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ON HÖLDER CONTINUITY OF SOLUTIONS OF CERTAIN INTEGRO-DIFFERENTIAL EQUATIONS

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In the paper of N.V. Krylov and M.V. Safonov [6] (see [5], [4] as well) estimates of Hölder norms are obtained for solutions of second-order parabolic and elliptic partial differential equations in nonvariational form with measurable coefficients. Although analytical methods are used in [6], an important role is played by the properties of corresponding diffusion processes, namely by the estimates of Green measures ([3], [5]).

In the present paper we estimate Hölder norms or the modulus of continuity of solutions of integro-differential equations with measurable coefficients associated with Ito's processes. Estimates of the Green measures of Ito's processes are used ([1], [7]).

In Section 1 of this paper we formulate the main result which is proved in Section 3. In Section 2 some auxiliary results are presented.

1. Statement of the problem

Let $\mathbf{R} = (-\infty, \infty)$, $\mathbf{R}^{d+1} = \{(t,x) : t \in \mathbf{R},\ x = (x_1, \dots, x_d) \in \mathbf{R}^d\}$. We denote

$$|x| = \Big\{\sum_{i=1}^d x_i^2\Big\}^{1/2}, \quad \rho(z,z') = |x-x'| + |t-t'|^{1/2},$$
$$z = (t,x), \quad z' = (t',x') \in \mathbf{R}^{d+1}.$$

If $Q \subset \mathbf{R}^{d+1}$, then we write $\overline{Q}$ for the closure, ∂Q for the boundary and 1_Q for the indicator function of the set Q.

Let $B(Q)$ be the set of all measurable functions u on Q such that $\|u\|_{\infty,Q} = \sup_{z\in Q}|u(z)| < \infty$. Let $L^p(Q)$, $p \geq 1$, be the set of measurable functions u on Q such that $\|u\|_{p,Q} = \big\{\int_Q |u(t,x)|^p \,dt\,dx\big\}^{1/p} < \infty$. We denote by $W_p^{1,2}$ the completion of the set $C_0^\infty(\mathbf{R}^{d+1})$ (of all smooth functions on $\mathbf{R}^{d+1}$ with compact support) with respect to the norm

$$\|u\|_{W_p^{1,2}} = \|\partial_t u\|_{p,\mathbf{R}^{d+1}} + \sum_{i,j=1}^d \|u_{x_i x_j}\|_{p,\mathbf{R}^{d+1}} + \sum_{i=1}^d \|u_{x_i}\|_{p,\mathbf{R}^{d+1}} + \|u\|_{p,\mathbf{R}^{d+1}}.$$

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For a domain $Q \subset \mathbf{R}^{d+1}$ we define $W_p^{1,2}(Q) = \{u|_Q : u \in W_p^{1,2}\}$ and the norm $\|u\|_{W_p^{1,2}(Q)}$ replacing in the above formula $\mathbf{R}^{d+1}$ by Q.

Let $\mathcal{S}_d^+$ be the set of all non-negative symmetric $d \times d$-matrices and let $\mathcal{M}$ be the set of non-negative Radon measures π on $E = \mathbf{R}^d \setminus \{0\}$ such that $\int |y|^2 \wedge 1\, \pi(dy) < \infty$.

Fix $K > \nu > 0$, $\delta \in (0,2]$ and denote by $\Gamma = \Gamma(\delta) = \Gamma(\delta, \nu, K)$ the set of measurable functions $\gamma : \mathbf{R}^{d+1} \longrightarrow \mathcal{S}_d^+ \times \mathbf{R}^d \times \mathcal{M}$, $\gamma(\cdot) = \big(a(\cdot), b(\cdot), \pi(\cdot)\big)$, such that

$$|a| + |b| + \int |y|^\delta \wedge 1\, \pi(\cdot, dy) \le K\,, \quad a \ge \nu I\,,$$

where I is the unit $d\times d$-matrix. Let $G = G(K)$ be the set of measurable functions $r : \mathbf{R}^{d+1} \to \mathbf{R}$ such that $|r| \le K$.

For $r \in G$, $\gamma = (a, b, \pi) \in \Gamma$ we introduce the operator $L = L(r, \gamma)$ acting on $u \in C_0^\infty(\mathbf{R}^{d+1})$:

$$\begin{aligned} Lu(t,x) &= \partial_t u(t,x) + \sum_{i,j=1}^d a_{ij}(t,x) u_{x_i x_j}(t,x) \\ &+ \sum_{i=1}^d b_i(t,x) u_{x_i}(t,x) + r(t,x) u(t,x) \\ &+ \int \Big[u(t, x+y) - u(t,x) - \sum_{i=1}^d u_{x_i}(t,x) y_i 1_{|y|\le 1}\Big]\, \pi(t, x, dy)\,. \end{aligned}$$

The main result of this paper is the following statement.

Theorem 1. *Let $Q' \subset Q \subset \mathbf{R}^{d+1}$ be open subsets, $\rho_0 = \inf\{|x - x'| + |t - t'|^{1/2} : (t', x') \in Q',\ (t, x) \in \partial Q,\ t' > t\} > 0$, $u \in B(\mathbf{R}^{d+1}) \cap W_{d+1}^{1,2}(Q)$, $L = L(r, \gamma)$, $r \in G$, $\gamma \in \Gamma(\delta)$.*

Then for each $z, z' \in Q'$

$$|u(z') - u(z)| \le \Phi_\delta(\rho(z', z))\big(\|u\|_{\infty, \mathbf{R}^{d+1}} + \|Lu\|_{d+1, Q}\big)\,,$$

where $\Phi_\delta(R) = N R^\alpha$ for some constants $\alpha = \alpha(d, \nu, \delta, K) > 0$, $N = N(d, \nu, K, \rho_0) > 0$, if $\delta < 2$; and Φ_2 depends only on d, ν, K, ρ_0 and $\Phi_2(R) \to 0$ as $R \downarrow 0$.

2. Auxiliary results

Let $D = D_{[-\infty,\infty)}(\mathbf{R}^d)$ be the set of $\mathbf{R}^d$-valued cadlag functions on $[-\infty, \infty)$ with canonical process X, $X_t(\omega) = \omega_t$, $\omega \in D$, $\mathcal{D} = \sigma\{X_u,\ u \in [-\infty, \infty)\}$, $\mathcal{D}_t^s = \sigma\{X_u,\ u \in [s,t]\}$, $\mathbf{D}^s = (\mathcal{D}_{t+}^s)_{t \ge s}$.

Let $\mathcal{L}(\delta) = \{L(0,\gamma) : \gamma \in \Gamma(\delta)\}$. For $L \in \mathcal{L}(\delta)$, $(s,x) \in \mathbf{R}^{d+1}$ we denote by $S_{s,x}(L)$ the set of probability measures P on $(D,\mathcal{D})$ such that $P\{X_u = x, \text{ for all } u \le s\} = 1$ and for each $u \in C_0^\infty(\mathbf{R}^{d+1})$ the process

$$u(t,X_t) - \int_s^t Lu(v,X_v)\,dv$$

is a $(\mathbf{D}^s, P)$-martingale. According to [1], $S_{s,x}(L) \neq \emptyset$ for each $(s,x) \in \mathbf{R}^{d+1}$, $L \in \mathcal{L}(\delta)$, $\delta \in (0,2]$.

For $R \in (0,1]$, $s \in \mathbf{R}$ we define the process

$$X_u^{R,s} = R^{-1} X_{[R^2(u-s)+s]\vee s}\,.$$

Remark 1. If $\gamma = (a,b,\pi)$, $L = L(0,\gamma) \in \mathcal{L}(\delta)$, $P \in S_{s,x}(L)$, then $X^{R,s}(P) \in S_{s,x}(L(0,\tilde{\gamma}))$ and $L(0,\tilde{\gamma}) \in \mathcal{L}(\delta)$.

In fact, it is easy to see that $\tilde{\gamma} = (\tilde{a},\tilde{b},\tilde{\pi})$, where $\tilde{a}(t,x) = a(R^2t,Rx)$, $\tilde{b}(t,x) = b(R^2t,Rx)R$, $\tilde{\pi} = (t,x,dy) = R^2\int 1_{dy}(z/R)\pi(R^2t,Rx,dz)$.

Put

$$S_{s,x}^{(2)} = \bigcup_{L\in\mathcal{L}(2)} S_{s,x}(L)\,.$$

In [1] the following statement is proved.

Lemma 1. *There is a constant* $N = N(\nu,K)$ *such that for each* $(s,x) \in \mathbf{R}^{d+1}$, $P \in S_{s,x}^{(2)}$, $f \in B(\mathbf{R}^{d+1})$

$$\mathbf{E}\int_s^\infty e^{-(u-s)} f(u,X_u)\,du \le N\|f\|_{d+1,\mathbf{R}^{d+1}}\,.$$

Corollary 1. *Let* Q *be a bounded domain in* $\mathbf{R}^{d+1}$, $\tau = \inf\{t : (t,X_t) \notin Q\}$, $P \in S_{s,x}^{(2)}$, $f \in B(Q)$. *Then*

$$\mathbf{E}\int_s^\tau f(u,X_u)\,du \le Ne^{\operatorname{diam} Q}\|f\|_{d+1,Q}\,.$$

For $T > 0$, $R > 0$, $z \in \mathbf{R}^{d+1}$ we denote $C_{T,R} = (0,T)\times\{x : |x| < R\} \subset \mathbf{R}^{d+1}$, $K_R^z = z + C_{R^2,R}$.

Corollary 2. *Let* $z = (t_1,x_1) \in \mathbf{R}^{d+1}$, $R \in (0,1)$, $Q = K_R^z$, $(s,x) \in Q$, $\tau = \inf\{t > s : (t,X_t) \notin Q\}$, $P \in S_{s,x}^{(2)}$, $f \in L^{d+1}(Q)$. *Then*

$$\mathbf{E}\int_s^\tau f(u,X_u)\,du \le NeR^{d/(d+1)}\|f\|_{d+1,Q}\,.$$

Proof. In fact, a change of variables $u = R^2(\tilde{u} - s) + s$ gives

$$J = \mathbf{E}\int_s^\tau f(u, X_u)\,du = R^2\mathbf{E}\int_s^{\tau'} f(R^2(u-s)+s, RX_u^{R,s})\,du,$$

where $\tau' = \inf\{u > s : (u, X_u^{R,s}) \notin (t_1, R^{-1}x_1) + C_{v,1}\}$, $v = [R^2 + (s - t_1)]R^{-2} + s$. Thus by Remark 1 and Corollary 1

$$|J| \le R^2 eN\Big\{\int_s^{t_1+v} \int_{|x - R^{-1}x_1|<1} \big|f(R^2(u-s)+s, Rx)\big|^{d+1} ds\,dx\Big\}^{1/(d+1)}$$
$$\le NeR^{d/(d+1)}\|f\|_{d+1,Q}.$$

If Q is a domain in $\mathbf{R}^{d+1}$, $\varepsilon > 0$, we define $Q^\varepsilon = \{(t,y) \in \mathbf{R}^{d+1} : |y - x| < \varepsilon,\ (t,x) \in Q\}$.

The paper [2] contains the following statement.

Lemma 2. *Let Q be a domain in $\mathbf{R}^{d+1}$, $\varepsilon > 0$. There is a constant $N = N(d)$ such that for each $u \in W_{d+1}^{1,2}(Q^\varepsilon)$*

$$\|T^\varepsilon u\|_{d+1,Q} \le N\|u\|_{W_{d+1}^{1,2}(Q^\varepsilon)},$$

where

$$T^\varepsilon u(t,x) = \sup_{|y|\le\varepsilon} |y|^{-2}|u(t, x+y) - u(t,x) - \sum_{i=1}^d u_{x_i}(t,x)y_i\,|.$$

Lemma 3. *Let Q be a bounded domain in $\mathbf{R}^{d+1}$, $(s,x) \in Q$, $\varepsilon > 0$, $u \in B(\mathbf{R}^{d+1}) \cap W_{d+1}^{1,2}(Q^\varepsilon)$, $\tau = \inf\{t : (t, X_t) \notin Q\}$, $P \in S_{s,x}(L)$, $L \in \mathcal{L}(2)$. Then the process*

$$u(t\wedge\tau, X_{t\wedge\tau}) - \int_s^{t\wedge\tau} Lu(r, X_r)\,dr$$

is a $(\mathbf{D}^s, P)$-martingale.

Proof. Let $u_n \in C_0^\infty(\mathbf{R}^{d+1})$, $u_n \to \hat{u}$ in $W_{d+1}^{1,2}$, $\hat{u}|_{Q^\varepsilon} = u|_{Q^\varepsilon}$. Set $\tilde{u}_n = u_n 1_{Q^\varepsilon} + u 1_{\mathbf{R}^{d+1}\setminus Q^\varepsilon}$. By Lemma 3 [7] the statement is true for $\tilde{u}_n$. Since $\|\tilde{u}_n - u\|_{\infty,\mathbf{R}^{d+1}} \longrightarrow 0$ as $n \to \infty$, the statement for u follows then from Lemma 2 and Corollary 1.

Lemma 4. *Let ε, $\theta \in (0,1)$, $R_0 > 0$, $p_0 > 1$, c_1, $c_2 > 0$, $z \in \mathbf{R}^{d+1}$, $u \in B(K_{R_0}^z)$, $w_R = \operatorname{osc}\{u; K_R^z\} = \sup\{u(y) : y \in K_R^z\} - \inf\{u(y) : y \in K_R^z\}$, $R < R_0$. Then:*

a) *if for some $p > 1$ and each $R \le R_0/p$*

(1) $$w_R \le \theta w_{pR} + c_1 R^\varepsilon,$$

then for each $R \le R_0$, $0 < \alpha < \alpha_0 \wedge \varepsilon$

$$w_R \le N(w_{R_0} + c_1)R^\alpha$$

(here $N = N(\theta, p, \varepsilon, R_0, \alpha)$, $\alpha_0 = -\log_p \theta$*),*

b) *if for some* $p \ge p_0$, $R \le R_0/p$

$$w_R \le \theta w_{pR} + c_1 R^\varepsilon + c_2 p^{-2}, \tag{2}$$

then for some $r_0 = r_0(p_0, \theta)$ *and each* $R \le r_0$

$$w_R \le N(w_{R_0} + c_1 + c_2)\theta^{\sqrt{\xi - 2\sqrt{\xi - \dots}}},$$

where $N = N(R_0, \theta, \varepsilon)$, $\xi = 2\log_{1/\theta}(R_0/R) - 1$.

Proof. a) Let $R_k = p^{-k}R_0$, $k = 0, 1, 2, \dots$. Iterating the inequality (1) we see, that

$$\begin{aligned} w_{R_k} &\le \theta^k w_{R_0} + c_1 \sum_{i=0}^{k-1} \theta^i R_{k-i}^\varepsilon \\ &= p^{-k\alpha_0} w_{R_0} + c_1 R_0^\varepsilon p^{-k\varepsilon} \sum_{i=0}^{k-1} p^{(\varepsilon - \alpha_0)i}. \end{aligned}$$

This implies that for $\alpha < \alpha_0 \wedge \varepsilon$

$$\begin{aligned} w_{R_k} &\le p^{-k\alpha}\big[p^{-k(\alpha-\alpha_0)} w_{R_0} + c_1 R_0^\varepsilon p^{-k(\varepsilon-\alpha)} \sum_{i=0}^{k-1} p^{(\varepsilon-\alpha_0)i}\big] \\ &\le p^{-k\alpha}\big[w_{R_0} + c_1 R_0^\varepsilon \sum_i p^{(\alpha-\alpha_0)i}\big] \le N_1(w_{R_0} + c_1)R_k^\alpha, \end{aligned}$$

with

$$N_1 = R_0^{-\alpha}\big[1 + R_0^\varepsilon(1 - p^{(\alpha-\alpha_0)})^{-1}\big].$$

Fix an arbitrary $R \le R_0$. There is $k \ge 1$ such that $R_{k+1} \le R \le R_k$. Hence

$$\begin{aligned} w_R &\le w_{R_k} \le N_1(w_{R_0} + c_1)R_k^\alpha \le N_1(w_{R_0} + c_1)R^\alpha \frac{R_k^\alpha}{R_{k+1}^\alpha} \\ &= N_1 p^\alpha (w_{R_0} + c_1)R^\alpha. \end{aligned}$$

b) Let $p \ge p_0$, $R_k = p^{-k}R_0$, $k = 1, 2, \dots$. For $k \ge 1$ we have by (2)

$$w_{R_k} \le \theta w_{R_{k-1}} + c_1 R_k^\varepsilon + c_2 p^{-2}.$$

Iterating this inequality we see that

$$\begin{aligned} w_{R_k} &\le \theta^k w_{R_0} + c_1 \sum_{i=0}^{k} \theta^i R_{k-i}^{\varepsilon} + c_2 p^{-2} \sum_{i=0}^{k} \theta^i \\ &\le \theta^k w_{R_0} + c_1 R_0^{\varepsilon} \sum_{i=0}^{k} \theta^i p^{-(k-i)\varepsilon} + \frac{c_2}{1-\theta} p^{-2} . \end{aligned}$$

By taking $p = \theta^{-k/2}$ for $k \ge k_0 = 2\log_{1/\theta} p_0$, we obtain

$$w_{R_k} \le \theta^k \big(w_{R_0} + \frac{c_2}{1-\theta} + c_1 c_3 \big) \le \theta^k N ,$$

where $N = (w_{R_0} + c_1 + c_2)(1 - (1-\theta)^{-1} + c_3)$, $c_3 = R_0^{\varepsilon} \sum_i \theta^{-i+\varepsilon i^2/2}$. Thus for $k \ge k_0$ we have

$$w_{R_k} \le N \theta^{\sqrt{-2\log_{1/\theta}(R_k/R_0)}} .$$

Fix $R \le R_{[k_0]+1}$. Then there is $k \ge [k_0]+1$ such that $R_{k+1} \le R \le R_k$. Hence

$$\begin{aligned} w_R \le w_{R_k} &\le N\theta^{\sqrt{-2\log_{1/\theta}(R_k/R_0)}} \le N\theta^{\sqrt{-2\log_{1/\theta}(RR_k/R_0R_{k+1})}} \\ &\le N\theta^{\sqrt{-2\log_{1/\theta}(R/R_0) - 2k - 1}} \le N\theta^{\sqrt{\xi - 2\sqrt{\xi - \dots}}} \end{aligned}$$

and the lemma follows.

3. Proof of Theorem 1

Let $A = L(r,\gamma) \in \mathcal{L}(\delta)$, $\delta \in (0,2]$, $\gamma = (a,b,\pi) \in \Gamma$, $r \in G$, $R_0 = 1 \wedge \rho_0$, $z \in Q'$, $R < R_0/4$, $R_1 > 0$. We shall estimate the oscillation of the function u on K_R^z. Let $\overline{u}_\rho = \sup\{u(z') : z' \in K_\rho^z\}$, $\underline{u}_\rho = \inf\{u(y) : y \in K_\rho^z\}$, $w_\rho = \overline{u}_\rho - \underline{u}_\rho$, $\rho > 0$. Introduce the processes

$$\xi_t^i = \int_{-\infty}^{t} \Big[\int_{(s,X_s+y) \notin K_{2R+R_1}^z} (c_i - u(s, X_s + y)) \pi(s, X_s, dy) - r(s,X_s) u(s,X_s) \Big] 1_{K_{2R}^z}(s, X_s)\, ds ,$$

$i = 1,2$, with $c_1 = \overline{u}_{2R+R_1}$, $c_2 = \underline{u}_{2R+R_1}$. Set

$$\begin{aligned} Q_R^{(1)} &= \{(t,x) \in K_{2R}^z : 2u(t,x) \le \overline{u}_R + \underline{u}_R\}, \\ Q_R^{(2)} &= \{(t,x) \in K_{2R}^z : 2u(t,x) \ge \overline{u}_R + \underline{u}_R\} . \end{aligned}$$

Two cases are possible:

(3) $$2\,\mathrm{mes}\, Q_R^{(1)} \geq \mathrm{mes}\, K_{2R}^z\,,$$

(4) $$2\,\mathrm{mes}\, Q_R^{(2)} \geq \mathrm{mes}\, K_{2R}^z$$

(here mes stands for the Lebesgue measure on $\mathbf{R}^{d+1}$).

Consider the case (3). Let $\overline{u}_R = u(t_0, x_0)$, $(t_0, x_0) \in \overline{K_R^z}$, $P \in S_{t_0,x_0}(L(0,\gamma))$, $\tau = \inf\{t > t_0 : (t, X_t) \notin K_{2R}^z\}$, $\tilde{u} = u 1_{K_{2R+R_1}^z} + \overline{u}_{2R+R_1} 1_{\mathbf{R}^{d+1}\setminus K_{2R+R_1}^z}$.

By Lemma 3 the process

(5) $$\tilde{u}(t\wedge\tau, X_{t\wedge\tau}) - u(t_0, x_0) - \int_{t_0}^{t\wedge\tau} Au(s, X_s)\,ds + \xi^i_{t\wedge\tau} - \xi^i_{t_0}$$

is a $(\mathbf{D}^{t_0}, P)$-martingale for $i = 1$.

Let $\tau_1 = \inf\{t > t_0 : (t, X_t) \in Q_R^{(1)}\}$, $\beta_1 = P(\tau > \tau_1)$. Because of (5)

$$\overline{u}_R \leq \frac{1}{2}\beta_1(\overline{u}_R + \underline{u}_R) + (1-\beta_1)c_1 + \mathbf{E}\left|\int_{t_0}^{\tau\wedge\tau_1} Au(s, X_s)\,ds\right| + \mathbf{E}\,|\xi^1_{\tau\wedge\tau_1} - \xi^1_{t_0}|\,.$$

By subtracting $\underline{u}_R$ from both sides of this inequality we easily see that

(6) $$w_R \leq (1-\beta_i/2)w_{2R+R_1} + 2\mathbf{E}\,|\xi^i_{\tau\wedge\tau_i} - \xi^i_{t_0}| + 2\mathbf{E}\left|\int_{t_0}^{\tau\wedge\tau_i} Au(s, X_s)\,ds\right|$$

for $i = 1$. By Corollary 2 [7] there is a constant $\delta = \delta(\nu, K) > 0$ such that $\beta_1 \geq \delta$. Thus

(7) $$w_R \leq \theta w_{2R+R_1} + 2\mathbf{E}\,|\xi^i_{\tau\wedge\tau_i} - \xi^i_{t_0}| + 2\mathbf{E}\left|\int_{t_0}^{\tau\wedge\tau_i} Au(s, X_s)\,ds\right|$$

for $i = 1$, $\theta = 1 - \delta/2$.

Consider now the case (4). Let $\underline{u}_R = u(t_0, x_0)$, $(t_0, x_0) \in \overline{K_R^z}$, $\tau_2 = \inf\{t > t_0 : (t, X_t) \in Q_R^{(2)}\}$, $\beta_2 = P(\tau > \tau_2)$, $\tilde{u} = u 1_{K_{2R+R_1}^z} + \underline{u}_{2R+R_1} 1_{\mathbf{R}^{d+1}\setminus K_{2R+R_1}^z}$. Then for $i = 2$ the process (5) is a $(\mathbf{D}^{t_0}, P)$-martingale. It is easy to see that

$$\underline{u}_R \geq (1-\beta_2)\underline{u}_{2R+R_1} + \frac{1}{2}\beta_2(\overline{u}_R + \underline{u}_R) - \mathbf{E}\,|\xi^2_{\tau\wedge\tau_2} - \xi^2_{t_0}| - \mathbf{E}\left|\int_{t_0}^{\tau\wedge\tau_2} Au(s, X_s)\,ds\right|$$

and (6), (7) are true for $i = 2$ as well.

It remains to estimate

$$I^i = \mathbf{E}\left|\int_{t_0}^{\tau\wedge\tau_i} Au(s, X_s)\,ds\right| + \mathbf{E}\,|\xi^i_{\tau\wedge\tau_i} - \xi^i_{t_0}|\,, \quad i = 1, 2.$$

From Corollary 2 we have

$$
\begin{aligned}
I^i &\le N(\nu,K)R^{d/(d+1)}\big(\|Au\|_{d+1,K^z_{2R}} + \|u\|_{\infty,\mathbf{R}^{d+1}}\,R^{(d+2)/(d+1)} \\
&\quad\cdot \|\pi(\cdot,\{|y|>R_1\})\|_{\infty,\mathbf{R}^{d+1}} + 1\big) \qquad (8)\\
&\le N(\nu,K)\big[R^{d/(d+1)}\|Au\|_{d+1,K^z_{2R}} + \|u\|_{\infty,\mathbf{R}^{d+1}}(1+R_1^{-\delta}\wedge 1)R^2\big]\,.
\end{aligned}
$$

If $\delta < 2$ we obtain from (7)

$$w_R \le \theta w_{3R} + N_1(\|Au\|_{d+1,K^z_{2R}} + \|u\|_{\infty,\mathbf{R}^{d+1}})R^{\varepsilon}\,,$$

by taking $R_1 = R$; here $\varepsilon = \big(d/(d+1)\big)\wedge(2-\delta) > 0$, $N_1 = N_1(\nu,K)$. The statement of the theorem for $\delta < 2$ follows then from the part a) of Lemma 4.

If $\delta = 2$ we obtain from (7), (8) for each $p \ge 3$, $R \le R_0/p$

$$w_R \le \theta w_{pR} + NR^{d/(d+1)}(\|Au\|_{d+1,K^z_{2R}} + \|u\|_{\infty,\mathbf{R}^{d+1}}\,p^{-2})\,,$$

by taking $R_1 = (p-2)R$; here $N = N(\nu,K)$. The statement of the theorem for $\delta = 2$ follows then from the part b) of Lemma 4.

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