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ON THE LEVY–PROKHOROV DISTANCE BETWEEN COUNTING PROCESSES

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Let (N,F) be a counting process with a deterministic compensator A and let (M,G) be another counting process. Suppose that B is the compensator of (M,G). Let us consider the restrictions of the processes to the interval $[0,T]$ and denote by $LP(M,N)$ the Levy–Prokhorov distance between the distributions of M and N on the Skorokhod space $D[0,T]$ (for the definition of the Levy–Prokhorov distance see for example [6]). We wish to find an upper bound for $LP(M,N)$.

We have in [5] derived an upper bound for $LP(M,N)$ in the case where the function B is continuous. Let us now assume that $A_t = t$ for all $t \geq 0$, i.e., N is a standard Poisson process and the compensator B is deterministic and continuous. Then our result from [5] gives

$$LP(M,N) \leq \sup_{t\leq T} |B_t - t| + |B_T - T|. \tag{1}$$

In the present note we extend this result to a more general class of compensators. Before stating our main theorem we note that, in what follows, we define $\Delta X_t = X_t - X_{t-}$ for a cadlag-process X.

Theorem. *Suppose that N is a standard Poisson process and the compensator B is deterministic (but not necessarily continuous). Then*

$$LP(M,N) \leq \sup_{t\leq T} |B_t - t| + |B_T - T| + \frac{3}{2}\sum_{t\leq T}(\Delta B_t)^2 + \Delta B_T. \tag{2}$$

Before the proof we discuss the upper bound in (2). Let $X_1,\ldots,X_n$ be independent Bernoulli random variables with $P\{X_i = 1\} = 1/n$, $i = 1,\ldots,n$. If

$$M_t = \sum_{i=1}^{[nt]} X_i,$$

then M is a counting process with compensator B, $B_t = [nt]/n$. Let $T = 1$. From (2) we get the result of Dudley [2] (see also Whitt [6])

$$LP(M,N) \leq \frac{7}{2n}.$$

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Dudley shows in [2] that this bound cannot be improved to order $o(n^{-1})$.

To continue our discussion about (2) we recall the following special case of results in Kabanov et al. [3]:

If (M^n) is a sequence of counting processes with deterministic compensators B^n such that

$$B_t^n \longrightarrow t \tag{3}$$

for every $t \geq 0$, then

$$M^n \xrightarrow{L(D)} N, \quad \text{as } n \to \infty$$

(here $\xrightarrow{L(D)}$ means weak convergence in the Skorokhod space $D[0,T]$). It is easy to check that if (3) holds the upper bound given by (2) tends to zero as $n \to \infty$. Kabanov et al. [3] show that (3) is a necessary condition for the above weak convergence.

Proof of the Theorem. Let $\{0 = t_0, \ldots, t_n = T\}$ be a partition of the interval $[0,T]$ such that $t_i = iT/n, i = 0, \ldots, n$. If X is a process, then we write $f^n(X)$ for the discretized process

$$f_t^n(X) = X_{t_i}, \quad \text{if } t_i \leq t < t_{i+1},$$

$i = 0, \ldots, n-1$, $f_T^n(X) = X_T$. If X is a cadlag-process, then it is not difficult to see that $f^n(X)$ converges weakly to X on $D[0,T]$ as $n \to \infty$ so that $LP(f^n(X), X) \longrightarrow 0$ as $n \to \infty$.

Let g be a continuous nondecreasing function such that $g(t_i) = B_{t_i}, i = 0, \ldots, n$ and denote by H the counting process $H_t = N_{g(t)}$. Then g is the compensator of the process H (with respect to the natural σ-field). Now we can estimate $LP(M,N)$ in the following way:

$$LP(M,N) \leq LP(M, f^n(M)) + LP(f^n(M), f^n(H)) + LP(f^n(H), H) + LP(H,N).$$

As noted above, $LP(M, f^n(M)) \longrightarrow 0$ as $n \to \infty$.

Before giving an upper bound for the term $LP(f^n(M), f^n(H))$ we need some notation. Put $\Delta_i^n(B) = B_{t_i} - B_{t_{i-1}}$ for $i = 1, \ldots, n$. Denote by $V(M,H)$ the variation distance between the distributions of M and H on $D[0,T]$ and by $V^n(M,H)$ the variation distance between the distributions of $(M_{t_0}, \ldots, M_{t_n})$ and $(H_{t_0}, \ldots, H_{t_n})$. Note that $V(f^n(M), f^n(H)) = V^n(M,H)$. Hence we have also $LP(f^n(M), f^n(H)) \leq V^n(M,H)$. According to a result of Brown [1] or Kabanov et. al [4]

$$V^n(M,H) \leq \sum_{i=1}^n |\Delta_i^n(B) - \Delta_i^n(g)| + \sum_{t \leq T} (\Delta B_t)^2. \tag{4}$$

But here $g(t_i) = B_{t_i}$ and so $|\Delta_i^n(B) - \Delta_i^n(g)| = 0$ in (4). Hence we have for the term $LP(f^n(M), f^n(H))$ the following upper bound:

$$LP(f^n(M), f^n(H)) \leq \sum_{t \leq T} (\Delta B_t)^2. \tag{5}$$

Note that H is a counting process with continuous compensator g. It is easily seen that

$$\sup_{t \leq T} |g(t) - t| \leq \sup_{t \leq T} |B_t - t| + \frac{T}{n}.$$

This and (1) yield

$$LP(H, N) \leq \sup_{t \leq T} |B_t - t| + \frac{T}{n} + |B_T - T|. \tag{6}$$

Next we derive an upper bound for $LP(f^n(H), H)$. We use the method of Dudley [2] (see also Whitt [6]). Denote by d_T the Skorokhod distance on $D[0, T]$. For any $\delta > 0$ we have

$$LP(f^n(H), H) \leq \max\{\delta\,,\, P\{d_T(f^n(H), H) \geq \delta\}\}.$$

Define

$$F = \{H_T - H_{t_{n-1}} \geq 1\} \quad \text{and} \quad G = \bigcup_{i=1}^{n} \{H_{t_i} - H_{t_{i-1}} \geq 2\}.$$

Put $C = F \cup G$. Dudley shows in [2] that on the complement of the set C we have $d_T(f^n(H), H) \leq T/n$. First we note the following inequality:

$$P(d_T(f^n(H), H) \geq \delta) \leq P(\{d_T(f^n(H), H) \geq \delta\} \cap C) + P(\{d_T(f^n(H), H) \geq \delta\} \cap C^c)$$

so that

$$LP(f^n(H), H) \leq P(F) + P(G) + \frac{T}{n}. \tag{7}$$

Because $g(t_i) = B_{t_i}$, we have in (7):

$$P(F) \leq B_T - B_{t_{n-1}} \quad \text{and} \quad P(G) \leq \frac{1}{2} \sum_{i=1}^{n} (\Delta_i^n(B))^2.$$

Letting now $n \to \infty$ in (7) we get

$$\limsup_{n} LP(f^n(H), H) \leq \Delta B_T + \frac{1}{2} \sum_{t \leq T} (\Delta B_t)^2. \tag{8}$$

The claim (2) follows now from (5), (6) and (8), by letting $n \to \infty$. This completes our proof.

References

[1] BROWN, T.C.: Some Poisson approximations using compensators. - Ann. Probab. 11, 1983, 726–744.

[2] DUDLEY, R.M.: Speeds of metric probability convergence, - Z. Wahrsch. Verw. Gebiete 22, 1972, 323–332.

[3] KABANOV, YU.M., R.SH. LIPTSER, and A.N. SHIRYAYEV: Some limit theorems for simple point processes (a martingale approach). - Stochastics 3, 1980, 203–216.

[4] KABANOV, YU.M., R.SH. LIPTSER, and A.N. SHIRYAYEV: Weak and strong convergence of distributions of counting processes. - Theor. Probability Appl. 28, 1983, 303–335 (English translation of Teor. Veroyatnost. i Primenen. 28, 2, 1983, 288–319).

[5] NIKUNEN M., and E. VALKEILA: Metric distances between counting processes. - In Statistics and control of random processes, Steklov seminar, 1984, edited by N.V. Krylov, R.Sh. Liptser and A.A. Novikov, Optimization Software Inc., New York, 1985, 377–388.

[6] WHITT, W.: On the quality of Poisson approximations. - Z. Wahrsch. Verw. Gebiete 28, 1973, 23–26.

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