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THE IMPACT OF LARS AHLFORS'S WORK IN VALUE–DISTRIBUTION THEORY

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Introduction

The major emphasis of Lars Ahlfors's research before the Second World War was connected with value-distribution theory, although with hindsight we can also see hints of later directions. Function theory already had a long tradition in Finland [11, p. 4], [32] and Ahlfors began his career just after the first waves from the theory developed by his teacher, Rolf Nevanlinna, had crested. In an age with no photocopying, limited telephone and transportation, fundamental contributions by Collingwood and Littlewood (in England), Bloch, Valiron and Cartan (France), and Shimizu (Japan) and others had quickly followed Nevanlinna's publication [58]; in [58, p. 91] we find special mention of the significant role of Collingwood and Littlewood in the very formulation of the second fundamental theorem. The main direction of Ahlfors's work was to return to the foundations of the subject, and view it from a more geometric (and differential geometric) point of view. This has provided the basis of almost all subsequent generalizations. We also find here the genesis of later work in quasiconformal mappings, conformal metrics and extremal length.

There may have been a less dramatic influence on the classical situation of meromorphic functions in the plane, where much research has been concentrated on problems already posed by Nevanlinna himself. For the most part, these were more refined questions whose very formulation would involve the order ρ of the function; in Ahlfors's work in value distribution, ρ rarely appears. However, his very first achievement was in a different direction, one in which ρ plays an essential role.

1. Denjoy's conjecture—Ahlfors's theorem

Ahlfors accompanied Nevanlinna to Zürich in 1928–29, where he became interested in Denjoy's conjecture: if f is entire and ($M(r)$ is the maximum modulus function) $\log M(r) = O(r^{\rho})$ as $r \to \infty$, then f can have at most 2ρ distinct finite asymptotic values. Many years later, Nevanlinna told me of his concern when his student disappeared for several weeks. Finally he appeared with the solution [2] in hand; Nevanlinna had by then been apprehensive for the young man's safety.

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This may appear as a specialized question, but "its general significance depends on the influence it has exerted on developing new methods" [62, p. 14]. Today, this work is probably viewed as belonging to harmonic measure–extremal length, so outside the scope of this article. It does display an elegant use of the length–area method, a technique which Ahlfors applied in many settings through his career. A full discussion of Ahlfors's work [2] and its enormous impact appears in Baernstein's article in this collection.

2. Nevanlinna's theory

This subject is well developed in the texts [41], [44], [60], [64]. We recall some basic notions. Let $f(z)$ be meromorphic in $B(R) = \{|z| < R\}$, where $0 < R \leq \infty$. Let $N(r,a) = \int_0^r n(t,a)t^{-1}\,dt$ be the integrated counting function of the a-values including multiplicity; similarly we define $\overline{N}(r,a)$ in terms of $\overline{n}(r,a)$, where multiplicity is ignored (these formulas must be modified when $f(0) = a$). The compensating term is

$$(2.1)\qquad m(r,a) = m\left(r, \frac{1}{f-a}\right) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta}) - a|}\,d\theta \qquad (a \neq \infty)$$

$(m(r,\infty) = m(r,f) = (2\pi)^{-1}\int_0^{2\pi} \log^+|f(re^{i\theta})|\,d\theta)$. Nevanlinna defines the increasing function $T(r) \equiv T(r,f) = m(r,\infty) + N(r,\infty)(0 \leq r < R)$ and obtains his first fundamental theorem by rewriting Jensen's formula:

$$(2.2)\qquad T(r,f) = T\left(r, \frac{1}{f-a}\right) + O(1) \qquad (\geq N(r,a) + O(1))$$

where the bounded term $O(1)$ depends on a.

We call f *admissible* if $R = \infty$ and f is nonconstant, or if $R < \infty$ and $\limsup_{r\to R} T(r)\left[\log\{1/(R-r)\}\right] = \infty$. The second fundamental theorem is considered to be far deeper than (2.2), and requires that f be admissible. The counting-function of the branch set is

$$N_1(r) \equiv 2N(r,\infty,f) - N(r,\infty,f') + N\left(r, 0, \frac{1}{f'}\right),$$

and the branching density is

$$\Theta(f) = \liminf_{r\to\infty} \frac{N_1(r)}{T(r)}.$$

The defect and index of multiplicity are defined by

$$(2.3)\qquad (0 \leq)\ \delta(a) = \liminf_{r\to\infty} \frac{m(r,a)}{T(r)}, \quad (0 \leq)\ \theta(a) = \liminf_{r\to\infty} \frac{N(r,a) - \overline{N}(r,a)}{T(r)},$$

so that $0 \le \delta(a) + \theta(a) \le 1$.

The second fundamental theorem asserts that ($\mathbf{C}^*$ is the Riemann sphere)

$$\sum_{a \in \mathbf{C}^*} \{\delta(a) + \theta(a)\} \le \sum \delta(a) + \Theta(f) \le 2, \tag{2.4}$$

in particular that

$$\sum \delta(a) \le 2. \tag{2.5}$$

Inequality (2.5) is Nevanlinna's generalization of Picard's theorem on omitted values, while the more precise (2.4) can be seen as a transcendental form of the Riemann–Hurwitz theorem. The hypothesis that f be *admissible* is essential for (2.4) and (2.5). For example, given $q \ge 2$, and complex numbers $\{a_j\}_1^q$, there exists a holomorphic universal cover map $f : B(1) \to \mathbf{C} \setminus \{a_j\}_1^q$ with [60, p. 149]

$$T(r, f) = \frac{1}{q-2} \log \frac{1}{1-r} + O\left(\log\log \frac{1}{1-r}\right). \tag{2.6}$$

Nevanlinna's proof of (2.4) and (2.5) was based on his analysis of the logarithmic derivative $(f'/f)(z)$. The key to his approach is the lemma of the logarithmic derivative (LLD)

$$m\left(r, \frac{f'}{f}\right) = O(\log rT(r)) \qquad \| \tag{2.7}$$

where, following H. Weyl, $\|$ indicates that the inequality may fail on a small r-set as $r \to \infty$. (Examples to show the exact size of the exceptional set in (2.7) have been given by A. Fernández [34].) The role of the LLD is that it allows all the proximity terms $m(r, a)(a \in \mathbf{C})$ to be bounded in terms of $m(r, 0, f')$.

That the sums in (2.4) and (2.5) have such simple upper bounds is also a consequence of the fact that definitions (2.3) involve $\liminf$, so the exceptional r-set in (2.7) can be avoided.

Ahlfors's approach was based on more geometric considerations, and avoided the LLD. While it has allowed this theory to be extended to more general settings, the approach based on (2.7) also retains its value.

It has been essential, for example, in a significant deepening of (2.5) for admissible functions in the plane, although the complete analogue to (2.4) remains unachieved at present. That is, the deficiency relation (2.5) remains valid when the sum is taken over all *small functions* $a(z)$, those for which $T(r, a(z)) = o(T(r, f))$ $(r \to \infty)$. This was shown some years ago for entire functions by C.T. Chuang [26]; a very long proof for meromorphic functions was obtained by Ch. Osgood

[66], and finally the full result was placed into a most elegant setting by N. Steinmetz [81]. Steinmetz's proof parallels Nevanlinna's original arguments. Given small functions $a_1(z)$, ..., $a_q(z)$, Steinmetz constructs a differential polynomial $P[f]$ such that $P[f] \not\equiv 0$, but $P[f - a_j(z)] \equiv P[f]$; the manner by which P is obtained is especially compelling. Nevanlinna's original proof follows when we choose $P[f] = f'(z)$.

More recently, W. Stoll [85] has made a partial extension to maps $f\colon \mathbf{C} \to \mathbf{CP}^n$.

3. Refinements of (2.2)

The distinction between the first and second fundamental theorems is not always clear. The general principle is that (2.2) is a relation which is valid for each value of r when f is nonconstant, while (2.4) and (2.5) apply when $R < \infty$ only for rapidly growing functions and use estimates such as (2.7) which fail for some values of r. The results discussed in this section only require that $T(r) \to \infty$ as $r \to R$.

Ahlfors's first work after [2] was based on the recent thesis of H. Cartan [17]. Cartan's starting point was to prove what now is known as the Boutroux–Cartan lemma: given $h > 0$,

$$(3.1) \qquad |P(z)| \equiv \prod_{i=1}^{n} |z - a_i| > h^n \qquad (z \notin E),$$

where E can be included in balls $B(z_j, r_j)$ with $\Sigma r_j \leq 2eh$; perhaps the factor e is not needed. (Inequality (3.1) had been asserted without proof by A. Bloch.) Cartan recognized that this conclusion could be greatly generalized (cf. [64, p. 146]), and even made some applications to Nevanlinna's theory.

Ahlfors pushed this much further, and in particular, put some tentative results of J. Littlewood and G. Valiron in precise form.

The principle of [3] is to apply the Cartan theory to each circle $\{|z| = r\}$, $0 < r < R$. Thus if h is increasing, $h(0) = 0$, $h(\infty) = 1$ and

$$(3.2) \qquad \int_0 h(t)t^{-1}\,dt < \infty,$$

then if $0 < r < R$, one of Ahlfors's generalizations of (3.1) is that for each r,

$$(3.3) \qquad m(r, a) \leq \int_{h=0}^{h=1} \log^+(1/t)\,dh(t) \qquad (a \notin E)$$

where E can be covered by balls $B(z_n, r_n)$ having $\Sigma h(r_n) \leq 6$. If for a fixed $\varepsilon > 0$ we replace $h(t)$ by $\varepsilon^{-1}h(t)$, then the right side of (3.3) remains finite. Hence

(3.3) may be applied on a relatively thick sequence $r_n \to R$, and the bounds for $m(r,a)$ with $r = r_n, r_{n+1}$ will hold universally as $r \to R$.

Inequality (3.3) is applied with specific choices of h, and shows that, for most a and r, $m(r,a)$ is small; thus no lim inf's are needed as in (2.3). For example, given $\eta > 0$, let F_η be the a-set such that

$$\limsup_{r\to R} \frac{\log m(r,a)}{\log T(r)} > \frac{1}{1+\eta}. \tag{3.4}$$

Ahlfors proved that if $\eta > 1$, then F_η has zero measure with respect to the Hausdorff function $\left(\log(1/t)\right)^{-\eta}$. Also, if $0 < \beta \le 2$ and E_β is the set of a with

$$\limsup_{r\to R} \frac{m(r,a)}{\log T(r)} > \frac{1}{\beta}, \tag{3.5}$$

then E_β has β-dimensional measure zero $\left(h(t) = t^\beta\right)$. (Littlewood [53] had settled the case $\beta = 2$.)

Another application was to the set V of Valiron defects. Let

$$\Delta(a) = \limsup_{r\to R} \frac{m(r,a)}{T(r)} \tag{3.6}$$

and

$$V = V(f) = \{a; \Delta(a) > 0\}. \tag{3.7}$$

Definition (3.6) was introduced by G. Valiron [88] who proved that if f has finite order then V is a planar null set. Ahlfors improved this: V has h-measure zero for any h-measure such that (3.2) holds (with no hypothesis on order). Finally, he showed that if $R < \infty$, $T(R) = \infty$ and

$$V^* = \left\{a; \lim_{r\to R} N(r,a) < \infty\right\}, \tag{3.8}$$

then V^* must also have h-measure zero whenever h satisfies (3.2).

Soon after, Frostman [37] developed his theory of logarithmic capacity, and instead of Hausdorff measures used the equilibrium measure in these questions. In this way he showed that in Ahlfors's situation V^* must have capacity zero. Nevanlinna then returned to the sets F_η of (3.4) and proved that F_η has zero capacity when $0 < \eta \le 1$. Sets of capacity zero just include those of zero h-measure with h as in (3.2).

These results are all essentially sharp. For the exceptional sets in (3.4) and (3.5), see W. K. Hayman [46] (for meromorphic functions) and J. Fernández [35] (for holomorphic functions). Both constructions, however, only yield functions defined in $B(1)$, rather than the full plane. It is not hard to see that capacity zero is the proper exceptional set corresponding to (3.7) and (3.8), for if E is any compact set of capacity zero with at least two points, then the universal cover map $f\colon B(1) \to \mathbf{C} \setminus E$ has $T(1) = \infty$, and $V = V^* = E$.

There is one construction when $R = \infty$ which warrants special mention. By an ingenious sewing of universal covering mappings, Hayman [45] showed that Nevanlinna's estimate for V is essentially best possible for functions in the plane: V can be any F_σ-set of capacity zero. Thus, the sets F_η in (3.4) are essentially independent of η when $\eta \le 1$. However, when f has finite order in the plane, then V is substantially smaller than capacity zero, as shown by Hyllengren [50]; Hyllengren's analysis exploits the absence of exceptional set in (2.7); conversely, its presence is the key to Hayman's example.

Much of this has been extended to higher dimensions. See, for example, [15] and the survey in [73]. Application to a different situation is in [1].

4. The second fundamental theorem and conformal metrics

Here we discuss [5] (and to a lesser extent [8]) and its influence on some of the generalizations of Nevanlinna theory. We consider equidimensional mappings in Section 5 and meromorphic curves in Section 6. That these papers have proved so influential seems due to the transparency of methods in [5] and the connections with geometry in [8].

The paper [5] gives an efficient, self-contained and rather complete account of the author's version of classical Nevanlinna theory, one that is often used in the standard texts (cf. [49], [87]). Ahlfors defines his proximity function in terms of the spherical metric $[a,b]$ on $\mathbf{C}^*$: $2\pi m(r,a) = \int_0^{2\pi} \log\bigl(1/[f(z),a]\bigr)\,d\theta$, $(z = re^{i\theta})$. Then if $S(t)$ is the spherical area of $f\bigl(B(t)\bigr)$, we set

$$T(r) = \int_0^r t^{-1} S(t)\,dt. \tag{4.1}$$

This definition of $T(r)$ differs from that in Section 2 by a bounded factor. Priority for definition (4.1) is due to T. Shimizu [78], although it is Ahlfors's analysis that is used today.

Ahlfors's proof of (2.2) using (4.1) is a direct consequence of the argument principle. Of more lasting significance is his approach to (2.4). Given a non-negative L^1 function $\rho(w)$ on $\mathbf{C}^*$ with

$$\int_{\mathbf{C}^*} \rho(w)\,d\omega(w) = 1, \tag{4.2}$$

($d\omega =$ spherical measure) Ahlfors makes an elementary but profound analysis of

$$\lambda(r) = \int_0^{2\pi} \frac{|w'|^2}{\left(1+|w|^2\right)^2}\rho(w)\,d\theta \qquad \bigl(w = f(z),\ z = re^{i\theta}\bigr). \tag{4.3}$$

By the arithmetic-geometric means inequality

$$(4.4) \qquad \int_0^{2\pi} \log \frac{|w'|^2}{(1+|w|^2)^2}\, d\theta + \int_0^{2\pi} \log \rho(w)\, d\theta \leq 2\pi \log \lambda(r) + \text{ const.}$$

The terms on the left side of (4.4) reflect different considerations, ones which carry through in other contexts: the differential geometry of the target space $\mathbf{C}^*$ and imposing a suitable metric on it. Also, it is not hard to see that the right side of (4.4) is negligible:

$$(4.5) \qquad \log \lambda(r) = O\big(\log rT(r)\big). \qquad \|$$

Indeed, we may average $\lambda(r)$ so that

$$H(r) \equiv \int_0^r t^{-1}\, dt \int_0^t \lambda(s) s\, ds \equiv \int_{\mathbf{C}^*} N(r,a)\rho(a)\, d\omega(a),$$

so according to (2.2), $H(r) \leq T(r) + O(1)$. The bound in (4.5) follows, since calculus ([43, p. 43], [64, p. 253]) asserts that since H is increasing, $H''(r) < r^K H^{1+\delta}(r)$ for most large r, where $K < \infty$ and $\delta > 0$ are fixed. Thus (4.5) follows on taking logarithms.

An appeal to (2.2) readily yields that the first integral in (4.4) is exactly $N_1(r) - 2T(r)+$ const. (this will yield the branching term in (2.4)) and is one explanation for the "2" in Picard's theorem. Thus all depends on the choice of $\rho(w)$ and a computation of

$$(4.6) \qquad \Lambda(r) \equiv \int_0^{2\pi} \log \rho(w)\, d\theta \qquad \big(w = f(z),\ z = re^{i\theta}\big).$$

Ahlfors chooses, for a fixed $\alpha > 1$,

$$(4.7) \qquad \log \rho(w) = 2 \sum_1^q \log\big(1/[w, a_\nu]\big) - \alpha \log\Big(\sum \log\big(1/[w, a_\nu]\big)\Big) + C$$

(C to guarantee (4.2)), and computes directly that $\Lambda(r)$ in (4.6) differs from $2\pi\Sigma m(r, a_\nu)$ by $O\big(\log T(r)\big)$. Hence (2.4) and so (2.5) follow at once from these estimates.

That elementary metrics such as (4.7) can achieve (2.4) made generalizations tempting and perhaps inevitable. Ahlfors comments that his motivation for (4.7) was F. Nevanlinna's proof of (2.4) based on the hyperbolic metric on $\mathbf{C}_q \equiv \mathbf{C} \setminus \{a_j\}_1^q$ ([57]; cf. [64, IX.4]). This metric is the pull-back to $\mathbf{C}_q$ of the Poincaré metric of constant negative curvature on $B(1)$, that is, a metric $\tau(w)|dw|^2$ on $\mathbf{C}_q$ with

$$\Delta \log \tau = -\tau^2, \tag{4.8}$$

Nevanlinna's use of this metric is in fact most appropriate for these questions: Picard's original proof of his theorem used in an essential way that $B(1)$ is the universal cover of $\mathbf{C}_q$. Although Nevanlinna's metric τ is highly nonexplicit, he carefully analyzed the behavior of $\tau(w)$ as w approaches the a_ν-points (this is reproduced in [64]), and this was enough for his analysis, which is parallel to that of (4.6), to succeed.

Ahlfors's insight was to write an explicit elementary metric ρ as in (4.7) such that $|\log(\rho/\tau)|$ is bounded near the a_ν, and then observe that the precise definition of ρ on compact portions of $\mathbf{C}_q$ plays no significant role, so long as ρ is strictly positive and smooth.

Paper [5] never discusses differential geometry, as (4.6) subject to (4.7) is computed directly. Later, Ahlfors [8] viewed these problems from a differential-geometric point of view, basing his approach on curvature and the Gauss–Bonnet theorem. Curvature as in (4.8) arises when computing expressions such as $\Lambda(r)$ in (4.6) since according to Green's identities, $r\Lambda'(r) = \int\int_{B(r)} \Delta \log \rho \, dx \, dy$. F. Nevanlinna's analysis explored this connection directly.

Paper [8] (and its abstracted form [7]) was the basis of Chern's interest twenty years later [24]. Chern's work in [24] was confined to Riemann surfaces, but began a new period of activity which achieved fundamental higher dimensional generalizations. An account of [24], in a presentation for classical function theorists, occupies the final chapter of [38]. The research monograph of Sario–Noshiro [70] reflects a similar orientation.

Chern's next contribution, [25] (submitted only a few months after [24]), started a new direction. It is phrased in the language of complex manifold theory and initiated a new theory of equidimensional holomorphic mappings $f\colon \mathbf{C}^m \to \mathbf{CP}^m$ (projective space). [In the early 1950's, W. Stoll also considered this problem, but Stoll's approach was based on [10], and so is deferred to Section 6.] Chern only obtains an (integrated) first main theorem, but that is sufficient to give a theorem of Casorati–Weierstrass type.

Appendix. Ahlfors's success with elementary, if singular metrics led to his short note [9], which must be among his most cited papers. If one is interested in results less refined than the Nevanlinna theorems (2.4) and (2.5), it is possible to go far with surprisingly elementary methods in the spirit of [9]. One useful account of this is in Minda–Schober [56].

5. The Carlson–Griffiths theory

The path suggested by [25] was pursued in the 1970's, by Ph. Griffiths and his co-authors, especially J. Carlson and J. King. A spirited guided tour is in Griffiths's lectures [43], and there now is a beautifully written expository monograph by B. Shabat [73].

We consider an entire holomorphic map

$$f\colon \mathbf{C}^m \to M \tag{5.1}$$

where M is a compact complex manifold, $\dim_{\mathbf{C}} M = m$. When the Jacobian determinant of f is not identically zero, f is called non-degenerate. The most ambitious program would be to study how often f covers objects D in M of various dimensions, but this seems out of reach. A particularly infuriating fact, for example, is that (Fatou–Bieberbach) f can miss an open set when $M = \mathbf{CP}^m$ with $m \geq 2$.

The viewpoint developed by Ahlfors, and, in particular the use of concrete metrics such as (4.7), can make a significant penetration of this problem, and it is possible to recognize Ahlfors's analysis of (4.3) beneath a great deal of sophisticated machinery.

The work [16] shows that, despite Fatou–Bieberbach, the theory does extend satisfactorily to analyse the value-distribution on $\mathbf{C}^m$ of certain *divisors* $D = \sum_{j=1}^{q} D_j$ on M. Recall that divisors are locally the zeros of holomorphic functions; in the classical theory they may be identified with points $w = a$. Here the D_j are smooth irreducible divisors which are required to meet transversally (have normal crossings). Thus in some neighborhood U of any point p of M through which exactly k divisors pass, we may choose holomorphic coordinates $w_1, \ldots, w_m$ so that $D \cap U = \{w; \prod_1^k w_k = 0\}$. In the classical situation, the hypothesis that D have normal crossings asserts that the points a_ν are distinct.

The theory of [16], and especially the hypothesis on normal crossings, allows the classical machinery be adapted to the situation (5.1) when two conditions are met, one a condition on M to get matters started, and another which ensures that D is large (the analogue of Picard's hypothesis that the omitted set have $q \geq 3$ points).

First, M is required to have a positive holomorphic line bundle L. This allows definition (4.1) to be transferred immediately, where now $S(t)$ is the integral on $\|z\| < t$ of the pull-back of the Chern form (curvature) of any holomorphic section s of L. The requirement that L be positive guarantees, for example, that S is the integral of a positive quantity. (That M possess such an L is a hypothesis of independent significance, as shown by K. Kodaira; cf. [93, Chapter VI].) It is then not difficult to obtain a first fundamental theorem and define deficiencies $\delta_f(D)$, $\delta_f(D_j)$ with $0 \leq \delta_f(D_j) \leq 1$, $q\delta_f(D) = \Sigma\delta_f(D_j)$ and $\delta_f(D_j) = 1$ if $f(\mathbf{C}^m)$ misses D_j.

The Carlson–Griffiths theory views second fundamental theorems as consequences of the existence on $M \setminus D$ of metrics of negative—or nearly negative—curvature [28, p. 94], and it is this requirement that forces D to be large. Thus, we let $\Psi_M(w)$ be the metric on M which arises from the canonical bundle; i.e. the bundle on M whose transition functions are the Jacobians of the coordinate changes in the intersections $U_{\alpha\beta}$ of domains in a covering of M. This reduces to the chordal metric when $M = \mathbf{C}^*$. Next, in terms of the components D_j of D, we may define sections $s_j(w)$ locally on M such that in any coordinate patch U, the divisor of s_j restricted to U is $D_j \cap U$. This leads to a globally defined Hermitian modulus $\|s_j\|$ [73, p. 78] which vanishes precisely on D_j. Then things proceed in a manner parallel to [5] if we pull back to $\mathbf{C}^m$, using f, the singular volume form on M

$$\psi_M(D) = \frac{\psi_M(w)}{\prod_1^q \|s_j\|^2 \left(\log(c\|s_j\|^2)\right)^2} \tag{5.2}$$

where c is some fixed and small positive constant. When we study $\log \psi_M(D)$ on $\mathbf{C}^m$, we find exact analogues of each of the terms on the left side of (4.4). The Ahlfors–Nevanlinna conditions on curvature become phrased in terms of the Ricci form if $\Psi_M(D)$ and apply when q is sufficiently large so that the Ricci form of $\Psi_M(D)$ is essentially positive [43, p. 49]. (The negative of the Ricci form is a generalization of curvature.)

Here are some applications of the Carlson–Griffiths theory. Suppose first that $M = \mathbf{CP}^m$ and the D_j arise from hyperplanes H_j in general position. The correspondence is $D_j \Leftrightarrow H_j \sim \{w = (w_0, \dots, w_m; \sum \alpha_{jk} w_k = 0\}$; see Section 6. The deficiency relation is $\sum \delta(H_j) \leq m+1$, a result established already in 1953 by Stoll [82] with a slightly more restrictive definition of nondegeneracy. Later [83], Stoll showed how to improve his degeneracy hypotheses.

A striking advantage of the Carlson–Griffiths theory is that it applies to nonlinear divisors. Thus if D is a nonsingular divisor on $\mathbf{P}^m$ of degree q, and $f\colon \mathbf{C}^m \to \mathbf{CP}^m$ is nondegenerate then

$$\sum \delta(D_j) \leq (m+1)/q. \tag{5.3}$$

When D does not have normal crossings, or is of codimension greater than one, the situation is not fully settled; cf. [73, II.6], [76], [27]. Thus while M. Green [42] has shown that if $f\colon \mathbf{C}^m \to \mathbf{CP}^m$ omits $(m+2)$ distinct hyperplanes [not necessarily in general position] then f must be degenerate, B. Shiffman [75] has shown that $\sum \delta(H) = \infty$ is possible if we sum over distinct (as opposed to general position) hyperplanes.

Y.T. Siu [79] has initiated a new study of non-equidimensional mappings which makes deeper connections with differential geometry. Siu also only considers value-distribution of divisors on M.

6. Meromorphic curves

The study of maps which decrease dimension can be reduced to that of equidimensional maps; cf. [73, p. 209]. Here we consider the other extreme of *meromorphic curves* $f = (f_0, \ldots, f_m)\colon \mathbf{C} \to \mathbf{C}^{m+1}$, so that each f_j is an entire function. Outside $f^{-1}(0, \ldots, 0), f$ induces $\tilde{f}\colon \mathbf{C} \to \mathbf{CP}^m$ using homogeneous coordinates, and unless $f \equiv 0, \tilde{f}$ may be extended to all of $\mathbf{C}$. Thus f and $\tilde{f}$ are essentially equivalent. This is the most fully worked-out multidimensional theory, where important advances continue to be made even in recent times.

The history of the subject was unclear for some years, but today two distinct lines emerge (cf. [51, Chapter VII]). The first, due to H. Cartan (announced in 1929 [21], published in [22]) is based on Nevanlinna's *LLD*, and studies how often the image of f meets hyperplanes in $\mathbf{CP}^m$, i.e. the image in $\mathbf{CP}^m$ of planes $H\colon \sum_{j=0}^{m} \alpha_j w_j = 0$ in $\mathbf{C}^{m+1}$.

Suppose that $f(\mathbf{C})$ is not contained in any hyperplane of $\mathbf{CP}^m$. Then Cartan defines $T(r, f) = (1/2\pi) \int_0^{2\pi} \max_j \{\log |f_j(re^{i\theta})|\}\, d\theta$, $n(r, H)$ the number of zeros of $\sum \alpha_j f_j(z)$ $(|z| < r)$ and $N(r, H) = \int_0^r t^{j-1} n(t, H)\, dt$. (This elegant definition of $T(r)$, which agrees with Nevanlinna's definition when $m = 1$, was suggested by A. Bloch; cf. [18].) The first fundamental theorem follows from Jensen's formula. Finally, if W is the Wronskian determinant of f, $W \not\equiv 0$, then Cartan obtains his analogue of (2.4) for hyperplanes $H_1, \ldots, H_q$ *in general position*:

$$(6.1) \qquad (q - m - 1)T(r) \le \sum_1^q N(r, H_j) - N(r, W) + o(T(r)) \qquad \|,$$

so that in particular

$$(6.2) \qquad \sum \delta(H_j, f) \le m + 1.$$

Cartan's elegant proof in [22] is reproduced in [51] and [77].

As one direct application of (6.2), we obtain E. Borel's generalization of Picard's theorem: if $f_0, \ldots, f_m$ are non-vanishing linearly-independent entire functions, then no non-trivial linear combination $f = \sum \alpha_j f_j$ can be zero free. (Indeed, were this true, $m + 2$ hyperplanes $(1, 0, 0, \ldots)$, $(0, 1, 0, \ldots)$, $(0, 0, \ldots, 1)$, $(\alpha_0, \cdots, \alpha_m)$ would have maximal deficiency one.)

Paper [22] was soon forgotten, and the next major work in the subject is due to H. and J. Weyl [94]. The Weyls had as goal to adapt Ahlfors's theory in [5] to meromorphic curves, and attempted to absorb the study of the *associated curves* in their theory. That is, if $\tilde{f}\colon \mathbf{C} \to \mathbf{C}^{m+1}$ is a representation of f, then for $0 \le k \le m$ the curve

$$\tilde{f}^{[k]} = \tilde{f} \wedge \cdots \wedge \tilde{f}^{(k)}\colon \mathbf{C} \to \mathbf{C}^M, \qquad M = \binom{m+1}{k+1}$$

projects to the kth associated curve $f^{[k]}: \mathbf{C} \to G(k;m)$ (the Grassmanian of linear k-spaces in $\mathbf{CP}^m$) $\approx \mathbf{CP}^{M-1}$. They also defined each $T_{(k)}(r)$, and studied the zero-distribution of the inner product $\tilde{f}^{[k]} \cdot H^{(k)}$, where $H^{(k)}$ is the image in $\mathbf{CP}^{M-1}$ of an $m-k$-dimensional linear subspace of $\mathbf{C}^M$.

While the Weyls raised the basic questions and proved the analogues of (2.2), they were unable to achieve (2.4), except when $k=0$. Meanwhile, Ahlfors had returned to Finland, and during the bleak and isolated period of the early 1940's resolved to complete their work.

Paper [10] confirms Ahlfors's success. This is a difficult paper and the way in which the computations and averaging through the various dimensions (especially in Section 6) are performed will humble most readers. This "constitutes, in my opinion, one of the most brilliant chapters in the annals of mathematics" [97, p. 180] ... "I can hardly believe where I got the patience to write it" [11, p. 364]. In particular, the deficiency sums (6.2) hold with M in place of $m+1$, and $f^{[k]}$ in place of f. The full relations [10, p. 27] give very sharp lower bounds for all sums $\sum_{j=1}^{q} N_k\left(r, H_j^{\ell}\right)$ where ℓ and k are arbitrary $\in \{1, \ldots, n\}$.

Ahlfors's work has been extended in several ways since 1941. Already in 1953 [82], W. Stoll had developed a general theory for $f: \mathbf{C}^n \to \mathbf{CP}^m$, which was based on [10] and [95]. In subsequent work, Stoll has developed a very general theory of mappings, for which we refer to the survey [84], and Professor Stoll's several well-known monographs.

It is now possible to base a proof of (6.1) and (6.2) for $f: \mathbf{C}^n \to \mathbf{CP}^m$ on the LLD, thus putting these results in the framework of [22] (where $m=1$). This has been shown by A. Vitter [90]. Vitter's proof was based on the Carlson–Griffiths theory as further developed by Griffiths.

We also cite the paper [28], in which the authors extend the Carlson–Griffiths theory to meromorphic curves. In order to recover Ahlfors's theory for the associated curves, they introduce a *family* of metrics ω_i for the various dimension encounted. Each ω_i has a form in the spirit of (4.7), but now the full collection is to be "negatively curved" [28, p. 107].

The role of non-degeneracy in these questions is not settled. For example in 1933, Cartan conjectured in that (6.2) should be replaced by

$$\sum \delta\left(H_j\right) \leq (2m-k+1) \tag{6.3}$$

if $f: \mathbf{C} \to \mathbf{CP}^m$ is k-non-degenerate: $f(\mathbf{C})$ is contained in a k-dimensional subspace of $\mathbf{CP}^m$, but none of lower dimension. (Note that (6.3) does not follow from (6.2) restricted to a $\mathbf{CP}^k$, since the hyperplanes $H_j \cap \mathbf{CP}^k$ need not be in general position.) Cartan's conjecture has recently been proved by E.I. Nochka [65]; an extension to the associated curves has now been achieved by Wanxi Chen [23].

Open problems and other related results are given in [77] and [73]. Ahlfors's hope [10, p. 1] to extend the methods of [10] to value-distribution of arbitrary algebraic manifolds also remains unfulfilled.

7. The inverse problem

For the rest of this report, the main emphasis will be on meromorphic functions, where more refined issues can be raised.

The function e^z shows that the bound "2" in (2.5) is sharp. The inverse problem asks for more: given data $\delta_i > 0, \theta_i > 0, 0 < \delta_i + \theta_i \leq 1$, such that (2.3) holds, and a sequence of complex numbers a_i, find a function $f(z)$ which is meromorphic in the plane with $\delta(a_i, f) = \delta_i$, $\theta(a_i, f) = \theta_i$, and $\delta(a, f) = \theta(a, f) = 0$ otherwise. R. Nevanlinna proposed this as far back as [60]. W.H.J. Fuchs and W.K. Hayman [40] (cf. [44, Chapter 4]) solved it (with no reference to the θ_i) for entire functions, and the whole matter was finally settled in 1977 by this author [30]. Since $\sum \delta(a_i)^{1/3} < \infty$ if f has finite order [92], the solution must in general have infinite order.

The first general, albeit partial, result in this problem was obtained by Nevanlinna himself in 1932, in which he introduced his class F_q of Riemann surfaces with finitely many logarithmic branch points and no algebraic branch points, and proved that the restricted inverse problems $\sum_{i=1}^n \delta_i = 2$, where $n < \infty$ and each δ_i is rational, has a solution in this class. That is, if we choose $F \in F_q$ to have branch points appropriately over the a_i, then F is parabolic, and the uniformizer $f \colon \mathbf{C} \to F$ will be the solution. If N is the l.c.d. of the δ_i, the solution $f(z)$ will have order $N/2$, mean type, and N branch points will be needed (not necessarily distinct). Nevanlinna's method depended on an analysis of $P(z) = \{f, z\}$, the Schwarzian derivative of f, and used the theory of differential equations at a singular point ($z = \infty$) to compute the order and value-distribution of f. Since f has no algebraic branch points, it is easy to check that in this situation P is a polynomial, which allows the differential equation to be solved asymptotically, and for many years [63], Nevanlinna hoped that a refinement of this approach would solve the general problem. An abbreviated account of this method is in Chapter XI of [64].

However, once f has algebraic branch points (in particular [31] if f has order ρ which is not an integral multiple of $1/2$) it seems hopeless to recover f concretely from its Schwarzian. Very precise information is needed to compute the deficiencies which arise when the plane is mapped to a given surface F; recall that in general the deficiencies will even depend on the choice of origin.

In [8], Ahlfors gave an alternate approach to Nevanlinna's solution; this paper appears directly after Nevanlinna's in the Acta Mathematica. With hindsight, it is clear that Ahlfors's method was in the direction in which the full solution was to be found. Ahlfors uses the same surfaces F as does Nevanlinna, but his method of uniformization goes back to the length–area principle that had been so valuable in [2]. He divides F into a finite number of pieces F_j (one corresponding to each logarithmic branch point). Then since the universal cover of $\mathbf{C} \backslash \{a_i\}_1^q$ is the upper half-plane H, he uses the universal cover map φ_j to map each properly-slit F_j

onto a subdomain G_j of H.

For sufficiently large ρ, he then considers the curve $\Gamma_\rho^j = G_j \cap \{|\zeta| = \rho\}$, and shows that the various Γ_ρ^j form a closed curve Γ_ρ on F. Then if ρ is large and $w = f(z)$ is the uniformizer of F, Γ_ρ will correspond to a closed Jordan curve in the z-plane which surrounds $z = 0$. By the length–area method, Ahlfors shows that

$$\rho \sim |z|^{2/N} \tag{7.1.}$$

Thus, the separate analytic mappings $\varphi_j^{-1}(\zeta)$ almost fit together to form a rigid correspondence between z and ζ near ∞. From this, it is not hard to show that our function f has order $N/2$, and the desired deficient values.

The lines by which Ahlfors proved (7.1) received an encore several years later in Teichmüller's theorem [86] that a quasiconformal mapping $w = \phi(z)$ of the plane whose dilatation $|\varphi_{\bar{z}}/\varphi_z|$ suitably tends to zero at ∞ satisfies

$$|w| \sim A|z| \qquad (z \to \infty,\ 0 < A < \infty). \tag{7.2}$$

The parallel is especially striking in Wittich's proof [96]; cf. [41, Chapter VII.3]. From this perspective, we could have replaced the *ad hoc* analysis in [4] by quasiconformally welding the various G_j together, and then obtain (7.1) directly by Teichmüller's theorem. Estimates (7.1) and (7.2) coincide when each half-plane H is replaced by a suitable sector.

All this anticipates [30], in which the solution is obtained by fitting together by quasiconformal mappings portions of certain surfaces F, and applying a generalization of Teichmüller's theorem due to P. Belinskii. Because of advances in the subject (with Ahlfors one of the key contributors) no explicit mention of Riemann surfaces is needed in [30].

The inverse problem is not settled in most of the more general theories. Dai Chong-jyi and Fuchs [39] have a partial solution to (2.5), when the data given are small functions $a(z)$. Quasiconformal mappings seem inappropriate for this more general problem.

8. The theory of covering surfaces

Key references: [6], [44], [87] and [64, Chapter XIII]. In his collected papers, Ahlfors well describes the setting for this work. It was an effort to produce a value–distribution theory which was more geometric, and more stable under small pertubations. For example, if f omits the values $w = a$ then the pullback of each ball $B(a, \varepsilon) = \{|w - a| < \varepsilon\}$ cannot be compact in the z plane for any $\varepsilon > 0$. Hence Picard's theorem suggests that if f is meromorphic and nonconstant in the plane (more generally, admissible as in Section 2), then there cannot exist three disjoint w-balls B_j whose preimages are all non-compact. However, such hypotheses are not natural in Nevanlinna's setting, but very appropriate here.

When Ahlfors first published this work, it created a sensation and was one of the key evidences for his being awarded one of the first Fields medals (Oslo, 1936) [14].

Nevanlinna's original approach to his theory was purely potential-theoretic, but Nevanlinna quickly observed [61] that (2.4) could be viewed as a transcendental form of the classical Riemann–Hurwitz relation. This is described very carefully in Chapter 12 of Nevanlinna's monograph [64] and provides an elegant if formal way of seeing the upper bound "2" in (2.4). Indeed, if $r(z)$ effects a rational map of degree n on the sphere (n is the "number of sheets"), the mean ramification of r is equal to

$$V = n^{-1} \sum_{p} b(p), \tag{8.1}$$

where the sum is over all branch points p of r, and b is the order of branching. The Riemann–Hurwitz formula asserts that $V = 2 - 2n^{-1}$; the main term 2 is the (negative of the) Euler characteristic of $\mathbf{C}^*$.

Nevanlinna motivates his second fundamental theorem (2.4) in the following way. Let f be meromorphic in the plane, and suppose there exists a sequence $r_m(z)$ of rational functions of degree $n_m \to \infty$ which tend appropriately to f. Consider the Riemann–Hurwitz relation for each r_m where in the sum (8.1) (with $n = n_m$) we group points p such that $r_m(p) = a$ for each w-value a. Then perhaps one could obtain (2.4) in the limit: for each a, the terms $\delta(a)$ would correspond to the density of branch points "of infinite order" lying over a of $f(z)$, and $\theta(a)$ the density of algebraic branch points "over" $w = a$. Clearly 2 is the limit of the corresponding quantities V_m.

Nevanlinna was fond of this idea, and returned frequently to it. It applies most elegantly to the special class of functions whose Schwarzian derivative is a polynomial, (see Section 7), but as far as I am aware this connection has been of no value in obtaining general results.

Ahlfors transformed this from a metaphor into a powerful analytical tool. A key insight was to replace the target surface $\mathbf{C}^*$ by the sphere with $q \geq 3$ small discs Δ_i removed; this surface generally is called F_0, whose Euler characteristic now is $q - 2 > 0$. Then for each $r > 0$, Ahlfors studies how F_0 is covered by $B(r)$ under $f(z)$.

Since we now have a bordered covering, estimates must be made, and these depend on a fixed metric I on F_0, which lifts to a metric on the covering of F_0 induced by f. This metric may be quite general; the key property is that a weak isoperimetric property hold locally. Then for each $r > 0$, one can compute $S = S(r)$ (the mean covering number over F_0 $[I(f(B(r))/I(F_0)]$) and $L(r)$ (the length of $f(\partial B(r)) \cap F_0$); also, if D and γ are nice subdomains or curves in F_0, we may define $S(D)$ and $S(\gamma)$ in an analogous manner. Elementary arguments yield that

$$|S - S(D)| < h(D)L(r) \tag{8.2}$$

$$|S - S(\gamma)| < h(\gamma)L(r), \tag{8.3}$$

where in general $h(\cdot)$ depends on $\cdot$ and F_0. Inequalities (8.2) and (8.3) are (unintegrated forms of) the first fundamental theorem.

For an unbranched covering of F_0, the Riemann–Hurwitz formula (see (8.1))is

$$\rho = N\rho(F_0) + \sum b(p), \tag{8.4}$$

where N is the degree of the map, ρ is the characteristic of $f^{-1}(F_0) \cap B(r)$, and the summation is over branch points. By an elementary but impressively powerful analysis, Ahlfors proved that one half of (8.4) always persists

$$\begin{aligned} \rho^+ &\equiv \max(p, 0) \geq S(r)\rho(F_0) - kL(r) \\ &\equiv (q-2)S(r) - kL(r), \end{aligned} \tag{8.5}$$

where k is a positive constant which depends only on F_0. The significance of (8.5) is that since $\rho(F_0) > 0$, a large $S(r)$ will tend to force ρ to be large: the pull-back of F_0 to $B(r)$ will be very multiply connected. Thus, there will be many compact components (islands) D_i in $B(r)$ "over" the Δ_i.

If f is nonconstant and meromorphic in the plane, then $S(r)$ is essentially the Ahlfors–Shimizu form of (the differentiated) Nevanlinna characteristic (as in (4.1)), and it is also easy to see (length–area!) that

$$\frac{L(r)}{S(r)} \to 0 \qquad (r \to \infty) \qquad \| \tag{8.6}$$

(the covering surface is *regularly exhaustible*) if f is meromorphic and nonconstant in the plane; when f is meromorphic in $B(R)$, then (8.6) (as $r \to R$) will follow if $\limsup_{r \to R} S(r)(1-r) = \infty$. As in Section 2, we observe that when $S(r) = O\left((1-r)^{-1}\right)$, (8.6) (and its consequence (8.7) below) may fail (universal cover maps).

Consider now the islands D_i in $\{|z| < r\}$ which lie over the various Δ_i. Let $n(r, D_i)$ be the multiplicity and $n_1(r, D_i)$ the total ramification of D_i. Then Ahlfors deduces from (8.5) and (8.6) that

$$\begin{aligned} \sum_{i=1}^{q} n(r, D_i) - \sum n_1(D_i) &\geq (q-2)S(r) - hL(r) \\ &\geq \left(q - 2 + o(1)\right)S(r) \qquad \|, \end{aligned} \tag{8.7}$$

which is a differentiated form of Nevanlinna's second fundamental theorem. In fact, since in (8.7) we only count a_i-values which are attained in islands D_i over the Δ_i, we have information that is impossible to imagine coming from Nevanlinna's own theory. This proved useful, for example, in [31].

Impact of the theory. This theory has played a decisive role in many areas of geometric function theory, and is certain to be of future use. However, it is technically quite formidable even today.

A. One generalization to several dimensions was made many years ago by Mme M.H. Schwartz [71], [72], but this work seems not to have been pursued. In more recent times, the viewpoint has been important in the extensions of S. Rickman and his school to value–distribution of quasiregular maps in $\mathbf{R}^n$. Rickman proves [68] that if $f\colon \mathbf{R}^n \to \mathbf{R}^n$ is K-quasiregular and nonconstant then f can omit only $q(n,K)$ points. In [69], he shows that $q(3,K) \to \infty$ as $K \to \infty$, so that Picard's theorem is of a different qualitative form in space than in the plane. It is striking that when M. Pesonen specializes the arguments of [68] to the plane [67], the resulting theory is slightly weaker than that of [6].

B. On pp. 193–194 of [6], Ahlfors asks if Nevanlinna's second fundamental theorem can be obtained from (8.7), and suggests that the presence of an exceptional r-set in (8.7) might preclude this.

However, J. Miles [54] showed that (2.5) and a slightly weaker form of (2.4) in fact do follow from (8.7). Miles in [55] and [47] (the latter with Hayman) also has used Ahlfors's covering lemmas to obtain two notable results concerning meromorphic functions in the plane. First, we have that a bound independent of q,

$$\sum_{i=1}^{q}\left(\frac{n(r,a_i)}{S(r)}-1\right)<K \tag{8.8}$$

with K an absolute constant, holds as $r \to \infty$ on a set of strictly positive logarithmic density. (Inequality (8.8) complements (8.7), but the best value of K or its geometric significance if any is unknown).

Using (8.8) and delicate comparisons involving $S(r)$ and the variation of $\arg\left(f\left(re^{i\theta}\right)\right)$, Miles and Hayman [47] proved that

$$\liminf_{r\to\infty}\frac{T(r,f)}{T(r,f^{(q)})}\le 3e,$$

a striking bound which is independent of the growth of f or the choice of derivative. The complementary inequality

$$\limsup_{r\to\infty}\frac{T(r,f)}{T(r,f^{(q)})}\le\frac{1}{q+1}$$

is an immediate consequence of Nevanlinna's second fundamental theorem.

C. Bloch-type theorems. Almost immediately after the first announcements of R. Nevanlinna's theory appeared, A. Bloch [13] (cf. also G. Valiron [89]) suggested that coverings of domains might provide a more uniform way of looking at these matters and lead to a theory less sensitive to small perturbations. It is Bloch [13, p. 94] who suggested a study of what we all call $S(r, f)$, and on p. 87 we find tantalizing hints of what was to become Ahlfors's five island theorem. The first positive theorems on coverings of domains are due to Cartan [19], [20], who was very influenced by Bloch.

Ahlfors published several primitive forms of his theory, but the definitive account is in [6]. Thus, he obtains directly from (8.7) that if each compact cover of Δ_i has multiplicity $\geq \mu_i$ with

$$\sum (1 - \mu_i)^{-1} > 2, \tag{8.9}$$

then the covering surface is not regularly exhaustible. As a corollary, we have that if $\mathcal{F}$ is regularly exhaustible, and there are three disjoint Δ_i (if f is entire) or five Δ_i (if f is meromorphic), then there must be a simple island (= "schlicht disk") over at least one Δ_i. For example, if f is entire and there were no simple islands over the three Δ_i, we should take Δ_4 disjoint from Δ_1, Δ_2, Δ_3, and choose $\mu_1 = \mu_2 = \mu_3 = 2, \mu_4 = \infty$, and contradict (8.9).

The character of the pull-back of disks has many significant consequences in function theory. A particularly compelling application of the existence of schlicht disks is in the Theorem at the end of this section. However, we mention here that a purely geometric characterization of certain standard classes of functions defined in $B(1)$, including the Bloch functions $\mathcal{B}$ (and their separable subclass $\mathcal{B}_0$) as well as the bounded mean oscillation functions BMOA can be made in terms of the pull-backs of $\{w; |f(z) - f(a)| < r\}$ as a and r vary [80].

We now discuss some other consequences of Ahlfors's theory.

Corollary 1. *If f is meromorphic in the disc $B(1)$, and*

$$|f'(0)|/(1 + |f(0)|^2) \geq C \tag{8.10}$$

for a sufficiently large C, and (8.9) holds, then the covering surface generated by f over F_0 is not regularly exhaustible.

Corollary 2. *Theorems of Schottky, Landau, Bloch* ([6], [44, Ch. 6]).

Corollary 3. *(Normal families.) Let $\mathcal{F}$ be a family of functions meromorphic in $B(1)$, and T be the Ahlfors–Shimizu characteristic. Then $\mathcal{F}$ is normal in a neighborhood of the origin if and only if there is an M such that for some $r_0 > 0$*

$$T(r, f) \leq M, \qquad f \in \mathcal{F}, \qquad 0 \leq r < r_0. \tag{8.11}$$

This is somewhat of a surprise, since the standard condition is that $\mathcal{F}$ is normal if and only if the spherical derivatives $f^\sharp, f \in \mathcal{F}$, are uniformly bounded on compacta. Thus (8.11) is clearly necessary, since $T(r,f)$ is an average of $f^\sharp(z)$, but there seems at first glance no reason why the converse should hold. However it is an elementary consequence of the ideas behind Corollary 1.

Condition (8.11) is the analogue to meromorphic families of the necessary condition $\log M(r,f) < \psi(r)(0 < r < 1, f \in \mathcal{F})$ for holomorphic families to be normal, but (8.11) is both necessary and sufficient. Note that (8.11) with T Nevanlinna's characteristic, need not imply normality.

D. *Growth and omitted values.* Suppose f is meromorphic in $B(1)$ and omits $q \geq 3$ values. Then (cf. [87, p. 215])

$$T(r,f) < \left\{\frac{1}{q-2} + o(1)\right\} \log \frac{1}{1-r} \qquad (0 < r < 1) \tag{8.12}$$

which is essentially best possible, by (2.6). (In fact, D.F. Shea and L.R. Sons [74] have used [5] to show that (2.6) must hold in this case.) If we consider (8.12) for $q = 3$ and formally differentiate, we would have that

$$S(r) = O\left((1-r)^{-1}\right) \tag{8.13}$$

a conclusion which in fact follows from the theory of [6] (the implicit constant is an absolute one also; cf. [44, Chapter 6]). With this evidence, W.K. Hayman conjectured that perhaps (8.12) could be differentiated in general, at least so that

$$S(r) = o\left((1-r)^{-1}\right) \qquad (r \to 1) \tag{8.14}$$

whenever f omits infinitely many values. Compelling evidence for this was obtained by Hayman, Patterson and Pommerenke [48] in 1977; (8.4) is true if f is a universal cover map of $\hat{\mathbf{C}} \setminus A$ with A an infinite set. Hence, it was a surprise when J. Fernández [36] showed that if A is any set of zero logarithmic capacity, and $f\colon B(1) \to \hat{\mathbf{C}} \setminus A$, then no more than (8.13) need hold universally as $r \to 1$. Fernández's theorem thus provides a bizarre characterization of sets of capacity zero, for if A has positive capacity then $\limsup_{r\to 1} T(r) \log\left((1-r)^{-1}\right) = 0$, so that trivially $S(r) = o\left((1-r)^{-1}\right)$. It also shows that bounds such as (8.14) are not inherited from those of the (superordinate) universal cover mappings.

E. *Some surprising hints for the future.* It appears that the term quasiconformal saw its birth on [6, p. 185], although the concept goes back several years to H. Grötzsch. Grötzsch proved that Picard's theorem holds for what are now called two-dimensional quasiregular mappings. Since length–area adapts well to this setting, Ahlfors observes in [6] that his full covering surface theory applies almost at once, giving another proof of Picard's theorem in this generality and much more.

Ahlfors also introduced [6, p. 160] a class of curves γ (now called Ahlfors-regular): there is to exist a constant C independent of z_0 and $r > 0$ such that $|\gamma \cap B(z_0, r)| < Cr$, uniformly, where $|\quad|$ is arc-length. It is to these curves that (8.3) applies. Rather recently, G. David [29] has shown that Ahlfors-regular curves are significant in a completely different context: it is precisely these curves γ for which the Cauchy integral defines a bounded operator on $L^2(\gamma)$.

F. *Iteration theory.* The classical theory of iteration of (non-linear) rational maps goes back to P. Fatou and G. Julia, and in [33], Fatou initiated the study of iteration of (non-linear) entire functions. This subject has in recent years been among the most active on one-variable function theory. Recall that this is the study of the family $\mathcal{F} = \{f_n\}$ of n-fold iterations of f. The Fatou set $F = F(f)$ consists of points at which $\mathcal{F}$ fails to be normal. A fixpoint α of order n satisfies $f_n(\alpha) = \alpha$, and is repulsive if $|f_n'(\alpha)| > 1$. Repulsive fix points are obviously in F, and Fatou and Julia proved before 1920 that such points are dense in F when f is rational. In [33], Fatou asked if this were also true in the entire case. I.N. Baker [12] used [6] in essential way to prove the

Theorem. *Let f be entire and $\mathcal{F} = \{f_n\}$. Then repulsive fixpoints are dense in $\mathcal{F}$.*

Remark. Before [12], it had been open whether such fixpoints had to exist. The proof here readily adapts to the rational function context. The theory of iteration often uses the "inverse function", a concept made for Ahlfors's theory.

Proof. Let $z_0 \subset F$ and N a disc about z_0. It is elementary that F is perfect, so we may choose three disjoint discs $B(a_i, \varepsilon) \subset N$ with $a_i \in F$. (We also want any Fatou exceptional point to be disjoint from these B's; this is always possible since there are most two such points). Since $\mathcal{F}$ is not normal in any $B(a_i, \delta)$, it is easy to see that given $C < \infty$ there is a point $b_i = b_i(n)$ in each $B(a_i, \varepsilon/3)$ such that the spherical derivative $f_n^\sharp(b_i(n)) > C\varepsilon^{-1}$ if $n > n_0$. This places us exactly in the situation (8.10), and so we are guaranteed that if n is sufficiently large, f_n restricted to $B(a_i, \varepsilon/3)$ will have a simple island over some $B(a_j, \varepsilon)$. By iterating this argument at most three times, we see that a high iterate of f, say f_m, will have a simple island over $B(a_i, \varepsilon)$ in $B(a_i, \varepsilon/)$.

By Schwarz's lemma, f_m^{-1}, restricted to $B(a_i, \varepsilon)$ has a strictly attractive fixed point, so that f_m has a repulsive fixed point, as asserted.

9. Conclusion

In the past several years, renewed interest in the subject has come from its formal analogies to Roth's theorem in number theory [51], [66], [91]. Vojta has even compiled a "dictionary" between the two disciplines.

This survey has omitted a great deal of relevant material. But the evidence presented here compels the conclusion that these ideas will continue to inspire our science for many years to come.

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