases. The first case deals with quasiconformal mappings of the unit ball B^n in $\mathbf{R}^n$ for which we generalize a classical theorem of Akira Mori (see [2], p. 47, or [16], p. 66). The second case deals with quasiconformal mappings of the whole space $\mathbf{R}^n$ which keep the x_1-axis pointwise fixed. In both cases our results will have the correct limiting behavior as $K \to 1$. Furthermore, all the estimates involved are explicitly computable. We shall also study conformal mappings onto quasidisks.

In 1956 the following theorem of A. Mori appeared [18].

1.1. Theorem. *A K-quasiconformal mapping f of the unit disk B^2 onto itself with $f(0)=0$ satisfies*

$$|f(x)-f(y)| \le 16|x-y|^{1/K} \tag{1.2}$$

for all $x,y \in B^2$. Furthermore, the constant 16 in (1.2) cannot be replaced by any smaller constant independent of K.

The main result of this paper is the following generalization of Theorem 1.1.

1.3. Theorem. *Let f be a K-quasiconformal mapping of B^n onto B^n, $n \ge 2$, with $f(0)=0$. Then*

$$|f(x)-f(y)| \le M_1(n,K)|x-y|^{\alpha} \tag{1.4}$$

for all $x,y \in B^n$ where $\alpha = K^{1/(1-n)}$ and the constant $M_1(n,K)$ has the following three properties:

$$\begin{aligned} &(1) \quad M_1(n,K) \to 1 \text{ as } K \to 1, \text{uniformly in } n;\\ &(2) \quad M_1(n,K) \text{ remains bounded for fixed } K \text{ and varying } n;\\ &(3) \quad M_1(n,K) \text{ remains bounded for fixed } n \text{ and varying } K. \end{aligned} \tag{1.5}$$

An n-dimensional version of Mori's theorem has already been given in [20]. In [11], Remark 1 on p. 235, it is said that this theorem holds with a constant

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satisfying (3) (namely $M_1(n,K) \le 4\lambda_n^2$ in our notation of Section 2), and in [14] the inequality (1.4) is also proved, but with a constant that does not satisfy any of these three properties. In Section 2 (Theorem 2.28) we shall give explicit bounds from above for the constant $M_1(n,K)$ which actually hold in a wider class of mappings of the unit ball (cf. (2.15)). For an extension of Mori's theorem to more general domains the reader is referred to [12], Corollary 3.30, and, for a recent application of it, to [8], Section 10.

In Section 3 we shall prove

1.6. Theorem. *Let $f : \mathbf{R}^n \to \mathbf{R}^n$ be a K-quasiconformal mapping which keeps the x_1-axis pointwise fixed. If $K > 1$, then*

$$|f(x)| \le \lambda_n^{2\beta-2} \frac{\beta^\beta}{(\beta-1)^{\beta-1}} |x| \tag{1.7}$$

for all $x \in \mathbf{R}^n$ where $\beta = K^{1/(n-1)}$ and λ_n is the Grötzsch ring constant (see Section 2).

This theorem is a sharpened version of Corollary 2.17 in [4]. Observe that the constant in (1.7) tends to one as $K \to 1$. Finally, in the last section, we apply these results to plane conformal mappings of the unit disk onto bounded K-quasidisks, again paying attention to the limiting behavior as $K \to 1$.

It is conjectured (cf. [16], p. 68) that the best constant in (1.2) is $16^{1-1/K}$, in place of 16. E. Reich has kindly informed us that his student G.P. Schwartz proved Mori's theorem (1.2) with the constant $360^{1-1/K}$ in place of 16, in an unpublished Ph.D. thesis in 1970. Schwartz' work relies heavily on the parametric representation of plane quasiconformal mappings and is therefore restricted to the two-dimensional case. A further improvement in the plane case has also been given in [19].

We shall adopt the relatively standard notation of [22], i.e., $e_1, \ldots, e_n$ denote the orthogonal unit basis vectors, $B^n(x,r)$ the ball with center x and radius $r > 0$, $S^{n-1}(x,r) = \partial B^n(x,r)$, $B^n(r) = B^n(0,r)$, $S^{n-1}(r) = \partial B^n(r)$, $B^n = B^n(1)$, $S^{n-1} = \partial B^n$ and ω_{n-1} the $(n-1)$-dimensional Lebesgue measure of S^{n-1}. In particular, we employ the definition of K-quasiconformal mapping given in [22], p. 42.

2. Mori's theorem

We shall next introduce some notation and some estimates necessary for the sequel.

A domain R in $\mathbf{R}^n$ is called a ring or a ring domain if its complement in $\overline{\mathbf{R}}^n$ consists of two components. Its conformal capacity is denoted by $\operatorname{cap} R$. By $R_{G,n}(t)$, $t > 1$, we denote the Grötzsch ring whose complementary components consist of the closed unit ball $\bar{B}^n$ and the ray $[te_1, \infty] = \{se_1 \colon s \ge t\}$, and by $R_{T,n}(t)$, $t > 0$, the Teichmüller ring whose complementary components are $[-e_1, 0] = \{ se_1 \colon -1 \le s \le 0 \}$ and $[te_1, \infty]$. For their capacities we write

$$\gamma_n(t) = \operatorname{cap} R_{G,n}(t),$$
$$\tau_n(t) = \operatorname{cap} R_{T,n}(t).$$

These functions are related by the functional identity

$$\gamma_n(t) = 2^{n-1}\tau_n(t^2 - 1) \tag{2.1}$$

(cf. [9], Lemma 6). Later we shall also use the estimation ([9], Lemma 8)

$$\gamma_n(t) \ge \omega_{n-1}(\log \lambda_n t)^{1-n}, \quad t > 1, \tag{2.2}$$

where $\lambda_n \in [4, 2e^{n-1}]$ is the Grötzsch ring constant (cf. [10]; for these estimations from above see [3] and from below [7], [13]; note also that $\lambda_2 = 4$ [16]).

For $K > 0$ we define a homeomorphism $\varphi_{K,n} \colon [0,1] \to [0,1]$ with $\varphi_{K,n}(0) = 0$, $\varphi_{K,n}(1) = 1$ and

$$\varphi_{K,n}(t) = \frac{1}{\gamma_n^{-1}\big(K\gamma_n(1/t)\big)}, \quad 0 < t < 1. \tag{2.3}$$

Throughout this paper we use α and β to denote the following numbers

$$\alpha = K^{1/(1-n)}, \quad \beta = 1/\alpha.$$

The following important estimates (due to Wang [26] for $n = 2$ and generalized to $n \ge 2$ in [4]) are essential for the sequel

$$\varphi_{K,n}(t) \le \lambda_n^{1-\alpha} t^{\alpha}, \tag{2.4}$$
$$\varphi_{1/K,n}(t) \ge \lambda_n^{1-\beta} t^{\beta}, \tag{2.5}$$

where $K \ge 1$. For $n = 2$, (2.4) is given also in [16] p. 65.

The Poincaré metric $\varrho(x,y)$ on B^n is defined by (cf.[6])

$$\tanh^2 \tfrac{1}{2}\varrho(x,y) = \frac{|x-y|^2}{|x-y|^2 + (1-|x|^2)(1-|y|^2)}. \tag{2.6}$$

It is easy to show (see [5], 3.2) that

$$|x - y| \le 2\tanh \tfrac{1}{4}\varrho(x,y) \tag{2.7}$$

for all $x, y \in B^n$.

The following theorem, a quasiconformal counterpart of the Schwarz lemma, is a conformally invariant formulation of Theorem 3.1 in [17] (cf. [23], 3.3).

2.8. Theorem. *Let f be a K-quasiregular mapping of the unit ball B^n into B^n. Then*

$$\tanh \tfrac{1}{2}\varrho(f(x), f(y)) \le \varphi_{K,n}\left(\tanh \tfrac{1}{2}\varrho(x,y)\right) \tag{2.9}$$

for all $x, y \in B^n$.

2.10. Corollary. *Let $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ be a K-quasiconformal mapping with $f(0) = 0$, $f(\infty) = \infty$ and $fB^n \subset B^n$. If $s > 1$ and $|x| \le s$, then*

$$\left|f(x)\right| \le \gamma_n^{-1}\left(\gamma_n(s)/K\right).$$

Proof. This inequality follows easily by inversion, application of Theorem 2.8 to the inverse mapping and formula (2.3).

As in [4] we define

$$H_n(K) = \sup \frac{\left|f(x)\right|}{\left|f(y)\right|} \tag{2.11}$$

where the supremum is taken over all K-quasiconformal mappings $f\colon \mathbf{R}^n \to \mathbf{R}^n$ with $f(0) = 0$ and over all pairs of points x, y in $\mathbf{R}^n$ with $|x| = |y| > 0$. From [24] and (2.5) we have

$$H_n(K) \le 1/\varphi_{1/K,n}(1/\sqrt{2})^2 \le \lambda_n^{2\beta-2} 2^\beta. \tag{2.12}$$

Since $\lambda_n \le 2e^{n-1}$ we get as in [4] a dimension-free bound for $H_n(K)$, namely $\lambda_n^{1-\alpha} \le 2^{1-1/K}K$, and hence $H_n(K) \le 2^{3K-2}K^{2K}$. Therefore this number remains bounded for fixed K and varying n. We also observe that $\lambda_n^{1-\alpha} \to 1$ as $K \to 1$, uniformly in n. Next we shall use the fact that

$$\lim_{K\to 1} H_n(K) = 1$$

for every $n \ge 2$. This can be concluded by a normal family argument. A quantitative inequality with this property has been given in [25], namely

$$H_n(K) \le \lambda_n^{2(\beta^2-1)} \exp\left(3K(K+1)\sqrt{K-1}\right), \tag{2.13}$$

for all $K \ge 1$ and $n \ge 2$. Hence, as $\lambda_n^{1-\alpha}$ and α, also $H_n(K)$ tends to one for $K \to 1$, uniformly in n.

Taking into account that a K-quasiconformal mapping $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ with $f(0) = 0$ and $f(\infty) = \infty$ maps the ball $B^n(s)$ into $B^n\left(H_n(K)|f(se_1)|\right)$, we note that by Corollary 2.10 we get

2.14. Corollary. *Let* $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ *be a* K*-quasiconformal mapping with* $f(0)=0$ *and* $f(\infty)=\infty$. *If* $|x| \le s|y|$, $s>1$, *then*

$$|f(x)| \le H_n(K)\gamma_n^{-1}(\gamma_n(s)/K)|f(y)|.$$

We observe that this estimation is similar to Theorem 2.12 in [4] where the constant is

$$1+\tau_n^{-1}(\tau_n(s)/K) \overset{(2.1)}{=} \left(\gamma_n^{-1}(\gamma_n(\sqrt{1+s})/K)\right)^2,$$

which is better in general and applies to all values $s>0$. However, the constant in Corollary 2.14 has the advantage that it tends to s for $K\to 1$, it is hence sharp. Finally, we want to add the remark that in [1] it is shown that for $n=2$ the sharp constant is $\tau_2^{-1}(\tau_2(s)/K)$ for all $s \ge 1$.

Every mapping satisfying the assumptions of Theorem 1.3 can be extended by reflection to a K-quasiconformal mapping of the whole space $\overline{\mathbf{R}}^n$. This leads us to the

2.15. Definition. $M_1(n,K)$ *is the smallest number such that* (1.4) *holds for all* $x,y\in B^n$ *and for all* K*-quasiconformal mappings* $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ *with* $f(0)=0$, $f(\infty)=\infty$ *and* $fB^n \subset B^n$.

We prove now that $M_1(n,K)$ satisfies (1.5) from which Theorem 1.3 then follows. Let f be as in the definition above and fix $x,y\in B^n$. First we prove part (3) of (1.5) (and in particular that $M_1(n,K)\le 3\lambda_n^2$). To this end we employ a fairly straightforward generalization of the 2-dimensional argument in [16], p. 66.

Proof of part (3). The proof is divided into two cases. Consider first the case when

$$|x-y|^2+(1-|x|^2)(1-|y|^2) \ge 1/16.$$

Then by (2.6)

$$\tanh \tfrac{1}{2}\varrho(x,y) \le 4|x-y|. \tag{2.16}$$

Furthermore, by (2.7) and Theorem 2.8

$$\begin{aligned}|f(x)-f(y)| &\le 2\tanh \tfrac{1}{4}\varrho(f(x),f(y)) \le 2\tanh \tfrac{1}{2}\varrho(f(x),f(y))\\ &\le 2\varphi_{K,n}\left(\tanh \tfrac{1}{2}\varrho(x,y)\right),\end{aligned}$$

and by (2.4) and (2.16)

$$|f(x)-f(y)| \le 2\lambda_n^{1-\alpha}\left(\tanh \tfrac{1}{2}\varrho(x,y)\right)^\alpha \le 2\lambda_n^{1-\alpha}4^\alpha|x-y|^\alpha. \tag{2.17}$$

In the remaining second case we have $|x-y|\le 1/4$ and $(1-|x|^2)(1-|y|^2)\le 1/16$. We may assume that $1-|x|^2 \le 1/4$, so $|x| \ge \sqrt{3}/2 > 0.85$. Hence

$$\tfrac{1}{2}|x+y| = \left|x+\tfrac{1}{2}(y-x)\right| \ge |x| - \tfrac{1}{2}|x-y| > 0.7.$$

Then the ring domain

$$A = \{ z \in \mathbf{R}^n \colon \tfrac{1}{2}|x-y| < |z - \tfrac{1}{2}(x+y)| < \tfrac{1}{2} \}$$

separates the origin and infinity from x and y, so fA separates 0 and ∞ from $f(x)$ and $f(y)$. By performing a spherical symmetrization we obtain by [9] (Lemma 2.6 in [4])

$$\operatorname{cap} fA \geq \tau_n \left(\frac{|f(y)|}{|f(x)-f(y)|} \right).$$

Furthermore, we have

$$\operatorname{cap} fA \leq K \operatorname{cap} A = K\omega_{n-1} \left(\log(1/|x-y|)\right)^{1-n}.$$

The functional identity (2.1) gives

$$2^{n-1}K\omega_{n-1} \left(\log(1/|x-y|)\right)^{1-n} \geq \gamma_n \left(\sqrt{\frac{|f(x)-f(y)|+|f(y)|}{|f(x)-f(y)|}} \right).$$

Then we use $|f(x)|$, $|f(y)| \leq 1$, the fact that γ_n is decreasing and (2.2) to infer that the right side is larger than

$$\omega_{n-1} \left(\log \left(\lambda_n \sqrt{\frac{3}{|f(x)-f(y)|}} \right) \right)^{1-n}.$$

Hence

$$\log \frac{1}{|x-y|} \leq 2\beta \log \left(\lambda_n \sqrt{\frac{3}{|f(x)-f(y)|}} \right)$$

and finally

$$|f(x)-f(y)| \leq 3\lambda_n^2 |x-y|^{\alpha}. \tag{2.18}$$

Since $4 \leq \lambda_n$, the inequality (2.18) holds in both cases (cf. (2.17)). Hence part (3) of (1.5) is proved with $M_1(n,K) \leq 3\lambda_n^2$.

Proof of part (1) and (2). Fix $s > 1$. Corollary 2.10 implies that f maps the ball $B^n(s)$ into $B^n(c)$ where $c = \gamma_n^{-1}(\gamma_n(s)/K)$. We define $g(z) = f(sz)/c$ and note that g maps $B^n(1/s)$ into $B^n(1/c)$. We put

$$a = \tfrac{1}{2}\varrho\left(\frac{x}{s}, \frac{y}{s}\right).$$

By (2.6) and $|x/s|,\ |y/s| \le 1/s$ we have

$$\tanh a \le \frac{|x/s - y/s|}{\sqrt{(1-|x/s|^2)(1-|y/s|^2)}} \le \frac{s|x-y|}{s^2-1}. \tag{2.19}$$

Application of Theorem 3.4 in [5] to the mapping g gives

$$\bigl|f(x)-f(y)\bigr| = c\bigl|g(x/s)-g(y/s)\bigr| \le c\frac{2\varphi_{K,n}(\tanh a)}{1+\sqrt{1-\varphi_{K,n}^2(\tanh a)}}. \tag{2.20}$$

From (2.3) and (2.5) we use

$$c \le \lambda_n^{\beta-1} s^{\beta}, \tag{2.21}$$

from (2.4) and (2.19)

$$\varphi_{K,n}(\tanh a) \le \min\left\{1, \lambda_n^{1-\alpha}\left(\frac{s}{s^2-1}\right)^{\alpha}|x-y|^{\alpha}\right\} \tag{2.22}$$

and hence we get

$$\bigl|f(x)-f(y)\bigr| \le \lambda_n^{\beta-1} s^{\beta} \frac{2\lambda_n^{1-\alpha}\bigl(s/(s^2-1)\bigr)^{\alpha}|x-y|^{\alpha}}{1+\sqrt{1-\min\{1,\lambda_n^{2-2\alpha}\bigl(s/(s^2-1)\bigr)^{2\alpha}|x-y|^{2\alpha}\}}}. \tag{2.23}$$

This inequality holds for all $s > 1$. We choose s (which depends on K) such that $s^{\beta+\alpha}/(s^2-1)^{\alpha}$ becomes minimal. This amounts to putting

$$s = \sqrt{\frac{\beta^2+1}{\beta^2-1}}.$$

A straightforward computation shows that we have proved that

$$M_1(n,K) \le \theta(n,K)\lambda_n^{\beta-\alpha}\frac{(\beta^2+1)^{\beta/2+\alpha/2}}{2^{\alpha}(\beta^2-1)^{\beta/2-\alpha/2}} \tag{2.24}$$

where

$$\theta(n,K) = \frac{2}{1+\sqrt{1-\min\{1,\lambda_n^{2-2\alpha}(\beta^4-1)^{\alpha}\}}}. \tag{2.25}$$

For $K \to 1$, $\lambda_n^{1-\alpha}$ tends to one, uniformly in n, as well as α and β do. Hence $\theta(n,K) \to 1$ uniformly in n and so does $M_1(n,K)$, since

$$(\beta^2-1)^{\beta/2-\alpha/2} = (\beta+1)^{\beta/2-\alpha/2}\beta^{1/2-\alpha/2}\sqrt{(\beta-1)^{\beta-1}(1-\alpha)^{1-\alpha}}.$$

Part (1) is proved and part (2) is now evident, since the following bounds do not depend on n:

$$\theta(n,K) \le 2, \qquad \lambda_n^{\beta-\alpha} \le (2^{1-1/K}K)^{K+1},$$

$$2^{-\alpha}(\beta^2+1)^{\beta/2+\alpha/2} \le (K^2+1)^{K/2+1/2}, \qquad (\beta^2-1)^{\alpha/2-\beta/2} \le \exp(-1/e).$$

The proof is complete.

Finally we derive a

2.26. Corollary. *Let* $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ *be* K*-quasiconformal with* $f(0)=0$, $f(e_1)=e_1$ *and* $f(\infty)=\infty$. *Then*

$$|f(x)-f(y)| \le M_2(n,K)|x-y|^{\alpha} \tag{2.27}$$

for all $x,y \in B^n$ *where the constant* $M_2(n,K)$ *has the properties*

(1) $M_2(n,K) \le H_n(K)M_1(n,K)$;
(2) $M_2(n,K) \to 1$ *as* $K \to 1$, *uniformly in* n;
(3) $M_2(n,K)$ *remains bounded for fixed* K *and varying* n.

Proof. Since such a mapping f maps B^n into $B^n(0,H_n(K))$, the former result applied to $f(z)/H_n(K)$ yields (2.27) and (1). From (1), the properties of $M_1(n,K)$, (2.12) and (2.13) we get (2) as well as (3).

Remark. For any constant $M_2(n,K)$ satisfying (2.27) clearly $M_2(n,K) \to \infty$ for fixed n and $K \to \infty$ as the example f_0 shows where f_0 is the identity in the right half space and an affine stretching in the left half space. A quantitative better lower bound is obtained by observing that (2.27) with $x=0$, $|y|=1$ implies that

$$M_2(n,K) \ge H_n(K) \ge \lambda(\beta)$$

where the last inequality is (1.14) in [4]. Here $\lambda(K)$ is a well-known transcendental function (cf.[16], p. 81). In fact

$$\lambda(K) = \left(\frac{\varphi_{K,2}(1/\sqrt{2})}{\varphi_{1/K,2}(1/\sqrt{2})}\right)^2$$

and hence $\lambda(K) \to 1$ as $K \to 1$ and $\lambda(K) \to \infty$ as $K \to \infty$. For the constant $M_1(n,K)$ we have the lower bound $4^{1-\alpha}$. This follows by rotation of the extremal plane quasiconformal mapping between the extremal Grötzsch ring domains (as it is done in the proof of Theorem 4.9 in [4]) and the fact that $\lim_{r\to 0} r^{-\alpha}\varphi_{1/\alpha,2}(r) = 4^{1-\alpha}$ (see p. 65 in [16]). In [16], p. 68, it is also shown that $M_1(2,K) \ge 16^{1-1/K}$.

We collect our results:

2.28. Theorem. *The constant* $M_1(n,K)$ *satisfies* $M_1(n,K) \le 3\lambda_n^2$ *and*

$$M_1(n,K) \le \theta(n,K)\lambda_n^{\beta-\alpha}\frac{(\beta^2+1)^{\beta/2+\alpha/2}}{2^{\alpha}(\beta^2-1)^{\beta/2-\alpha/2}}$$

where $\theta(n,K)$ *is given by* (2.25), *in particular,* $\theta(n,K) \in [1,2]$ *and* $\theta(n,K) \to 1$ *for* $K \to 1$.

Remark. It is not known if there exists an upper bound for $M_1(n,K)$ which is independent of n and K.

We close this section by considering the special case $n=2$ and improving Theorem 2.28 slightly.

2.29. Theorem. *The constant* $M_1(2,K)$ *satisfies* $M_1(2,K) \le 16$ *and*

$$M_1(2,K) \le \left(1+\varphi_{K,2}\left(\frac{K^2-1}{K^2+1}\right)\right) 2^{2K-3/K}\frac{(K^2+1)^{(K+1/K)/2}}{(K^2-1)^{(K-1/K)/2}}.$$

Proof. $M_1(2,K) \le 16$ is the original content of Theorem 1.1, and from its proof in [16], p. 67, it is clear that this constant also holds for our definition (2.15) of the constant $M_1(2,K)$. For the second part we use the same notation as in the preceding proof and recall (2.20) for $n=2$:

$$|f(x)-f(y)| \le c\frac{2\varphi_{K,2}(\tanh a)}{1+\sqrt{1-\varphi^2_{K,2}(\tanh a)}}. \tag{2.30}$$

For two points $u,v \in B^n(r)$ with $0 \le r \le 1$ we have

$$\tanh \tfrac{1}{2}\varrho(u,v) \le \frac{2r}{1+r^2}, \tag{2.31}$$

because $\varrho(u,v)$ is maximized in $\bar{B}^n(r)$for opposite points on $S^{n-1}(r)$ (where its value is $2\log((1+r)/(1-r))$ and $\tanh\log[(1+r)/(1-r)] = 2r/(1+r^2)$. Next we use the functional identity

$$\frac{2}{1+\sqrt{1-\varphi^2_{K,2}(2r/(1+r^2))}} = 1+\varphi_{K,2}(r^2). \tag{2.32}$$

This can be derived from the identities

$$\varphi_{K,2}(r) = \sqrt{1-\varphi^2_{1/K,2}(\sqrt{1-r^2}\,)}, \tag{2.33}$$

$$\varphi_{K,2}(r) = \frac{1-\varphi_{1/K,2}((1-r)/(1+r))}{1+\varphi_{1/K,2}((1-r)/(1+r))} \tag{2.34}$$

which follow from [16], (2.7) and (2.9) on p. 61, by applying the function $\mu(r) = 2\pi/\gamma_2(1/r)$ to (2.33) and (2.34) and recalling that $\varphi_{K,2}(r) = \mu^{-1}(\mu(r)/K)$.

Since $|x/s|$, $|y/s| \le 1/s$ we have by (2.31) with $r = 1/s$

$$\varphi_{K,2}(\tanh a) \le \varphi_{K,2}\left(\frac{2/s}{1+1/s^2}\right)$$

and with (2.30) and (2.32)

$$|f(x) - f(y)| \le c(1 + \varphi_{K,2}(1/s^2))\varphi_{K,2}(\tanh a),$$

finally (2.4), (2.19) and $c \le 4^{K-1}s^K$ give

$$|f(x) - f(y)| \le 4^{K-1}s^K\,(1 + \varphi_{K,2}(1/s^2))\,4^{1-1/K}\left(\frac{s}{s^2-1}\right)^{1/K}|x-y|^{1/K}$$

and, as before, the choice $s = \sqrt{(K^2+1)/(K^2-1)}$ yields the desired bound.

3. Mappings keeping an axis pointwise fixed

In this section we study the distortion of K-quasiconformal mappings $f\colon \overline{\mathbf{R}}^n \to \overline{\mathbf{R}}^n$ with the property

(3.1) $$f(te_1) = te_1 \qquad \text{for all } t \in \mathbf{R}.$$

Proof of Theorem 1.6. Let f be a K-quasiconformal mapping of $\mathbf{R}^n$ satisfying (3.1). In order to study the quantity $|f(x)|/|x|$ we may evidently assume that $|x| = 1$ and that $f(x)$ is in the right half space (first co-ordinate non-negative). Then we fix $s > 0$ and consider the ring R' whose complement consists of $[-se_1, 0]$ and $\{\,f(x) + t(f(x) + e_1)\colon t \ge 0\,\}$. We put $a = |f(x) + se_1|$, and hence $a^2 \ge |f(x)|^2 + s^2$. By Lemma 2.58 in [23], which is due to Gehring (Lemma 2.7 in [4]), we have

(3.2) $$\operatorname{cap} R' \le \tau_n\left(\frac{a}{s} - 1\right).$$

On the other hand, we put $R = f^{-1}(R')$ and conclude by [9] (Lemma 2.6 in [4])

(3.3) $$\operatorname{cap} R \ge \tau_n\left(\frac{1}{s}\right).$$

(3.2), (3.3) and $\operatorname{cap} R \le K \operatorname{cap} R'$ then yield

(3.4) $$1 + \tau_n^{-1}\left(\tau_n\left(\frac{1}{s}\right)/K\right) \ge \frac{a}{s} \ge \sqrt{1 + |f(x)|^2/s^2}.$$

From the functional identity (2.1) and the definition (2.3) we infer that the left side in (3.4) is equal to

$$c := 1/\varphi_{1/K,n}^2\left(1/\sqrt{1+1/s}\,\right).$$

Hence

$$|f(x)| < s\sqrt{c^2-1}. \tag{3.5}$$

If we choose $s = 1$, then this proof reduces to the one given in [4], and (3.5) reduces to the bound given there. To get a bound that gives the right behavior for $K \to 1$ we use from (3.5)

$$|f(x)| \le sc$$

and then (2.5) to get

$$|f(x)| \le \lambda_n^{2\beta-2} s \left(\frac{s+1}{s}\right)^{\beta}.$$

This holds for any $s > 0$. The best choice is $s = \beta - 1$ which yields

$$|f(x)| \le \lambda_n^{2\beta-2} \frac{\beta^\beta}{(\beta-1)^{\beta-1}}$$

and Theorem 1.6 is proved.

In the special case $n = 2$ the set of values which can be taken by K-quasiconformal mappings satisfying (3.1) is known for any subset of $\mathbf{R}^2$. Namely, f then maps the upper half plane onto itself keeping the boundary points fixed, so Teichmüller's Verschiebungssatz [21] then provides the answer. This result easily shows (see [15]) that the set of values $f(x)$ of such mappings f at a given point x is a hyperbolic disk with center x and radius

$$\varrho(K) = 2\arctan \mu^{-1}\Big(\log((\sqrt{K}+1)\ /\ (\sqrt{K}-1))\Big),$$

where $\mu(r) = 2\pi/\gamma_2(1/r)$ as above. Hence the possible set of values attained on S^1 is the set of all points x in the upper half plane with hyperbolic distance to S^1 less or equal $\varrho(K)$ as well as its mirror image in the lower half plane and the points 1 and -1. The euclidean distance of x and $f(x)$ is hence maximal for $x = i$ and $f(x) = i\exp\varrho(K)$. Therefore (1.7) holds with the sharp constant $\exp\varrho(K) = (1+\mu^{-1}(t))\ /\ (1-\mu^{-1}(t))$ instead of $\lambda_n^{2\beta-2}\beta^\beta/(\beta-1)^{\beta-1}$ where $t = \log((\sqrt{K}+1)/(\sqrt{K}-1))$.

4. Conformal mappings of the unit disk onto quasidisks

A plane domain D is called a K-quasidisk if there exists a K-quasiconformal mapping $g\colon \overline{\mathbf{R}}^2 \to \overline{\mathbf{R}}^2$ with $gD = B^2$. We first prove

4.1. Lemma. *Let D be a K-quasidisk with $0 \in D$ and $\max\{|z|\colon z \in \partial D\} = 1$. If $r = \min\{|z|\colon z \in \partial D\}$ then there is a number $K_1 = K_1(r,K)$ such that there is a K_1-quasiconformal mapping $g_1\colon \overline{\mathbf{R}}^2 \to \overline{\mathbf{R}}^2$ with $g_1(0) = 0$, $g_1(\infty) = \infty$ and $g_1 D = B^2$ where $K_1 \to 1$ as $K \to 1$ and $r \to 1$.*

Remark. To achieve $K_1 \to 1$ it is necessary to let r tend to one, too.

Proof. Let g be the K-quasiconformal mapping in the definition of the bounded K-quasidisk D. By a Möbius transformation we may assume that $g(\infty) = \infty$. First we find an upper bound for $|g(0)|$ (and may hence assume that $g(0) \neq 0$).

Let R' be the ring with complementary components $[g(0), g(0)/|g(0)|]$ and $[-\infty, -g(0)/|g(0)|]$. Then

$$\operatorname{cap} R' = \tau_2\left(\frac{1+|g(0)|}{1-|g(0)|}\right). \tag{4.2}$$

Next we put $R = g^{-1}(R')$ and conclude as in the preceding section by [9] or Lemma 2.6 in [4]

$$\operatorname{cap} R \geq \tau_2\left(\frac{1}{r}\right). \tag{4.3}$$

From $\operatorname{cap} R \leq K \operatorname{cap} R'$, and (4.2) and (4.3) we derive that

$$\frac{1+|g(0)|}{1-|g(0)|} \leq \tau_2^{-1}\left(\frac{1}{K}\tau_2\left(\frac{1}{r}\right)\right). \tag{4.4}$$

By Teichmüller's Verschiebungssatz there is a K^*-quasiconformal mapping g^*: $B^2 \to B^2$ with $g^*(z) = z$ for $z \in \partial B^2$ and $g^*(g(0)) = 0$ with

$$\varrho(K^*) = \log\frac{1+|g(0)|}{1-|g(0)|}$$

where $\varrho(K)$ is as in Section 3. Hence the explicit formula is

$$K^* = \left(\frac{\exp\mu(|g(0)|)+1}{\exp\mu(|g(0)|)-1}\right)^2. \tag{4.5}$$

The desired mapping g_1 is now defined by $g_1(z) = g(z)$ for $z \in \overline{\mathbf{R}}^2 \setminus D$ and $g_1(z) = g^*(g(z))$ for $z \in \bar{D}$. Its maximal dilatation $K_1 \leq KK^*$ has the required property by (4.4) and (4.5).

Remark. Explicit estimates for $K_1(r,K)$ can be derived from (4.4) and (4.5) and

$$\begin{aligned}\tau_2^{-1}\left(\frac{1}{K}\tau_2\left(\frac{1}{r}\right)\right) &= \frac{1-\varphi_{1/K,2}^2\left(\sqrt{r/(r+1)}\right)}{\varphi_{1/K,2}^2\left(\sqrt{r/(r+1)}\right)} = \frac{\varphi_{K,2}^2\left(\sqrt{1/(r+1)}\right)}{\varphi_{1/K,2}^2\left(\sqrt{r/(r+1)}\right)} \\ &\leq 4^{2(1-1/K)}\left(\frac{1}{r+1}\right)^{1/K} 4^{2(K-1)}\left(\frac{r+1}{r}\right)^K \\ &= 16^{K-1/K}\frac{(r+1)^{K-1/K}}{r^K}\end{aligned}$$

where (2.1), (2.3), (2.33) and finally (2.4) and (2.5) have been used.

4.6. Theorem. *Let D be a bounded K-quasidisk, normalized such that $0 \in D$ and $1 = \max\{\,|z|\colon z \in \partial D\,\}$. Let $f\colon B^2 \to D$ be a conformal mapping with $f(0) = 0$. If $r = \min\{\,|z|\colon z \in \partial D\,\}$, then*

$$|f(x) - f(y)| \le M_1(2, K_1^2)|x - y|^{1/K_1^2}$$

for all $x, y \in B^2$ where $K_1 = K_1(r, K)$ is the constant from Lemma 4.1, in particular, the constant $M_1(2, K_1^2)$ tends to one for K and r tending to one.

Proof. Let g_1 be as in Lemma 4.1. Then f has a K_1^2-quasiconformal extension to $\overline{\mathbf{R}}^2$ which keeps ∞ fixed, namely $g_1^{-1} \circ i \circ g_1 \circ f \circ i$ where i denotes inversion. By Definition 2.15 the inequality follows, since this extension fixes 0 and ∞ and sends B^2 into itself.

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