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HYPERBOLICITY IN COMPLEX ANALYSIS

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1. The Schwarz–Pick lemma

Everyone who takes a course in Complex Analysis learns the Schwarz lemma. The most familiar form of the lemma states that a holomorphic function f with $f(0) = 0$ and $|f(z)| \leq 1$ in the unit disk $|z| < 1$ satisfies the inequalities $|f'(0)| \leq 1$ and $|f(z)| \leq |z|$. It is usually proved by applying the maximum principle to the function $f(z)/z$.

Carathéodory [1912] first published this form of the lemma and its proof. Carathéodory calls this result "The Schwarz lemma." He further comments that most authors have used the Harnack theorem for purposes similar to his, but notes that the "Schwarz lemma" is a purely function-theoretic equivalent, with the advantage of being quite elementary in nature. Carathéodory attributes the first use of this lemma in conformal mapping to Hermann Amandus Schwarz [1869], who, in his treatment of the Riemann mapping theorem, uses the fact that a holomorphic function f in the closed unit disk with $\bigl|f(z)\bigr| \leq \delta$ there must have $\bigl|f'(0)\bigr| \leq \delta$. Schwarz establishes this by looking at the Cauchy integral formula for $f'(0)$. Carathéodory seems to have been the first to recognize the importance of this lemma for function theory and that one need not presuppose any regularity of the function f on the boundary of the unit disk.

Poincaré [1881] introduced the non-Euclidean metric for the unit disk and noted that it is invariant under Möbius transformations of the disk onto itself. In [1894] he established the version of the lemma which asserts $\bigl|f(z)\bigr| \leq |z|$ and used it to prove that every conformal map of the unit disk onto itself is given by a Möbius transformation.

Lindelöf [1907] proved a general theorem which states that, if f is a holomorphic map of a domain D into a domain D', then the Green's functions G and G' of D and D' satisfy $G'\bigl(f(z), f(z_0)\bigr) \leq G(z, z_0)$. This is equivalent to the Schwarz lemma when both D and D' are the unit disk.

Georg Pick [1916] reformulated the Schwarz lemma to state that every holomorphic map of the unit disk into itself is distance decreasing in the Poincaré non-Euclidean metric. He expressed this both in the *integrated* form, which states that the Poincaré distance between two points is greater than or equal to the

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Poincaré distance between their images, and in the *differential* form which states that the element of arc length in the Poincaré metric does not increase under a holomorphic map of the unit disk into itself. This latter formulation is equivalent to the statement that the non-Euclidean length of the image of an arc under a holomorphic map of the disk into itself is at most equal to the non-Euclidean length of the arc. The differential form of the Poincaré metric for the unit disk is

$$ds = \frac{|dz|}{1-|z|^2},$$

and the differential form of the Schwarz-Pick lemma takes the following elegant form

$$\frac{|df|}{1-|f|^2} \le \frac{|dz|}{1-|z|^2},$$

or, equivalently,

$$\frac{|f'(z)|}{1-|f(z)|^2} \le \frac{1}{1-|z|^2}.$$

The integrated form of the inequality assumes the form

$$\frac{|f(z_1)-f(z_2)|}{|1-f(z_1)\overline{f(z_2)}|} \le \frac{|z_1-z_2|}{|1-z_1\bar{z}_2|}.$$

This brief paper of Pick's contains another proof of the Schwarz lemma, based on the theorem of Carathéodory which asserts that a function f holomorphic in the unit disk with a non-negative real part there must have $|f'(0)| \le 1/2$. Pick also uses his formulation of the Schwarz lemma in connection with the elliptic modular function to obtain the best bound in $|z| \le \rho \le 1$ for a function f which is holomorphic in the unit disk and is never 0 or 1 there.

The Schwarz lemma became a mainstay of the theory of functions of a complex variable in the following years. It was widely used and greatly popularized by Carathéodory, who showed its utility in a number of problems of conformal mapping. As a result of the general uniformization theorem developed by Poincaré and Koebe, one extends the Poincaré non-Euclidean metric to those Riemann surfaces whose universal covering surface is the unit disk. The Pick formulation generalizes so that any holomorphic mapping of the disk into such a surface must be distance decreasing from the Poincaré metric of the disk to this metric

2. The Carathéodory metric

Carathéodory [1926] introduced, for each bounded domain D in $\mathbf{C}^2$, a metric with the property that all holomorphic maps of D into itself are distance decreasing with respect to this metric. This generalizes the Pick formulation of the Schwarz lemma to several complex variables. The Carathéodory metric $\rho_D(p,q)$ of the domain D is defined by

$$\rho_D(p,q) = \sup \rho_\Delta\big(\varphi(p),\varphi(q)\big),$$

where the supremum is taken over all holomorphic maps $\varphi : D \to \Delta$ of D into the unit disk Δ furnished with the Poincaré metric ρ_Δ. It follows from the Schwarz–Pick lemma that the Carathéodory metric for the unit disk Δ is just the Poincaré metric ρ_Δ. If p, q, and r are three points in D, and φ any holomorphic mapping of D into the unit disk we have

$$\rho_\Delta\big(\varphi(p),\varphi(q)\big) \leq \rho_\Delta\big(\varphi(p),\varphi(r)\big) + \rho_\Delta\big(\varphi(r),\varphi(q)\big) \leq \rho_D(p,r) + \rho_D(r,q).$$

Taking the supremum over all such φ gives

$$\rho_D(p,q) \leq \rho_D(p,r) + \rho_D(r,q),$$

and we see that the triangle inequality holds for ρ_D. Since D is bounded, suitable non-zero multiples of the coordinate functions map D into the unit disk, and we see that $\rho_D(p,q) > 0$ for $p \neq q$. Thus ρ_D is a metric.

If $f : D \to D'$ is a holomorphic map of D into a domain D' and $\varphi : D' \to \Delta$ a holomorphic map of D' into the unit disk Δ, then $\varphi \circ f$ is a holomorphic map of D into Δ, and so

$$\rho_\Delta\Big(\varphi\big[f(p)\big],\varphi\big[f(p)\big]\Big) \leq \rho_D(p,q).$$

Taking the supremum over all such maps gives

$$\rho_{D'}\big(f(p),f(q)\big) \leq \rho_D(p,q),$$

and we see that f is distance decreasing from the Carathéodory metric for D to the Carathéodory metric for D'.

This metric became a powerful tool in the hands of Carathéodory and Henri Cartan. It was quite useful for expressing and deriving results about normal families for holomorphic mappings in several variables. Cartan used these ideas in his characterization of the holomorphic automorphisms of a bounded domain which have a fixed point. Carathéodory introduced the indicatrix at a point for such a metric (essentially the unit ball for the infinitesimal form of the metric) and showed that an automorphism of D which takes p into q must effect a linear mapping of the indicatrix at p onto that at q. He also showed that the indicatrix at the center of a convex circled domain is the domain itself.

The definition of the Carathéodory metric can be extended to unbounded domains, complex manifolds, and even analytic varieties. In these cases, however, it may happen that we only get a pseudometric, that is, we may have $\rho(p,q) = 0$ without $p = q$. This leads us to the important concept of a hyperbolic manifold or variety: A manifold D is said to be (Carathéodory) hyperbolic if its Carathéodory metric is a metric rather than a pseudometric.

There is an infinitesimal (or differential-geometric) version of the Carathéodory metric for a manifold M: We define a norm $\|\xi\| = G(\xi,p)$ on the tangent vectors ξ at p by setting

$$G(\xi,p) = \sup\{|\varphi_*\xi| : \varphi : M \to \Delta,\ \varphi(p) = 0\}.$$

This infinitesimal metric was first investigated in depth by Reiffen [1963], who showed that it gives the same length for arcs as the Carathéodory metric, and is thus the inner metric corresponding to the Carathéodory metric. This metric, which is properly called the Carathéodory–Reiffen metric, is, in its integrated form, equivalent to the Carathéodory metric. Since it is the inner metric derived from the Carathéodory metric, it is always at least as large as the Carathéodory metric, but it is actually larger in many cases. The Carathéodory–Reiffen metric also decreases under holomorphic maps, both in its integrated and in its differential form. This fact for the differential form of the metric is a result quite similar in expression to the usual differential form of the Schwarz–Pick lemma.

The problem of finding the Carathéodory (or Carathéodory–Reiffen) metric explicitly in a particular case is usually difficult, and there are few domains in several complex variables where we know it explicitly. The determination of this metric for a multiply connected plane domain leads to extremal problems for holomorphic maps of the domain into the unit disk. These were solved in fairly explicit terms, however, by Lars Ahlfors [1947], who showed that the extremal function for a domain of connectivity n is given by a holomorphic map f which maps the domain exactly n-to-one onto the unit disk. This map is now called the Ahlfors map. There are some generalizations to domains of infinite connectivity, regarded as the complement in the Riemann sphere $\hat{\mathbb{C}}$ of a closed bounded set E. In this case the infinitesimal form of the Carathéodory metric at ∞ is usually referred to as the Ahlfors capacity or the analytic capacity of the set E. Garabedian [1949] and Ahlfors [1950] applied the techniques of dual extremal problems to similar problems on finite Riemann surfaces.

In the case of one complex variable we can apply the principle of Lindelöf, which is formulated in terms of Green's functions. For simply connected regions this is equivalent to the Schwarz lemma, but for multiply connected regions it yields stronger inequalities, although not as strong as the exact inequalities obtained from Ahlfors' theory.

3. The Ahlfors–Schwarz lemma

Pick's formulation of the Schwarz lemma in terms of the Poincaré metric for the disk was well known and often exploited during the twenties and thirties. It was also well known that the Poincaré metric (as used by Pick) had constant negative Gaussian curvature equal to -4 everywhere, but no one supposed there was a causal connection between these facts until Ahlfors [1938] provided one. This seminal paper by Ahlfors established the connection between curvature and hyperbolicity: He showed that if a conformal metric

$$ds = \lambda|dz|$$

on the unit disk has Gauss curvature which is everywhere less than or equal to -4, then every holomorphic map f of the disk into itself is distance decreasing from the Poincaré metric to the λ-metric, i.e.,

$$\lambda(f(z))|f'(z)\,dz| \leq \frac{|dz|}{1-|z|^2}.$$

If we take λ to be the Poincaré metric, so that $\lambda = 1/(1-|z|^2)$, then the Ahlfors form gives the usual Pick formulation. For a conformal metric $ds = \lambda|dz|$, the Gauss curvature K is given by the formula

$$K = \frac{-\Delta \log \lambda}{\lambda^2}.$$

Once formulated, the Ahlfors–Schwarz lemma is not difficult to prove, and the original proof in Ahlfors [1938] (republished in Ahlfors [1973]) is simple and elegant. The ingenuity of Ahlfors consists in believing that a lemma of this sort might be true and that there was a connection between the curvature of a differential metric and the distance decreasing property expressed by the Schwarz–Pick lemma.

The great utility of the Ahlfors version lies in the fact that, for specific domains or Riemann surfaces, it is usually far easier to construct a metric which has curvature everywhere less than or equal to -4 than to construct one whose curvature is everywhere equal to -4. Ahlfors himself used this formulation of the Schwarz lemma to give a new estimate for the Bloch constant. Although the use of a conformal metric whose curvature is sometimes less than -4 will not give the sharpest possible bound for the derivative of the mapping function from one region to another, it will give some bound, and that is good enough to establish that the family of mappings from one domain into another is a normal family. Thus an elementary construction in the triply-punctured sphere of a conformal metric whose curvature is bounded from above by a negative constant gives a short proof of Montel's theorem that the family of functions which omit three given values is normal.

In connection with his formulation in [1938], Ahlfors introduced the concept of a supporting metric: The conformal metric $ds = \lambda_0|dz|$ is said to be a supporting metric for the metric $ds = \lambda|dz|$ at the point p if λ_0 is defined and smooth in a neighborhood U of p with $\lambda_0(p) = \lambda(p)$ and $\lambda(q) \geq \lambda_0(q)$ for all q in U. He then observes that we need not assume smoothness for λ so long as the metric $\lambda|dz|$ has at each point a supporting metric whose curvature is at most -4. The advantage of this formulation is that, if λ_1 and λ_2 each have curvature less than K, then the metric obtained by taking the supremum of the two at each point has a supporting metric at each point with curvature at most K.

Grauert and Reckziegel [1965] observed that the Ahlfors form of the Schwarz lemma could be extended to give inequalities for holomorphic maps of the unit disk into higher dimensional complex manifolds when the manifold has a differential metric with negative curvature in a suitable sense. Let us define the Gauss curvature $K(\lambda)$ of a continuous conformal metric $ds = \lambda(\zeta)|d\zeta|$ at 0 to be the infimum of the curvatures at 0 of all the smooth metrics which are supporting metrics of λ at 0. By a differential metric on a complex manifold M we mean a continuous function $G(\xi, z)$ on the tangent bundle of M which assigns a non-negative "length" to each tangent vector ξ at the point z, such that $G(\alpha\xi, z) = |\alpha|G(\xi, z)$. If φ is a holomorphic map of a domain in $\mathbf{C}$, the pullback of G, defined by $\varphi^*G = G(\varphi'(\zeta), \varphi(\zeta))|d\zeta|$, is a conformal metric on Δ. We define the holomorphic sectional curvature of the metric G at a point z in M by

$$K(G) = \sup\{K(\varphi^*G) : \varphi(0) = z,\ \varphi'(0) = \xi\}$$

where the supremum is taken over all holomorphic maps φ of a neighborhood of 0 in $\mathbf{C}$ and the curvature of φ^*G is taken at $\zeta = 0$. If G is any differential metric on M whose holomorphic sectional curvature is everywhere less than or equal to -4 and $f : \Delta \to M$ is holomorphic, then f^*G is a differential metric on Δ with curvature everywhere less than or equal to -4 in the supporting sense. Thus the Ahlfors version of the Schwarz–Pick lemma asserts that

$$G(f'(\zeta), f(\zeta))|d\zeta| \leq \frac{|d\zeta|}{1 - |\zeta|^2}.$$

This gives us a generalization of the Ahlfors version of the Schwarz–Pick lemma for maps of the disk into a complex manifold in higher dimensions.

4. The Kobayashi and other hyperbolic metrics

Although Carathéodory states in one of his papers that he is not sure how useful his metric will be in the future study of functions of several complex variables, it found numerous applications by him and by Cartan. Kobayashi [1967a, b], however, introduced a new metric for complex manifolds which has proved even more useful. We shall see that the Carathéodory metric is the smallest of the class of metrics which generalize the Schwarz lemma and that the new metric of Kobayashi is the largest. These metrics are particularly useful in the hyperbolic case, that is, when they are actually metrics rather than pseudometrics. The advantage of the Kobayashi metric lies in that, being larger, it will be an actual metric for some manifolds on which the Carathéodory metric is only a pseudometric.

Let us define a hyperbolic metric ρ to be a functor which assigns a metric ρ_M to each complex manifold (in some class of manifolds) such that for the unit disk ρ_Δ is the Poincaré non-Euclidean metric for the disk and such that each holomorphic map f from M to N is distance decreasing from ρ_M to ρ_N :

$$\rho_N(f(p), f(q)) \leq \rho_M(p, q).$$

Let ρ be a hyperbolic metric in this sense. If φ is any map of the manifold M into the disk Δ, we must have

$$\rho_\Delta(\varphi(p), \varphi(q)) \leq \rho_M(p, q).$$

But the supremum of the left hand side of this equation is, by definition, the Carathéodory metric $\rho_M^C(p, q)$. Thus

$$\rho_M^C(p, q) \leq \rho_M(p, q),$$

and we see that the Carathéodory metric ρ^C is the smallest possible hyperbolic metric.

On the other hand we get the largest possible hyperbolic metric on M by considering holomorphic maps of the unit disk into M: Define the unreduced Kobayashi distance $\delta_M(p, q)$ between two points p and q of M by

$$\delta_M(p_1, p_2) = \inf \rho_\Delta(\zeta_1, \zeta_2),$$

where φ ranges over all holomorphic maps of Δ into M with $\varphi(\zeta_j) = p_j$. If φ is any such map and ρ is any hyperbolic metric, then

$$\rho_M(p_1, p_2) \leq \rho_\Delta(\zeta_1, \zeta_2),$$

taking the infimum over φ shows that

$$\rho_M(p_1, p_2) \leq \delta_M(p_1, p_2).$$

The distance $\delta_M(p_1, p_2)$ does not satisfy the triangle inequality and so does not give us a metric (or pseudometric). But, as Kobayashi observed, we do get a metric if we take

$$\rho_M^K(p,q) = \inf \sum_{i=1}^{n} \delta_M(p_i, p_{i+1}),$$

where the infimum is taken over all finite chains $p = p_1, p_2, \ldots, p_n = q$. This is Kobayashi's definition of the metric for M. We observe that, if f is any holomorphic map of M into N and φ any holomorphic map of Δ into M with $\varphi(\zeta_i) = p_i$ for $i = 1, 2$, then $f \circ \varphi$ is a holomorphic map of Δ into N. Hence

$$\delta_N\big(f(p_1), f(p_2)\big) \leq \rho_\Delta(\zeta_1, \zeta_2).$$

Taking the infimum over all such φ, gives

$$\delta_N\big(f(p_1), f(p_2)\big) \leq \delta_M(p_1, p_2).$$

From this we see that

$$\rho_N^K\big(f(p_1), f(p_2)\big) \leq \rho_M^K(p,q).$$

Thus the Kobayashi metric is a hyperbolic pseudometric and is clearly the largest possible one.

We can reformulate the preceding consideration in terms of infinitesimal metrics. By a *hyperbolic* infinitesimal metric we mean an assignment of an infinitesimal metric $G_M(\xi, p)$ to each complex manifold M (in some class of complex manifolds) such that for the unit disk Δ

$$G_\Delta(\xi, z) = \frac{|\xi|}{1 - |z|^2}$$

and such that for any two manifolds M and N of our class and any holomorphic map $f : M \to N$ we have

$$G_N\big(f_*\xi, f(p)\big) \leq G_M(\xi, p).$$

Let φ be any holomorphic map of the manifold M to the disk δ with $\varphi(p) = 0$. Then, since the infinitesimal form of the Poincaré metric in Δ at $z = 0$ is the Euclidean norm, and since φ must be distance decreasing from G_M to the Poincaré metric, we must have

$$|\varphi_*\xi| \leq G_M(\xi, p).$$

Taking the supremum over all such maps φ gives us

$$G_M^C(\xi, p) \leq G_M(\xi, p),$$

where $G_M^C(\xi, p)$ is the Carathéodory–Reiffen infinitesimal metric. This shows that the Carathéodory–Reiffen metric is the smallest possible hyperbolic infinitesimal metric.

Let us define an infinitesimal metric F_M on the manifold M by setting

$$F_M(\xi, p) = \inf\{a^{-1} : \varphi : \Delta_a \to M,\ \varphi(0) = p,\ \varphi'(0) = \xi\}.$$

where Δ_a is the disk of radius a centered at 0 and the infimum is taken over all holomorphic maps φ which map some Δ_a into M. Suppose that $f : M \to N$ is a holomorphic map of the complex manifold M to the complex manifold N and that φ is a map of Δ_a into M with $\varphi(0) = p$ and $\varphi'(0) = \xi$. Then $f \circ \varphi$ maps Δ into N, and we have

$$F_N(f_*\xi, f(p)) \leq a^{-1}.$$

Taking the infimum over all such maps φ, gives us

$$F_N(f_*\xi, f(p)) \leq F_M(\xi, p).$$

This shows that F_M is a hyperbolic infinitesimal metric.

If G_M is any infinitesimal hyperbolic metric and φ a map of Δ_a into M with $\varphi(0) = p$ and $\varphi'(0) = \xi$, then the distance decreasing property of hyperbolic metrics asserts that

$$G_M(\xi, p) \leq a^{-1}.$$

Taking the infimum over all such maps φ shows that

$$G_M(\xi, p) \leq F_M(\xi, p),$$

and that $F_M(\xi, p)$ is the largest infinitesimal hyperbolic metric.

If we define the arc length of a differentiable curve $x(t)$ in M to be the upper Riemann integral

$$R\overline{\int} G_M(x'(t), x(t))\, dt$$

and define $\rho_M(p, q)$ to be the infimum of all the lengths of all differentiable curves joining p to q, then ρ_M is a hyperbolic metric. Since the Kobayashi metric ρ^K is the largest hyperbolic metric,

$$\rho_M(p, q) \leq \rho_M^K(p, q).$$

It was shown in Royden [1971] that, in fact, we have equality:

$$\rho_M(p, q) = \rho_M^K(p, q).$$

Consequently, the Kobayashi metric is the integrated form of the infinitesimal metric F_M. This means that the Kobayashi metric is an inner metric (i.e., one defined by arc length).

We say that a complex manifold is (Kobayashi) *hyperbolic* if the Kobayashi metric on M is a metric rather than a pseudometric. If M and N are two complex manifolds and N is hyperbolic, then all holomorphic maps of M into N are equicontinuous. Thus a family of holomorphic maps of M into N will be a normal family as soon as we have a suitable compactness condition on the images $\{f(p)\}$. There is a generalization of the Hopf–Rinow theorem which asserts that, if an inner metric on a locally compact space is complete, then the closed bounded sets of the space are compact. Thus if the complex manifold N is complete hyperbolic, i.e., is such that the Kobayashi metric is complete, then the family of all holomorphic maps of M into N is a normal family. Applications of these notions can be found in Kobayashi [1970], [1973] and Wu [1967].

5. The Ahlfors–Schwarz lemma in several variables

The extensions to several complex variables discussed so far have been done by defining the metrics, Carathéodory, Kobayashi, etc., in such a way that the distance decreasing property for holomorphic maps follows directly from the definition. The Grauert–Reckziegel results are a partial exception: Although the holomorphic sectional curvature of a differential metric on a complex manifold is defined here in a manner that allows us to apply the Ahlfors–Schwarz lemma directly to the pull-back of the differential metric to the disk, the concept of holomorphic sectional curvature for Kähler metrics was known earlier. If $R_{\alpha\bar{\beta}\gamma\bar{\delta}}$ is the curvature tensor and ξ^α is a unit tangent vector at the point p, then the holomorphic sectional curvature in the direction ξ is given by the formula

$$K(\xi) = 2R_{\alpha\bar{\beta}\gamma\bar{\delta}}\xi^\alpha\xi^{\bar{\beta}}\xi^\gamma\xi^{\hat{\delta}}.$$

In the case of a Kähler metric this is just the Riemann sectional curvature of the section determined by the vectors ξ and $i\xi$. It is also the Gauss curvature of a one (complex) dimensional curve through p which is geodesic to first order at p and whose tangent vector at p is a multiple of ξ. In the case of a Hermitian metric the holomorphic sectional curvature in the direction ξ is still given by the formula above, provided R is the curvature of the connection which is compatible with the metric and with the complex structure of the manifold. If the metric is not Kähler, however, the holomorphic sectional curvature is no longer described by the Riemann sectional curvature of the metric. It is also possible to give explicit formulae for the holomorphic sectional curvature in the case of a regular complex Finsler metric (cf. Royden [1986]).

Chern [1968] showed that in certain cases a holomorphic mapping $f : M \to N$ must be volume decreasing in terms of Kähler metrics on M and N if the

Ricci curvature of N is bounded above by a suitable negative constant and the scalar curvature of M is bounded from below by a suitable negative constant. Chern assumed M to be compact or to be a polydisk with the product of the Poincaré metrics of the factors. Lu [1968] extended these results to show that a holomorphic mapping was distance decreasing in these cases if the holomorphic sectional curvature of the Kähler metric on N was bounded above by a suitable negative constant and the Ricci curvature of M was bounded from below by a suitable negative constant. Again M is assumed to be compact or the polydisk with the standard metric.

For results such as these to hold some restriction (other than those on curvature) must be placed on the manifold M where the mapping f is defined, since the disk Δ_a of radius $a \leq 1$ with the non-Euclidean metric of the unit disk Δ has curvature -4 everywhere and can be mapped onto the unit disk Δ with derivative a^{-1} at the origin. In fact, if we took the spherical metric of $\hat{\mathbf{C}}$ for the metric on Δ_a, we would have a metric with curvature $+4$ everywhere and still have a non-constant map from Δ_a to Δ. Alfred Huber [1957] showed, however that if M is a Riemann surface with a *complete* conformal metric of non-negative curvature, then there is no non-constant conformal map of M into the unit disk.

Yau [1978] was the first to show that one could get bounds on the gradient of a holomorphic map $f : M \to N$ in terms of a lower bound on the Ricci curvature of M and a negative upper bound on the biholomorphic sectional curvature of M, provided that M is *complete.* Chen, Cheng, and Look [1979] proved that a holomorphic map $f : M \to N$ from a complete Kähler manifold M whose holomorphic sectional curvature is bounded from below by a negative constant $-a^2$ into a Kähler manifold N with holomorphic sectional curvature bounded above by a negative constant $-b^2$ must satisfy $\|df\| \leq a/b$. They must also assume that the Riemann sectional curvature of the metric on M is bounded from below by some constant. Some improvements of these results were obtained in Royden [1980].

We illustrate these results by proving the theorem for the case of smooth differential metrics on M and N when M is compact:

Theorem. *Let $f : M \to N$ be a holomorphic map from a compact complex manifold M to a complex manifold N, and suppose that M has a differentiable metric whose holomorphic sectional curvature is everywhere greater than or equal to a negative constant $-a^2$ and N has one whose holomorphic sectional curvature is everywhere less than or equal to a negative constant $-b^2$. Then*

$$\|df\| \leq \frac{a}{b}.$$

Proof. Since M is compact, there is a point $p \in M$ where $\|df\|$ attains its maximum value m and a tangent vector ξ at p such that $\|df(\xi)\| = m\,\|\xi\|$. Since the holomorphic sectional curvature of the metric on M is greater than or equal to

$-a^2$, we can find a holomorphic mapping φ of the disk Δ into M with $\varphi(0) = p$ and $\varphi'(0)$ a multiple of ξ so that the curvature of the metric $\lambda(\zeta) = \|\varphi'(\zeta)\|$ on Δ has curvature at the origin which is greater than $-a^2 - \epsilon$. Since the holomorphic sectional curvature of the metric on N is less than or equal to $-b^2$ at $f(p)$, the metric $\mu = \|(f \circ \varphi)'(\zeta)\|$ on Δ has curvature at the origin less than or equal to $-b^2$. Now $\mu/\lambda \leq m$ in Δ with equality at the origin, and so $\log \mu/\lambda$ has a maximum at the origin. Thus we must have

$$0 \geq \Delta \log \mu - \Delta \log \lambda \geq b^2\mu^2 - (a^2 + \epsilon)\lambda^2.$$

at the origin. Hence

$$m = \frac{mu^2}{\lambda^2} \leq \frac{a^2 + \epsilon}{b^2},$$

and the theorem follows by letting ϵ go to zero. □

Although we have assumed the metrics smooth for the sake of convenience, we can take curvatures in the sense of supporting metrics and modify the proof as in Ahlfors' original paper. If M is not compact, we can still carry out a modified version of the above proof provided we have a positive function u which is proper (the sets where $u \leq c$ are all compact), which has bounded gradient and whose complex Hessian $u_{\alpha\bar{\beta}}$ is bounded from above. In this case we look at a point where $(1 - \epsilon u)\,\|df\|$ has its maximum and apply similar considerations. If M is complete and either Kähler with biholomorphic sectional curvature bounded from below or Hermitian with bounded torsion[1] and Riemann sectional curvature bounded from below, then we may take u to be the distance from a point p_0, after we smooth it in a neighborhood of p_0.

Bibliography

[1938] AHLFORS, L.V.: An extension of Schwarz' lemma. - Trans. Amer. Math. Soc. 43, 1938, 359–364.

[1947] AHLFORS, L.V.: Bounded analytic functions. - Duke Math. J. 14, 1947, pages.

[1950] AHLFORS, L.V.: Open Riemann surfaces and extremal problems on compact subregions. - Comment. Math. Helv. 24, 1950, 100–134.

[1958] AHLFORS, L.V.: Extremalprobleme in der Funktiontheorie. - Ann. Acad. Sci. Fenn. Ser. A I Math. 249/1, 1958, pages.

[1973] AHLFORS, L.V.: Conformal invariants: Topics in geometric function theory. - McGraw-Hill, New York, 1973.

[1] The condition that the torsion should be bounded in the Hermitian case was unfortunately omitted from Royden [1980]. The statements of Proposition 2 and Theorem 2 there should require this for the Hermitian case. I do not know whether this hypothesis or the hypothesis that the Riemann sectional curvature is bounded from below are needed for Theorem 2 or are merely a requirement of the proof given.

[1950] AHLFORS, L. V., and A. BEURLING: Conformal invariants and function-theoretic nullsets. - Acta Math. 83, 1950, 101–129.

[1911] CARATHÉODORY, C.: Über den Variabilitätsbereich der Fourierschen Konstanten von positiven harmonischen Funktionen. - Rend. Circ. Mat. Palermo 32, 1911, 193–218.

[1912] CARATHÉODORY, C.: Untersuchen über die konformen Abbildungen von festen und veränderlichen Gebieten. - Math. Ann. 72, 1912, 107–144.

[1914] CARATHÉODORY, C.: Elementar Beweis für den Fundamentalsatz der konformen Abbildungen. - Mathematische Abhandlungen, Herman Amandus Schwarz zu seinem fünzigjährigen Doktorjubiläum am 6 August 1914 gewidmet, 1914, 19–41.

[1926] CARATHÉODORY, C.: Über das Schwarzsche Lemma bei analytischen Funktionen von zwei komplexen Veränderlichen. - Math. Ann. 97, 1926, 76–98.

[1927] CARATHÉODORY, C.: Über eine spezielle Metrik, die in der Theorie der analytischen Funktionen auftritt. - Atti Pontificia Accad. Sci. Nouvi Lincei 80, 1927, 135–141.

[1928] CARATHÉODORY, C.: Über die Geometrie der analytischen Abbildungen, die durch analytische Funktionen von zwei Veränderlichen vermittelt werden. - Abhandlungen aus dem Mathematischen Seminar der Hamburgischen Universität 6, 1928, 96–145.

[1932] CARATHÉODORY, C.: Über die Abbildungen, die durch Systeme von analytischen Funktionen von mehreren Veränderlichen erzeugt werden. - Math. Z. 34, 1932, 758–792.

[1930] CARTAN, H.: Sur les fonctions de deux variables complexes. Les transformations d'un domaine borné D en un domaine intérieur à D. - Bull. Soc. Math. France 58, 1930, 199–219.

[1931] CARTAN, H.: Les fonctions de deux variables complexes et le problème de la représentation analytique. - J. Math. 10, 1931, 1–114.

[1979] CHEN, C.H., S.-Y. CHENG, and K.H. LOOK: On the Schwarz lemma for complete Kähler manifolds. - Acta Sci. Sinica Ser. A 22, 1979, 1238–1247.

[1968] CHERN, S.S.: On holomorphic mappings of Hermitian manifolds of the same dimension. - Proc. Sympos. Pure Math. 11, Amer. Math. Soc., Providence, R.I., 1968, 157–170.

[1949] GARABEDIAN, P.R.: Schwarz's lemma and the Szegő kernel function. - Trans. Amer. Math. Soc. 67, 1949, 1–35.

[1965] GRAUERT, H., and H. RECKZEIGEL: Hermitsche Metriken und Normale Familien holomorpher Abbildungen. - Math. Z. 89, 1965, 108–125.

[1971] GREENE, R.E., and H.-H. WU: Curvature and complex analysis. - Bull. Amer. Math. Soc. 77, 1971, 1045–1049.

[1979] GREENE, R.E., and H.-H. WU: Functions on manifolds which possess a pole. - Lecture Notes in Mathematics 699. Springer-Verlag, Berlin–Heidelberg, 1971.

[1957] HUBER, A.: On subharmonic functions and differential geometry in the large. - Comment. Math. Helv. 32, 1957, 13–72.

[1967a] KOBAYASHI, S.: Invariant distances on complex manifolds and holomorphic mappings. - J. Math. Soc. Japan 19, 1967, 460–480.

[1967b] KOBAYASHI, S.: Distance, holomorphic mappings and the Schwarz lemma. - J. Math. Soc. Japan 19, 1967, 481–485.

[1968] KOBAYASHI, S.: Volume elements, holomorphic mappings and the Schwarz lemma. - Proc. Sympos. Pure Math. 11, Amer. Math. Soc., Providence, R.I., 1968, 253–260.

[1970] KOBAYASHI, S.: Hyperbolic manifolds and holomorphic mappings. - Marcel Dekker, New York, 1970.

[1973] KOBAYASHI, S.: Some problems on distances and measures. - C. Carathéodory Symposium, The Greek Mathematical Society, 1973, 306–317.

[1976] KOBAYASHI, S.: Intrinsic distances, measures and geometric function theory. - Bull. Amer. Math. Soc. 82, 1976, 357–416.

[1907] KOEBE, P.: Über die Uniformizierung beliebiger analytischer Kurven. - Nachrichten der Gesellschaft der Wissenschaft zu Göttingen, 1907.

[1910] KOEBE, P.: Über die Uniformizierung beliebiger analytischer Kurven. - J. Math. 138, 1910.

[1907] LINDELÖF, E.: Mémoire sur certaines inégalités dans la théorie des fonctions monogènes et sur quelques propiétés nouvelles de ces fonctions dans le voisinage d'un point singulier essentiel. - Acta Soc. Sci. Fenn. 35:7, 1907.

[1968] LU, V.C.: Holomorphic mappings of complex manifolds. - J. Differential Geom. 3, 1968, 57–78.

[1916a] PICK, G.: Über eine eigenschaft der konformen Abbildungen kreisförmiger Bereiche. - Math. Ann. 77, 1916, 1–6.

[1916b] PICK, G.: Über die Beschränkungen analytischer Funktionen, welche durch vorgegebene Funktionwerte bewirkt werden. - Math. Ann. 77, 1916, 7–23.

[1918] PICK, G.: Über die Beschränkung analytischer Funktionen durch vorgegebene Funktionwerte. - Math. Ann. 78, 1918, 270–275.

[1881] POINCARÉ, H.: Théorie des groupes Fuchsiens. - Acta Math. 1, 1881.

[1884] POINCARÉ, H.: Sur les groupes des équations linéaires. - Acta Math. 4, 1884, 201–312.

[1907] POINCARÉ, H.: Sur l'uniformisation des fonctions analytiques. - Acta Math. 31, 1907, 1–63.

[1963] REIFFEN, H.J.: Die differentialgeometrischen Eigenschaften der invarianten Distanzfunktion von Carathéodory. - Schriftenreihe Math. Inst. Univ. Münster 26, 1963.

[1965] REIFFEN, H.J.: Die Carathéodorysche Distanz und ihre zugehörige Differentialmetrik. - Math. Ann. 161, 1965, 315–324.

[1962] ROYDEN, H.L.: The boundary values of analytic and harmonic functions. - Math. Z. 78, 1962, 1–24.

[1971] ROYDEN, H.L.: Remarks on the Kobayashi metric. - Several complex variables II. Conference proceedings, University of Maryland 1970; Lecture Notes in Mathematics 185. Springer-Verlag, Berlin–Heidelberg, 1971.

[1980] ROYDEN, H.L.: The Ahlfors–Schwarz lemma in several complex variables. - Comment. Math. Helv. 88, 1980, 547–558.

[1986] ROYDEN, H.L.: Complex Finsler metrics. - In Complex differential geometry and nonlinear differential equations, Brunswick Maine conference, 1984. Contemp. Math. 49, Amer. Math. Soc., Providence, R.I., 1986.

[1869] SCHWARZ, H.A.: Zur Theorie der Abbildungen. - Programm der eidgenössischen polytechnischen Schule in Zürich für das Schuljahr 1869-70. Gesammelte Abhandlungen II, 1969, 109.

[1967] WU, H.-H.: Normal families of holomorphic mappings. - Acta Math. 119, 1967, 193–233.

[1978] YAU, S.-T.: A general Schwarz lemma for Kähler manifolds. - Amer. J. Math. 100, 1978, 197–203.

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