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SOME COEFFICIENT ESTIMATIONS IN THE CLASS Σ_b' OF MEROMORPHIC UNIVALENT FUNCTIONS

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1. Introduction

Let Σ_b be the class of meromorphic univalent functions omitting a disc:

$$\Sigma_b = \{H \mid H(z) = z + \sum_0^\infty A_\nu z^{-\nu}, \quad |z| > 1, \quad |H(z)| > b \in (0,1)\}.$$

This is closely connected with the class $S(b)$ of bounded univalent functions:

$$S(b) = \{f \mid f(z) = b(z + \sum_2^\infty a_\nu z^\nu), \quad |z| < 1, \quad |f(z)| < 1, \quad b \in (0,1)\}.$$

The one-to-one connection between f and H reads

$$H(z) f(z^{-1}) = b, \qquad |z| > 1. \tag{1}$$

From $w = H(z)$ we obtain the inverse relationship

$$z = I(w) = w + \sum_0^\infty E_\nu w^{-\nu}.$$

In [6] the inverse coefficients E_n were all maximized by the radial-slit mapping I_r defined by

$$I_r + I_r^{-1} = w - 2(1-b) + b^2 w^{-1}.$$

The side-condition $A_0 = -a_2 = 0$ yields the subclass $\Sigma_b' \subset \Sigma_b$:

$$\Sigma_b' = \{H \mid H(z) = z + \sum_1^\infty A_\nu z^{-\nu}, \quad |z| > 1, \quad |H(z)| > b \in (0,1)\}.$$

The special case $b = 0$, i.e., $\Sigma_0' = \Sigma'$, has been extensively considered by Schober [4]. Especially the initial odd inverse coefficients were succesfully estimated.

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In the present paper we are dealing with Σ'_b and will generalize some of the results of [4] for odd inverse coefficients. Similarly, some initial A_ν-coefficients will be considered.

The connections between H, I and f yield the corresponding coefficient connections. Thus

$$\begin{aligned}
A_1 &= -E_1,\\
A_2 &= -E_2,\\
A_3 &= -E_3 - E_1^2,\\
A_4 &= -E_4 - 3E_2E_1,\\
A_5 &= -E_5 - 4E_3E_1 - 2E_2^2 - 2E_1^2.
\end{aligned}$$

Clearly, the letters A and E can here also be interchanged. In a_ν-coefficients we have further

$$\begin{aligned}
A_1 &= -a_3, & E_1 &= a_3,\\
A_2 &= -a_4, & E_2 &= a_4,\\
A_3 &= -a_5 + a_3^2, & E_3 &= a_5 - 2a_3^2,\\
A_4 &= -a_6 + 2a_4a_3, & E_4 &= a_6 - 5a_4a_3,\\
A_5 &= -a_7 + 2a_5a_3 + a_4^2 - a_3^2; & E_5 &= a_7 - 6a_5a_3 - 3a_4^2 + 7a_3^2.
\end{aligned}$$

2. A_1, A_2 by aid of coefficient bodies

For lower a_ν-coefficients we have lots of connections and inequalities which yield sharp information also in the special case $A_0 = -a_2 = 0$ in question.

The first non-trivial coefficient body (a_2, a_3) is studied in [9] (cf. p. 241 as well as pp. 264–265). Thus, for $a_2 = 0$ we have $|a_3| \le 1 - b^2$, i.e.,

$$|A_1| = |E_1| \le 1 - b^2.$$

The equality holds for the function f of the type 2:2 (the notation is explained in [9], p. 149) with the image of two symmetric radial slits and defined by

$$f(1+f^2)^{-1} = bz(1+z^2)^{-1}.$$

The second coefficient body (a_2, a_3, a_4) is also described in [9] with sufficient accuracy for our present purposes. However, there exist also direct estimations, given by Grunsky-type inequalities, which are sharp in the special case $a_2 = 0$. For example, from [7] there follows

$$\operatorname{Re} a_4 \le \frac{2}{3}(1 - b^3) - \frac{|a_3|^2}{4(1-b)} \le \frac{2}{3}(1 - b^3).$$

The final equality holds for $a_2 = a_3 = 0$ and for a symmetric radial-slit mapping 3:3 for which

$$f(1-f^3)^{-2/3} = bz(1-z^3)^{-2/3}.$$

Thus

$$|A_2| = |E_2| \leq \frac{2}{3}(1-b^3).$$

3. $E_{2\nu+1}$ ($\nu = 0,\ldots,4$) by aid of FitzGerald–Launonen inequality

For inverse coefficients the FitzGerald inequality appears to be effective. It is advisable to apply it in the integral form given by Launonen [2]. In [5] this method is used in estimating the coefficients of functions inverse to odd Σ_b-functions—call them briefly "odd inverse coefficients". We are going to apply the Siejka-method to them in the case Σ_b' and will thus test the possibilities of generalizing the estimations of Schober in [4].

For the inverse Σ_b-function the FitzGerald–Launonen condition (20) of [5] holds. This implies inequalities for odd E_ν-coefficients. These are obtained as in [5] from (22) and can be expressed by aid of the numbers α_ν of

$$I(w)^{-1} = w^{-1} + \alpha_2 w^{-2} + \alpha_3 w^{-3} + \cdots,$$

where $\alpha_2 = -a_2 = 0$, $\alpha_3 = -E_1$, $\alpha_4 = -E_2$, $\alpha_5 = -E_3 + E_1^2$, $\alpha_6 = -E_4 + 2E_2E_1, \ldots$. The inequalities in question are those in (24) of [5] (write $D_{2\nu-1} = E_{2\nu-1}$). Thus, for $\nu = 1$ and $\nu = 2$ we obtain immediately

$$|E_1| \leq 1-b^2, \qquad |E_3| \leq |\alpha_2|^2 + 1 - b^2 = 1 - b^2.$$

For the first condition the equality case was obtained from the coefficient body (a_2, a_3) and was found to be the symmetric 2:2-case. In what follows, call the corresponding extremal $S(b)$-function f_o. The equality in the second condition is more problematic. In Section 4 we will show that again the previous mapping f_o is the only extremal function.

For $\nu = 3$ the condition (22) of [5] yields

$$\begin{aligned}|E_5| &\leq |\alpha_3|^2 + (4-b^2)|\alpha_2|^2 + 1 - b^2 \\ &= |E_1|^2 + 1 - b^2 \leq (1-b^2)^2 + 1 - b^2 = 2 - 3b^2 + b^4.\end{aligned}$$

The maximum is reached with that of $|E_1|$, i.e., by the function f_o.

The first non-trivial estimation occurs for $\nu = 4$:

$$\begin{aligned}|E_7| &= |\alpha_4|^2 + |2\alpha_3 + \alpha_2^2|^2 - b^2|\alpha_3|^2 + (9-4b^2)|\alpha_2|^2 + 1 - b^2 \\ &= |A_2|^2 + (4-b^2)|A_1|^2 + 1 - b^2.\end{aligned}$$

In order to estimate this we apply the area inequality for $S(b)$ ([9], p. 182):

$$\sum_1^\infty \nu|\tilde{\alpha}_\nu - b^2 a_\nu|^2 \le 1,$$

where

$$bf(z)^{-1} = z^{-1} + \sum_0^\infty \tilde{\alpha}_\nu z^\nu.$$

Because of (1)

$$z^{-1} + \sum_1^\infty A_\nu z^\nu = H(z^{-1}) = bf(z)^{-1}.$$

Thus $\tilde{\alpha}_\nu = A_\nu$ and the area inequality reads

$$\sum_1^\infty \nu|A_\nu - b^2 a_\nu|^2 \le 1. \tag{2}$$

For E_7 we use the consequence

$$|A_1 - b^2|^2 + 2|A_2|^2 \le 1, \tag{3}$$

yielding

$$\begin{aligned}|E_7| &= \frac{3}{2} - b^2 + (4 - b^2)|A_1|^2 - \frac{1}{2}|A_1 - b^2|^2 \\ &= \frac{3}{2} - b^2 - \frac{1}{2}b^4 + (\frac{7}{2} - b^2)|A_1|^2 + b^2 \operatorname{Re} A_1.\end{aligned}$$

The rotated function $\tau^{-1}I(\tau w) = w + \Sigma_1^\infty \tau^{-\nu-1}E_\nu w^{-\nu}$, $|\tau| = 1$, preserves $|E_7|$ and allows the normalization $E_1 \ge 0$, i.e., $A_1 = -E_1 \le 0$. In the variable

$$x = A_1 \in \big[-(1-b^2), 0\big]$$

we thus have

$$\begin{aligned}|E_7| &\le \frac{3}{2} - b^2 - \frac{1}{2}b^4 + P(x); \\ P(x) &= (\frac{7}{2} - b^2)x^2 + b^2 x.\end{aligned}$$

Require

$$P\big(-(1-b^2)\big) = (1-b^2)(b^4 - \frac{11}{2}b^2 + \frac{7}{2})_\circ \ge P(0) = 0.$$

This yields

$$\max P = P\big(-(1-b^2)\big) \qquad \text{for} \quad 0 \le b \le b_\circ$$

where

$$b_o = \frac{1}{2}(11 - \sqrt{65}\,)^{1/2} = 0.856\,992\,160$$

is the zero of $(\;)_o$. Hence for $0 \le b \le b_o$ the sharp estimation holds:

$$|E_7| \le \frac{3}{2} - b^2 - \frac{1}{2}b^4 + P\big(-(1-b^2)\big) = 5 - 10b^2 + 6b^4 - b^6.$$

Again, with $|A_1|$ the coefficient $|E_7|$ is maximized by the function f_o.

The case $\nu = 5$ can be treated similarly.

$$\begin{aligned}|E_9| &\le |\alpha_5|^2 + (4-b^2)|\alpha_4|^2 + (9-4b^2)|\alpha_3|^2 + 1 - b^2\\ &= |E_1^2 - E_3|^2 + (4-b^2)|A_2|^2 + (9-4b^2)|A_1|^2 + 1 - b^2.\end{aligned}$$

Because $|E_1| \le 1 - b^2$, $|E_3| \le 1 - b^2$ we obtain, by using (3):

$$\begin{aligned}|E_9| &\le \big[(1-b^2)^2 + 1 - b^2\big]^2 + \frac{1}{2}(4-b^2)\big[1 - |A_1 - b^2|^2\big]\\ &\quad + (9-4b^2)|A_1|^2 + 1 - b^2\\ &= (1-b^2)^2(2-b^2)^2 + 3 - \frac{3}{2}b^2 - (2 - \frac{1}{2}b^2)b^4 + Q(x);\end{aligned}$$

$$Q(x) = (7 - \frac{7}{2}b^2)x^2 + (4-b^2)b^2x;$$

$$x = A_1 \in [-(1-b^2), 0].$$

Require

$$Q\big(-(1-b^2)\big) = \frac{1}{2}(1-b^2)(9b^4 - 29b^2 + 14) \ge Q(0) = 0.$$

This holds for

$$0 \le b \le b_o = \Big[\frac{1}{18}(29 - \sqrt{337}\,)\Big]^{1/2} = 0.768\,925\,667$$

where the following sharp estimation is thus valid:

$$\begin{aligned}|E_9| &\le (1-b^2)^2(2-b^2)^2 + 3 - \frac{3}{2}b^2 - (2 - \frac{1}{2}b^2)b^4 + Q\big(-(1-b^2)\big)\\ &= 14 - 35b^2 + 30b^4 - 10b^6 + b^8.\end{aligned}$$

Again, with $|A_1|$ also $|E_9|$ is maximized by f_o.

The remaining interval $(b_o, 1)$ is left open by the above method. The structure of the FitzGerald inequalities shows that with increasing index the role of A_ν-coefficients increases. Already for A_2 the function f_o is not the extremal one and, as will be seen, similar situation holds for A_3. Hence, there is no hope to proceed very far in the $E_{2\nu+1}$-estimations by using the above method. It is not excluded that f_o actually loses its extremal role, at least for some b-intervals, for higher odd inverse coefficients.

For E_{11} and E_{13} the above estimation technique fails if $b > 0$ and thus remains succesful only at $b = 0$ as was proved by Schober in [4].

Collect the sharp results found:

Theorem. *In $\sum_b'$ the coefficients E_1, E_3, E_5 are for the whole interval $b \in [0,1]$ maximized by the $S(b)$-function f_o of the symmetric $2:2$-type defined by*

$$f(1+f^2)^{-1} = bz(1+z^2)^{-1}.$$

For E_7, f_o preserves the extremal role at least for $0 \le b \le 0.856\,992\,160$ and for the coefficient E_9 the same holds at least for $0 \le b \le 0.768\,925\,667$.

The coefficient $|A_2| = |E_2|$ is maximized by the $S(b)$-function of the symmetric $3:3$-type defined by

$$f(1-f^3)^{-2/3} = bz(1-z^3)^{-2/3}.$$

The uniqueness of the extremal function for E_3 is proved at the beginning of Section 4.

4. E_3 for $b \in [0,1]$ and A_3 for $b \in [e^{-2}, 1]$

In (85) p. 473 of [3] there is the Grunsky-type inequality for a_5 which for $a_2 = 0$ yields

$$2\mathrm{Re}\,(a_5 - 2a_3^2) - (1-b^4) \le -\mathrm{Re}\,(a_3^2) - \frac{2(\mathrm{Re}\,a_3)^2}{\ln b^{-1}}.$$

Denote here $a_3 = u + iv$:

$$2\mathrm{Re}\,(a_5 - 2a_3^2) - (1-b^4) \le v^2 - (1 + 2/\ln b^{-1})_o u^2. \tag{4}$$

Because $(\ )_o > 0$ for $b \in [0,1)$ we have

$$2\mathrm{Re}\,(a_5 - 2a_3^2) - (1-b^4) \le v^2 \le (1-b^2)^2$$

with the equality for $u = 0$, $|v| = 1 - b^2$, because $|a_3| \le 1 - b^2$ for $a_2 = 0$. Thus

$$\mathrm{Re}\,(a_5 - 2a_3^2) \le 1 - b^2,$$

i.e., $|E_3| \leq 1 - b^2$ where equality holds exactly for $a_2 = 0$, $|a_3| = 1 - b^2$, i.e., for the function f_o.

Turn to the combination $-A_3 = a_5 - a_3^2$. For it (4) assumes the form

$$2\text{Re}\,(a_5 - a_3^2) - (1 - b^4) \leq (1 - 2/\ln b^{-1})u^2 - v^2 \leq 0 \tag{5}$$

provided $1 - 2/\ln b^{-1} \leq 0$. This implies two cases

$$\begin{array}{llll} 1) & 1 - 2/\ln b^{-1} < 0 & \Leftrightarrow & e^{-2} < b < 1; \\ 2) & 1 - 2/\ln b^{-1} = 0 & \Leftrightarrow & b = e^{-2}. \end{array}$$

In the case 1) the final equality in (5) holds for $u = v = a_3 = 0$. In the case 2) equality requires $v = 0$ but u is left as a free parameter.

Consider the equality cases more closely and apply the rotation $\tau^{-1}f(\tau z)$, $|\tau| = 1$, to yield $a_5 - a_3^2 \Rightarrow \tau^4(a_5 - a_3^2)$, $a_3 \Rightarrow \tau^2 a_3$. Thus we can normalize

$$-A_3 = |A_3| = \text{Re}\,(a_5 - a_3^2) \geq 0, \qquad \text{Re}\, a_3 \leq 0.$$

In order to study equality in the cases 1)–2) put $a_2 = 0$ in the original inequality (82) p. 472 of [3]:

$$\text{Re}\,(\ln b \cdot x_o^2 + a_3 x_1^2 + a_5 - 3/2 a_3^2 + 2a_3 x_o + 2a_4 x_1) \leq (1 - b^2)|x_1|^2 + \frac{1}{2}(1 - b^4).$$

Here x_o and x_1 are free complex parameters. In the normalized equality case of (5) $a_5 = 1(1 - b^4)/2 + u^2$, $v = 0$, which implies

$$\text{Re}\,(\ln b \cdot x_o^2 + 2u\, x_o - \frac{u^2}{2} + u\, x_1^2 + 2a_4 x_1) \leq (1 - b^2)|x_1|^2.$$

In the case 1) $u = 0$. By choosing $x_o = 0$ we obtain

$$2\text{Re}\,(x_1 a_4) \leq (1 - b^2)|x_1|^2.$$

Putting $x_1 = |x_1|e^{i\phi}$ and letting $0 < |x_1| \to 0$ we find

$$2\text{Re}\,(e^{i\phi}a_4) \leq (1 - b^2)|x_1| \quad \Rightarrow \quad \text{Re}\,(e^{i\phi}a_4) \leq 0, \quad 0 \leq \phi \leq 2\pi$$

which implies $a_4 = 0$.

In the case 2) $b = e^{-2}$, $u < 0$ and the above condition assumes the form

$$\text{Re}\,\left\{-2\left(x_o - \frac{u}{2}\right)^2 + u x_1^2 + 2a_4 x_1\right\} \leq (1 - b^2)|x_1|^2.$$

Choose now $x_o = u/2$ and put, as before, $x_1 = |x_1|e^{i\phi}$, $0 < |x_1| \to 0$:

$$\text{Re}\left(u|x_1|^2 e^{i2\phi} + 2e^{i\phi}a_4\right) \le (1-b^2)|x_1|$$

$$\Rightarrow \qquad \text{Re}\,(e^{i\phi}a_4) \le 0, \qquad 0 \le \phi \le 2\pi$$

and hence $a_4 = 0$.

In the normalized extremal case all the coefficients up to a_5 are thus real. From the Power inequality it then follows that we may use the general condition derived for the extremal function in the real class. This is the condition (35) p. 488 in [8]:

$$\begin{aligned} &2x_o \ln f + b^2(f^2 - f^{-2}) = 2x_o \ln z + z^2 - z^{-2};\\ &2x_o = a_3 = u \le 0. \end{aligned} \tag{6}$$

In the case 1) in (6) $u = x_o = 0$. The image $f(U)$, $U : |z| < 1$, is of the type 4:4 with four symmetrically located radial slits.

In the case 2), $b = e^{-2}$, the image $f(U)$ can be studied by aid of the boundary correspondence. Thus, put

$$z = e^{i\phi}, \qquad f(e^{i\phi}) = r(\phi)e^{i\psi(\phi)}$$

in (6):

$$\begin{aligned} &\cos\, 2\psi = -e^4 u \frac{\ln r}{r^2 - r^{-2}} \to -ue^4/4 \qquad \text{for} \quad r \to 1;\\ &u\psi + e^{-4}(r^2 + r^{-2})\sin\, 2\psi = u\phi + 2\sin\, 2\phi. \end{aligned}$$

The first condition implies the limitation for u:

$$-0.073\,262\,556 = -4e^{-4} \le u \le 0. \tag{7}$$

(6) determines f':

$$z\frac{f'}{f} = \frac{z^2 + x_o + z^{-2}}{bf^2 + x_o + b^2 f^{-2}}.$$

Thus, the pre-image $z = e^{i\phi}$ of the tip of the slit is determined by

$$2\cos 2\phi + x_o, \qquad x_o = u/2$$

and the starting point of $f = e^{i\psi}$ of the slit satisfies

$$2\cos 2\psi + b^{-2}x_o, \qquad b = e^{-2}.$$

In Figure 1 there are some slits connected with $f(U)$ of the type 4:4 in the case $b = e^{-2}$.

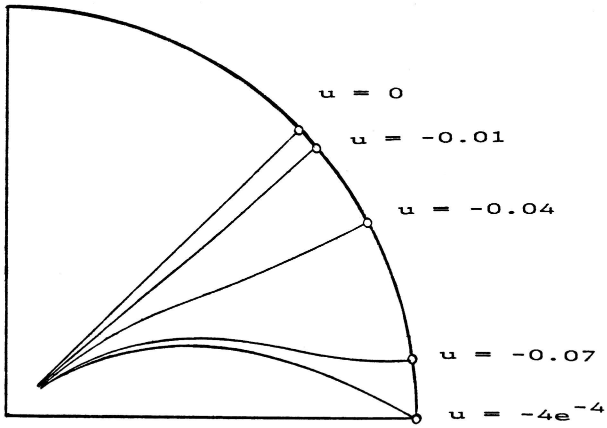

Figure 1.

5. A_3 in the odd subclass of Σ'_b

On the interval $0 < b < e^{-2}$ the problem of maximizing $-A_3 = a_5 + a_3^2$ remains open in Σ'_b. The corresponding question in the *odd* subclass of Σ'_b appears to be more reasonable and can be solved by aid of inequalities holding in $S(b)$.

As is well known $f \in S(b)$ determines an odd $\tilde{f} = \sqrt{f(z^2)} \in S(b^{1/2})$. Conversely, any odd $\tilde{f} = \tilde{f}(z) \in S(b)$ determines $f \in S(b^2)$ through the connection

$$\tilde{f}(z)^2 = f(z^2).$$

Any condition true for $f \in S(b)$ can be transformed to $\tilde{f}(z) = \tilde{b}(z + \tilde{a}_3 z^3 + \cdots) \in S(\tilde{b})$ by the alteration

$$z, f, b \quad \Rightarrow \quad z^2, \tilde{f}^2, \tilde{b}^2.$$

More closely, if $f(z) = b(z + a_2 z^2 + \cdots)$ we have the coefficient connections

$$b = \tilde{b}^2, \quad a_2 = 2\tilde{a}_3, \quad a_3 = 2\tilde{a}_5 + \tilde{a}_3^2.$$

Take the optimized Power inequality true for $S(b)$-functions, [10], p. 7:

$$\text{Re}\,(a_3 - a_2^2) \le 1 - b^2 + U^2/\ln b, \quad U = \text{Re}\,a_2.$$

Equality here can be reached if $|U| \le 2b|\ln b|$. For odd $S(\tilde{b})$-functions this implies

$$\begin{aligned} &2\text{Re}\,(\tilde{a}_5 - \tilde{a}_3^2) - (1 - \tilde{b}^4) \le \text{Re}\,(\tilde{a}_3^2) + 4(\text{Re}\,\tilde{a}_3)^2/\ln \tilde{b}^2 \\ &= (1 + 4/\ln \tilde{b}^2)\tilde{u}^2 - \tilde{v}^2 \le (1 + 4/\ln \tilde{b}^2)\tilde{u}^2 = \widetilde{M}(\tilde{u}), \end{aligned}$$

with equality for $\tilde{v} = 0$, where $\tilde{a}_3 = \tilde{u} + i\tilde{v}$.

If $e^{-2} \le \tilde{b} < 1$ we obtain from this our former estimation $\mathrm{Re}\,(\tilde{a}_5 - \tilde{a}_3^2) \le (1-\tilde{b}^4)/2$.

If $0 < \tilde{b} < e^{-2}$, $1 + 4/\ln \tilde{b}^2 > 0$ and $|\tilde{u}| \le \tilde{b}^2|\ln \tilde{b}^2|$ we obtain the non-sharp estimation

$$\widetilde{M}(\tilde{u}) \le (1 + 4/\ln \tilde{b}^2)\tilde{b}^4 \ln^2\tilde{b}^2 = \tilde{b}^4 \ln \tilde{b}^2(4 + \ln \tilde{b}^2) \tag{8}$$

which will be needed later.

The above inequality can be sharpened for the values $|U| \ge 2b|\ln b|$, [10], p. 17:

$$\mathrm{Re}\,(a_3 - a_2^2) \le 1 - b^2 - 2|U|\sigma + 2(\sigma - b)^2;$$

$$\sigma \ln \sigma - \sigma + b + |U|/2 = 0, \qquad U = \mathrm{Re}\, a_2, \qquad \sigma \in [b, 1].$$

We shift this for $\tilde{f}$ and denote in this connection

$$\sigma = \tilde{\sigma}^2 \in [\tilde{b}^2, 1] \;\Rightarrow \tilde{\sigma} \in [\tilde{b}, 1];$$

$$2\mathrm{Re}\,(\tilde{a}_5 - \tilde{a}_3^2) - (1 - \tilde{b}^4) \le \mathrm{Re}\,(\tilde{a}_3^2) - 4|\mathrm{Re}\,\tilde{a}_3|\tilde{\sigma}^2 + 2(\tilde{\sigma}^2 - \tilde{b}^2)^2;$$

$$|\mathrm{Re}\,\tilde{a}_3| = -(\tilde{\sigma}^2 \ln \tilde{\sigma}^2 - \tilde{\sigma}^2 + \tilde{b}^2).$$

Denoting $\tilde{a}_3 = \tilde{u} + i\tilde{v}$ we obtain

$$\begin{aligned} 2\mathrm{Re}\,(\tilde{a}_5 - \tilde{a}_3^2) - (1 - \tilde{b}^4) &\le \tilde{u}^2 - \tilde{v}^2 - 4|\tilde{u}|\tilde{\sigma}^2 + 2(\tilde{\sigma}^2 - \tilde{b}^2)^2 \\ &\le \tilde{u}^2 - 4|\tilde{u}|\tilde{\sigma}^2 + 2(\tilde{\sigma}^2 - \tilde{b}^2)^2; \end{aligned}$$

$$|\tilde{u}| = -(\tilde{\sigma}^2 \ln \tilde{\sigma}^2 - \tilde{\sigma}^2 + \tilde{b}^2),$$

with the equality for $\tilde{v} = 0$. This estimation is to be used for

$$\tilde{b}^2|\ln \tilde{b}^2| \le |\tilde{u}| \le 1 - \tilde{b}^2.$$

For brevity, return to the variable σ in estimating the upper bound found:

$$2\mathrm{Re}\,(\tilde{a}_5 - \tilde{a}_3^2) - (1 - \tilde{b}^4) \le (\sigma \ln \sigma - \sigma + b)^2 + 4\sigma(\sigma \ln \sigma - \sigma + b) + 2(\sigma - b)^2 = M(\sigma),$$

$$\tilde{b}^2 = b \le \sigma \le 1.$$

According to [10] p. 15 $\sigma = \sigma(\tilde{u}) \in [b, 1]$ is uniquely determined by $|\tilde{u}| \in [\tilde{b}^2|\ln \tilde{b}^2|, 1 - \tilde{b}^2]$. Thus, M as a function of $\tilde{u}$ is maximized by maximizing $M(\sigma)$ for $\sigma \in [b, 1]$. Because

$$\frac{dM(\sigma)}{d\sigma} = 2\ln \sigma(\sigma \ln \sigma + 3\sigma + b)_1$$

we see that the root $\sigma \in [e^{-4}, 1]$ of $(\ )_1 = 0$ yields the maximum in question. Because the maximum of $\widetilde{M}(\tilde{u})$ in (8) equals $M(b)$ we have maximized Re $(\tilde{a}_5 - \tilde{a}_3^2)$ and hence $|\tilde{a}_5 - \tilde{a}_3^2|$.

Collect the results concerning A_3. All the extremal functions to be mentioned are the essential ones, to which rotation can be added.

Theorem. *In Σ_b' the coefficient A_3 is maximized as follows.*

$e^{-2} < b < 1 : |A_3| \leq (1-b^4)/2$. *Equality holds for the symmetric 4:4-mapping and the corresponding $S(b)$-function f is obtained from (6) for $x_o = 0$.*

$b = e^{-2}$: *The above maximum remains to hold for a one-parametric family of 4:4-mappings the $S(b)$-function f of which is determined by (6) with $u \in [-4e^{-4}, 0]$ as a parameter.*

$0 \leq b < e^{-2}$: *In the odd subclass of Σ_b' $|A_3| \leq (1-b^4)/2 + (\sigma^2 - b^2)^2$ where $\sigma \in [e^{-2}, 1]$ is the root of*

$$\sigma^2 \ln \sigma^2 + 3\sigma^2 + b^2 = 0.$$

In the limit case $b = 0$, $\sigma = e^{-3/2}$ and $\max|A_3| = 1/2 + e^{-6}$. The extremal $S(b)$-function f is of the type 2:4 obtained (through f_σ) from,

$$\begin{aligned} &-4\sigma^2 \ln f_\sigma + \sigma^2(f_\sigma^2 - f_\sigma^{-2}) = -4\sigma^2 \ln z + z^2 - z^{-2}, \\ &f + f^{-1} = \sigma/b(f_\sigma + f_\sigma^{-1}). \end{aligned}$$

The condition for the 2:4-function follows from [9], p. 72, where the extremal function of the type 1:2 for a_3 in $S(b)$ is defined.

In the limit cases $b = e^{-2}$ and $b = 0$ the maxima of odd functions are known to hold also in the general class Σ_b'. For $b = 0$ the result was proved by Garabedian and Schiffer [1]. Thus, it seems to be most probable that a similar state of things remains to hold also for the intermediate values $0 < b < e^{-2}$.

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