Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 13, 1988, 225–229

ON A LIMIT THEOREM FOR DEPENDENT RANDOM VARIABLES

V.K. Malinovskiĭ

1. Let $X_i, i = 1, 2, \ldots$ be a regenerative sequence with finite regeneration times $\nu_i, i = 0, 1, 2, \ldots$. An example of such a sequence is a homogeneous Markov chain with stationary transition probabilities having a recurrent state (see [2], [6]).

For $i = 1, 2, \ldots$ let $\tau_i = \nu_i - \nu_{i-1}, \tau_0 = \nu_0$ be the regeneration cycles and

$$Z_i = \sum_{j=\nu_{i-1}+1}^{\nu_i} X_j, \qquad Z_0 = \sum_{j=1}^{\nu_0} X_j$$

be the increments on such cycles. It is well known that the random vectors (Z_i, τ_i) are independent for $i = 0, 1, 2, \ldots$ and identically distributed for $i = 1, 2, \ldots$. Let us denote $a = \mathbf{E}\,\tau_1 > 0$, $N = [n/a]$ (= the integer part of n/a), $t(n) = \max\{k \geq 0 : \nu_k \leq n\}$ and

$$R = \sum_{j=\nu_{t(n)}+1}^{n} X_j, \qquad T_i = \sum_{j=1}^{i} Z_j \,.$$

A.N. Kolmogorov and W. Doeblin were the first to propose a method of proving limit theorems for sums $S_n = \sum_{i=1}^n X_i$. The idea amounts to the following (see [2]). First,

$$S_n = Z_0 + T_{t(n)} + R \,.$$

Second, the sum $T_{t(n)}$ is asymptotically the same as the sum of N independent random variables Z_i because

$$\frac{t(n)}{n} \longrightarrow \frac{1}{a} \text{ in probability as } n \to \infty$$

under quite general assumptions. Finally, R and Z_0 are small in comparison with $T_{t(n)}$ when n is large. Therefore, a limit theorem for S_n can be reduced to a corresponding assertion for a sum of independent random variables T_N. In some problems a more detailed analysis of the difference $\Delta_n = S_n - T_N$ is important. The aim of the present paper is to find the asymptotical distribution of the properly standardized random vector (T_N, Δ_n) as well as estimates of Berry–Esseen type.

doi:10.5186/aasfm.1988.1314

The improved techniques of the Kolmogorov–Doeblin regenerative method proposed in [1] and developed in [4], [5] are used in the present paper to prove this result. These proofs rely on the limit theorems for sums of independent m-lattice random vectors obtained in [3]. Further remarks on the Kolmogorov–Doeblin regenerative method and related references may be found in [4], [5].

2. Let the random variables $\tau_i, i = 1, 2, \dots$ be nondegenerate and take values in the lattice of integers with span 1. Let $\sigma_z^2 = \mathbf{E}\, Z_1^2, \sigma_\tau^2 = \mathbf{E}\,(\tau_1 - a)^2, \kappa_{z\tau} = (\sigma_z \sigma_\tau)^{-1} \mathbf{E}\, Z_1 \tau_1$,

$$V = \begin{pmatrix} 1 & \kappa_{z\tau} \\ \kappa_{z\tau} & 1 \end{pmatrix}, \quad \Phi_V(x, y) = \int_{-\infty}^{x} \varphi_V(z, y)\, dz,$$

where $\varphi_V(x, y)$ stands for the density of the two-dimensional normal distribution with zero expectation vector and covariance matrix V. Moreover, $\Phi(x)$ and $\varphi(x)$ denote the distribution function and the density of the standard normal distribution, respectively.

Theorem. *Suppose that for the regenerative sequence $X_i, i = 1, 2, \dots$, defined above,*

1) $\mathbf{E}\, \nu_0 < \infty, \quad \mathbf{E}\, \tau_1^3 < \infty,$

2) $\mathbf{E} \sum_{i=1}^{\nu_0} |X_i| < \infty, \quad \mathbf{E} \Big(\sum_{i=\nu_0+1}^{\nu_1} |X_i| \Big)^3 < \infty,$

3) $\sigma_z > 0$ *and* $\mathbf{E}\, Z_1 = 0$.

Then, as $n \to \infty$

$$\sup_{x,y \in \mathbf{R}} \left| P\Big(\frac{\sqrt{a}}{\sigma_z \sqrt{n}} T_N \le x, \frac{a^{3/4}}{\sigma_z \sqrt{\sigma_\tau} n^{1/4}} \Delta_N \le y\Big) - \int \Phi_V(x, z) \Phi\Big(\frac{y}{\sqrt{|z|}}\Big)\, dz \right| = O(n^{-1/4})$$

with integration over the range $(-\infty, 0) \cup (0, \infty)$.

Proof. The techniques presented here have been obtained by improving those used in [1] and [4], [5]. Therefore we shall omit the details of calculations whenever it cannot cause any misunderstanding.

Applying the formula of total probability with respect to a partition of the sample space defined by using the regeneration times ν_i, $i = 1, 2, \dots$, we can easily show that

$$P\Big(\frac{\sqrt{a}}{\sigma_z \sqrt{n}} T_N \le x, \frac{a^{3/4}}{\sigma_z \sqrt{\sigma_\tau} n^{1/4}} \Delta_n \le y\Big) = \tag{1}$$

$$\sum_{r=1}^{n} \sum_{l=1}^{n} \iint \Big[\sum_{m=1}^{N-1} P\Big(\frac{\sqrt{a}}{\sigma_z \sqrt{n}} \sum_{i=1}^{N} Z_i \le x, \frac{a^{3/4}}{\sigma_z \sqrt{\sigma_\tau} n^{1/4}} \sum_{i=m+1}^{N} (-Z_i) \le y_{u,v}, \sum_{i=1}^{m} \tau_i = n_{r,l}\Big)$$

$$+ \sum_{m=N+1}^{n} P\Big(\frac{\sqrt{a}}{\sigma_z \sqrt{n}} \sum_{i=1}^{N} Z_i \le x, \frac{a^{3/4}}{\sigma_z \sqrt{\sigma_\tau} n^{1/4}} \sum_{i=N+1}^{m} Z_i \le y_{u,v}, \sum_{i=1}^{m} \tau_i = n_{r,l}\Big)\Big]$$

$$\cdot P(Z^r \in du, \nu_0 = r) P(Z^l \in dv, \tau_{m+1} > l),$$

where

$$Z^i = \frac{a^{3/4}}{\sigma_z \sqrt{\sigma_\tau} n^{1/4}} \sum_{j=1}^{i} X_j, \quad y_{u,v} = y - u - v, \quad n_{r,l} = n - r - l.$$

The random vectors $(Z_i/\sigma_z, (\tau_i - a)/\sigma_\tau)$, $i = 1, 2, \ldots$, are independent and identically distributed having zero expectation vectors and covariance matrix V. We use now the approximation provided by the lemma of Section 3. It is easy to show that the essential step in the completion of the proof, after the substitution of this approximation into (1), consists in simplifying the expression

$$\begin{aligned} &I(x,y,n) = \\ &\sum_{r=1}^{n}\sum_{l=1}^{n} \iint \Bigg[\sum_{m=1}^{N-1} \int \Phi_V\Big(x\sqrt{\frac{n}{am}} - z\sqrt{\frac{N-m}{m}},\ \lambda_{r,l,m}\Big) \mathrm{Ind}\Big\{z : z \geq -\frac{n^{1/4}\sqrt{\sigma_\tau} y_{u,v}}{a^{3/4}\sqrt{N-m}}\Big\} \\ &\cdot \varphi(z) \frac{dz}{\sigma_\tau \sqrt{m}} + \sum_{m=N+1}^{n} \sum_{k=1}^{n} \Phi_V\Big(x, \frac{k-aN}{\sigma_\tau\sqrt{N}}\Big) \Phi_V\Big(\frac{n^{1/4}\sqrt{\sigma_\tau} y_{u,v}}{a^{3/4}\sqrt{m-N}}, \frac{n-(m-N)k}{\sigma_\tau\sqrt{m-N}}\Big) \\ &\cdot \frac{1}{\sigma_\tau^2 \sqrt{(m-N)N}} \Bigg] P(Z^r \in du,\ \nu_0 = r) P(Z^l \in dv,\ \tau_1 > l), \end{aligned}$$

which is obtained after the substitution, here

$$\lambda_{r,l,m} = \frac{n - r - l - am}{\sigma_\tau \sqrt{m}}, \qquad \lambda_m = \lambda_{0,0,m};$$

and the estimate for the remainder term

$$\begin{aligned} J(n) = &\sum_{r=1}^{n}\sum_{l=1}^{n}\Bigg[\sum_{m=1}^{N-1} (C_1 m^{-1} + C_2 (N-m)^{-1/2})\big(1 + |\lambda_{r,l,m}|^3\big)^{-1} \\ &+ C \sum_{m=N+1}^{n} \sum_{k=1}^{n} \Big[N(m-N)\Big(1 + \Big|\lambda_{r,l,m} - \frac{k}{\sigma_\tau\sqrt{m-N}}\Big|^3\Big)\Big(1 + \Big|\frac{k-aN}{\sigma_\tau\sqrt{N}}\Big|^3\Big)\Big]^{-1}\Bigg] \\ &\cdot P(\nu_0 = r) P(\tau_1 > l). \end{aligned}$$

The proof that $J(n)$ is of order $O(n^{-1/4})$ is accomplished by some elementary calculations in a similar fashion as in the corresponding part of [1] and we shall omit the details.

It is easy to show using similar arguments as in [1] (see the estimate (3.6)) that the expression $I(x,y,n)$ differs from the sum of the quantities

$$I_1(x,y,n) =$$
$$\sum_{m=1}^{N-1} \int \Phi_V\Big(x\sqrt{\frac{n}{am}} - z\sqrt{\frac{N-m}{m}}, \lambda_m\Big) \mathrm{Ind}\Big\{z : z \geq -\frac{n^{1/4}\sqrt{\sigma_\tau} y}{a^{3/4}\sqrt{N-m}}\Big\} \varphi(z) \frac{a}{\sigma_\tau\sqrt{m}}\, dz\,,$$

$$I_2(x,y,n) =$$
$$\sum_{m=N+1}^{n} \sum_{k=1}^{n} \Phi_V\Big(x, \frac{k-aN}{\sigma_\tau\sqrt{N}}\Big) \Phi_V\Big(\frac{n^{1/4}\sqrt{\sigma_\tau} y}{a^{3/4}\sqrt{m-N}}, \frac{n-(m-N)k}{\sigma_\tau\sqrt{m-N}}\Big) \frac{a}{\sigma_\tau^2\sqrt{(m-N)N}}$$

only by terms of order $O(n^{-1/4})$ uniformly for $x, y \in \mathbf{R}$. The proof of this rests on the moment conditions of the theorem.

The next essential step in the completion of the proof of the theorem amounts then to the evaluation of the integral in $I_1(x,y,n)$ and the sum over k in $I_2(x,y,n)$ and to the determination of the main terms of the resulting expressions. This part of the proof is similar to the evaluation of the estimate (3.7) in [1]. It is easy to show that the expression $I_1(x,y,n)$ differs from the expression

$$I_3(x,y,n) = \sum_{m=1}^{N-1} \Phi_V\Big(x\sqrt{\frac{n}{am}}, \lambda_m\Big) \Phi\Big(\frac{n^{1/4}\sqrt{\sigma_\tau} y}{a^{3/4}\sqrt{N-m}}\Big) \frac{a}{\sigma_\tau\sqrt{m}},$$

and, respectively, $I_2(x,y,n)$ from

$$I_4(x,y,n) = \sum_{m=N+1}^{n} \Phi_V\Big(x, \frac{a(m-N)}{\sigma_\tau\sqrt{N}}\Big) \Phi\Big(\frac{n^{1/4}\sqrt{\sigma_\tau} y}{a^{3/4}\sqrt{m-N}}\Big) \frac{a}{\sigma_\tau\sqrt{N}}$$

only by quantities of order $O(n^{-1/4})$ uniformly for $x, y \in \mathbf{R}$. When dealing with $I_2(x,y,n)$ we rewrite the sum over k as an integral sum and approximate it by the corresponding integral.

Finally, we rewrite the expressions I_3 and I_4 as integral sums over m. Using the identities

$$\lambda_m - \lambda_{m+1} = \frac{a}{\sigma_\tau\sqrt{m}} + \lambda_{m+1}\Big(\sqrt{1+\frac{1}{m}} - 1\Big) \quad \text{and} \quad N - m = \frac{\sigma_\tau\sqrt{m}}{a}\lambda_m\,,$$

we find (see the estimate (3.7) in [1]) that such expressions are approximated by

$$\int_0^\infty \Phi_V(x,z)\Phi\Big(\frac{y}{\sqrt{z}}\Big)\, dz \quad \text{and} \quad \int_{-\infty}^0 \Phi_V(x,|z|)\Phi\Big(\frac{y}{\sqrt{|z|}}\Big)\, dz\,,$$

respectively, with accuracy $O(n^{-1/4})$ uniformly for $x, y \in \mathbf{R}$.

Remark. An obvious extension of the techniques used above yield the asymptotical expansions in this theorem (see [4]).

3. We are stating in this section a version of a nonuniform limit theorem for sums of independent identically distributed so-called m-lattice random vectors, proved in a general form in [3].

Let (ξ_i^1, ξ_i^2), $i = 1, 2, \ldots$, be independent identically distributed random vectors with zero expectation vector and covariance matrix V. The random variables ξ_i^2 take values in the lattice

$$\{ih + c,\quad c > 0,\quad i = 0, \pm 1, \ldots\}$$

with a maximal span $h > 0$.

Lemma. *Suppose that for the vectors defined above*

$$\mathbf{E}\,|\xi_1^1|^3 < \infty, \quad \mathbf{E}\,|\xi_1^2|^3 < \infty .$$

Then, for any integer k,

$$\sup_{x \in \mathbf{R}} \left| \frac{\sqrt{n}}{h} P\Big(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \xi_i^1 \le x,\ \sum_{i=1}^{n} \xi_i^2 = kh + cn\Big) - \Phi_V\Big(x, \frac{kh + cn}{\sqrt{n}}\Big) \right|$$
$$\le \text{const}\,\sqrt{n}\Big(1 + \Big|\frac{kh + cn}{\sqrt{n}}\Big|^3\Big)^{-1} .$$

References

[1] Bolthausen, E.: The Berry–Esseen theorem for functionals of discrete Markov chains. - Z. Wahrsch. Verw. Gebiete 54, 1980, 59–73.

[2] Chung, K.L.: Markov chains with stationary transition probabilities. - Springer-Verlag, Berlin–Göttingen–Heidelberg, 1960.

[3] Dubinskaĭte, Ĭ.: Limit theorems in $\mathbf{R}^k$, I–III. - Lithuanian Math. J. 22, 1982, 129–140; 24, 1984, 256–265, and 325–334 (English translation of Litovsk. Mat. Sb. 22, 1982, 2, 51–68; 24, 1984, 3, 120–132, and 4, 68–80).

[4] Malinovskiĭ, V.K.: Limit theorems for Harris Markov chains, I. - Theory Probab. Appl. 31, 1987, 269–285 (English translation of Teor. Veroyatnost. i Primenen. 31, 1986, 315–332).

[5] Malinovskiĭ, V.K.: Asymptotic expansions in the central limit theorem for recurrent Markov reneval processes. - Theory Probab. Appl. 31, 1987, 523–526 (English translation of Teor. Veroyatnost. i Primenen. 31, 1986, 594–598).

[6] Nummelin, E.: A splitting technique for Harris recurrent Markov chains. - Z. Wahrsch. Verw. Gebiete 43, 1978, 309–318.

Steklov Mathematical Institute
Vavilova 42
117966 Moscow GSP-1
USSR