Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 14, 1989, 3–12

REPRESENTATION THEOREMS FOR ANALYTIC FUNCTIONS WITH QUASIMEROMORPHIC EXTENSIONS

Zerrin Göktürk

This paper is concerned with normalized quasimeromorphic functions of the extended plane $\bar{\mathbf{C}}$ which are analytic in a domain D of the plane, and have a pole only at one point. For these functions, which are strictly finitely multivalent in $\bar{\mathbf{C}}$, we generalize representation theorems concerning normalized quasiconformal homeomorphisms of $\bar{\mathbf{C}}$ which are conformal in D. The representation formulas yield estimates for the power series coefficients.

1. Definitions

A function f is called k-quasimeromorphic in a plane domain D, if f is spherically continuous and a generalized L^2-solution of a Beltrami differential equation $f_{\bar{z}} = \mu f_z$ in D, where the complex dilatation μ satisfies the condition $\|\mu\|_\infty \leq k < 1$.

We introduce the class F_k^p of k-quasimeromoprhic functions f of $\bar{\mathbf{C}}$ whose restrictions to $D^* = \{z \mid |z| > 1\}$ are meromorphic and of the form

$$f(z) = \sum_{n=0}^{p} a_n z^{p-n} + \sum_{n=1}^{\infty} a_{p+n} z^{-n}, \qquad a_0 = 1,\ p \in N, \tag{1.1.}$$

and $f(z) = \infty$ only for $z = \infty$.

$\sum_k^p$ denotes the subclass of F_k^p consisting of functions f whose singular part at ∞ reduces to z^p [2]:

We denote by $\sum_k$ the class of k-quasiconformal homeomorphisms f of $\bar{\mathbf{C}}$ which are conformal in D^* with a development of the form

$$f(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}. \tag{1.2}$$

The research was supported by the Academy of Finland.

doi:10.5186/aasfm.1989.1407

2. Representation theorems

Theorem 2.1. *A function $f \in F_k^p$ has the representation*

$$f = P \circ h \tag{2.1}$$

where $h \in \sum_k$ and P is a polynomial of degree p with leading coefficient 1.

Proof. Let $f \in F_k^p$ and μ be its complex dilatation. From the existence and uniqueness theorems for the Beltrami equation it follows that there is a unique quasiconformal homeomorphism $h \in \sum_k$ with complex dilatation $\mu_h = \mu$ a.e. The function $P = f \circ h^{-1}$ has then the complex dilatation zero a.e. in $\mathbf{C}$. Since it has L^1-derivatives, it is meromorphic in $\bar{\mathbf{C}}$, and hence rational. Because f has the only pole at $z = \infty$, P is a polynomial. If follows from the normalization (1.1) that it is a polynomial of degree p with leading coefficient 1.

Let $f \in \sum_k^p$. Then the polynomial P in (2.1) is the Faber polynomial of degree p of h, since the only polynomial of degree p such that the singular part of $P\big[h(z)\big]$ at ∞ reduces to z^p is the p^{th} Faber polynomial of h.

Remark 2.1. Let $L_{0,k}^\infty$ denote the set of complex valued measurable functions μ satisfying $\|\mu\|_\infty \le k < 1$ and having support in the closure of the unit disc D. A function $\mu \in L_{0,k}^\infty$ determines uniquely the element $h \in \sum_k$ whose complex dilatation μ_h equals μ a.e. but not the element of F_k^p. For, if P is an arbitrary polynomial of degree p with leading coefficient 1, then $f = P \circ h \in F_k^p$ with $\mu_h = \mu$ a.e. However there is a one-to-one correspondence between the functions $\mu \in L_{0,k}^\infty$ and the elements $f \in \sum_k^p$. In this case the uniqueness (and the existence) of $f \in \sum_k^p$ follows from the uniqueness (the existence) of the Faber polynomials.

It is an immediate consequence of Theorem 2.1 that a function $f \in F_k^p$ takes each value in $\bar{\mathbf{C}}$ exactly p times. In particular, a function $f \in F_k^1$ is a homeomorphism, a translation of its basic homeomorphism $h \in \sum_k$. Moreover, since 0-quasimeromorphic functions are meromorphic, F_0^p is the set of all polynomials of degree p with leading coefficient 1. We use the same notation F_k^p for the class of the restrictions $f|_{D^*}$ of all $f \in F_k^p$. Then every F_k^p , $0 \le k < 1$, is contained in the class F^p of functions f which take every value at most p times in D^* and have a development of the form (1.1).

A function g which is analytic in the interior of $C_R = h\big(|z| = R\big)$ for some $R \in (1, \infty)$ can be expanded into a series of Faber polynomials belonging to h, i.e., the function g has the representation

$$g(w) = \sum_{m=0}^{\infty} c_m P_m(w)$$

in the interior of C_R, where P_m denotes the m^{th} Faber polynomial of h, and

$$c_m = \frac{1}{2\pi} \int_{|z|=\varrho} g\big(h(z)\big) z^{-m-1}\, dz, \qquad \varrho \in (1, R),\ m = 0, 1, \ldots. \tag{2.2}$$

The representation is unique [8], [11].

Theorem 2.2. *Let $f \in F_k^p$. Then f has the representation*

$$f = f_p + a_1 f_{p-1} + \cdots + a_{p-1} f_1 + a_p,$$

where $f_m = P_m \circ h \in \sum_k^m$, $m = 1, \ldots, p$, and a_m are the power series coefficients of f.

Proof. Let $f \in F_k^p$. By Theorem 2.1 f has a representation of the form (2.1). Expanding P into series in terms of Faber polynomials P_m of h we obtain

$$P(w) = \sum_{m=0}^{p} c_m P_m(w)$$

where the c_m are given by (2.2). It is clear that $c_m = a_{p-m}$, $m = 0, \ldots, p$. Hence, $f = P \circ h = \left(\sum c_m P_m\right) \circ h$ and the assertion follows.

Consequently, for a given polynomial $G(z) = \sum_{n=0}^{p} a_n z^{p-n}$, $a_0 = 1$, and a function $\mu \in L_{0,k}^{\infty}$ there exists a unique quasimeromorphic function $f \in F_k^p$ which has complex dilatation $\mu_f = \mu$ a.e., and G as its principal part at $z = \infty$.

Let T be the operator defined by

$$Tw(z) = -\frac{1}{\pi} \iint_{\mathbf{C}} \frac{w(\zeta)}{\zeta - z}\, d\xi\, d\eta, \qquad \zeta = \xi + i\eta$$

and H the two-dimensional Hilbert transformation ([9] Chapter III.7). It is well known that using T and H, a function of $\sum_k$ can be represented with the aid of its complex dilatation. The proof of the representation formula ([9], Chapter V.5) applies, with obvious modifications, to the functions of F_k^p also.

Theorem 2.3. *Let $f \in F_k^p$. Then*

$$f(z) = G(z) + \sum_{n=1}^{\infty} T\phi_n(z), \qquad z \in \mathbf{C}, \tag{2.3}$$

where G is the principal part of f at $z = \infty$ and the functions ϕ_n are defined by $\phi_1 = \mu G'$, $\phi_n = \mu H \phi_{n-1}$, $n = 2, 3, \ldots$. The series is uniformly convergent .

Just as in the case $p = 1$, formula (2.3) gives asymptotic estimates for the coefficients of the functions in F_k^p.

Theorem 2.4. *Let f belong to F_k^p and have the expansion (1.1). Then*

(2.4.) $$|a_{p+n}| \le 2k \sum_{m=0}^{p-1} \frac{p-m}{p+n-m} |a_m| + O(k^2), \qquad n = 1, 2, \ldots.$$

The estimate is sharp.

The proof given in [10], pp. 432–433, for the counterpart of (2.4) in $\sum_k$ can be repeated as such, with the only difference that now $\phi_1 = \mu G'$.

Equality holds for the functions

$$f_n = f_{n,p} + a_1 f_{n,(p-1)} + \cdots + a_{p-1} f_{n,1} + a_p$$

where the functions $f_{n,m}$ are defined by

$$f_{n,m}(z) = \begin{cases} \left(z^{(m+n)/2} + kz^{-(m+n)/2}\right)^{2m/(m+n)} & \text{for } |z| > 1, \\ \left(z^{(m+n)/2} + k\bar{z}^{(m+n)/2}\right)^{2m/(m+n)} & \text{for } |z| \le 1, \end{cases}$$

for $m = 1, \ldots, p$.

For the special case $G(z) = z^p$, inequality (2.4) yields the sharp estimate

$$|a_{p+n}| \le \frac{2pk}{p+n} + O(k^2), \qquad n = 1, 2, \ldots.$$

Let $F_k^p(\zeta)$ denote the class of functions f in F_k^p which take the value zero only at the point ζ. If $f \in F_k^p(\zeta)$, then it follows from Theorem 2.1 that

$$f = \left(h - h(\zeta)\right)^p \tag{2.5}$$

where $h \in \sum_k$.

Theorem 2.5. *Let $f \in F_k^p(0)$. Then*

$$|a_1| \le 2pk. \tag{2.6}$$

Equality holds only for the functions

$$f(z) = \begin{cases} \left(z^{1/2} + ke^{i\theta} z^{-1/2}\right)^{2p} & \text{for } |z| > 1, \\ \left(z^{1/2} + ke^{i\theta} \bar{z}^{1/2}\right)^{2p} & \text{for } |z| \le 1. \end{cases}$$

Proof. The estimate (2.6) follows from $f = \left(h - h(0)\right)^p$ when we take into account Kühnau's result $|h(0)| \le 2k$ ([5]).

In Section 3 we shall derive the above estimates from a general inequality.

Let $f = \left(h - h(\zeta)\right)^p \in F_k^p(\zeta)$ and let b_n, $n = 1, 2, \ldots$, denote the power series coefficients of h. We see that $a_1 = a_2 = \cdots = a_N = 0$ if and only if $h(\zeta) = 0$ and

$$b_1 = b_2 = \cdots = b_{N-1} = 0. \tag{2.7}$$

In this case,

$$a_n = pb_{n-1}, \qquad n = N+1, \ldots, 2N+1. \tag{2.8}$$

Theorem 2.6. *Let $f \in F_k^p(0)$. If $a_n = 0$, $n = 1, 2, \ldots, N$ $(N \geq 1)$, then*

$$|a_n| \leq \frac{2kp}{n}, \qquad n = N+1, \ldots, 2N+1. \tag{2.9}$$

Equality holds for the functions

$$f(z) = \begin{cases} \left(z^{n/2} + ke^{i\theta} z^{-n/2}\right)^{2p/n} & \text{for } |z| > 1, \\ \left(z^{n/2} + ke^{i\theta} \bar{z}^{n/2}\right)^{2p/n} & \text{for } |z| \leq 1. \end{cases}$$

Proof. Again, we make use of (2.5). Because (2.7) is true, $|b_n| \leq 2k/(n+1)$, $n = N, N+1, \ldots, 2N$ (Kühnau [6]). Hence, (2.9) follows from (2.8).

In particular, for $N = 1$ we have $|a_2| \leq kp$ and $|a_3| \leq 2kp/3$.

In [3] we proved that $|a_1| \leq 4k/3$ in $\sum_k^2$, which can also be deduced from (2.9). In $\sum_k^p$, the estimate $|a_p| \leq k$ holds true ([2]).

3. Majorant principle for the class $F_k^p(\zeta)$

In this section we establish a counterpart of Lehto's majorant principle [10] for the class $F_k^p(\zeta)$, $\zeta \in \bar{D}$, from which we obtain estimates for the power series coefficients a_n of a function $f \in F_k^p(\zeta)$. The estimate for $|a_1|$ leads to a distortion theorem for $|h|$, $h \in \sum_k$ in $\bar{D}$.

We denote by $F^p(0)$ the class of functions f in F^p which do not assume the value zero in D^*. Then every restricted class $F_k^p(\zeta)$, $\zeta \in \bar{D}$, is contained in the class $F^p(0)$.

The classes $\sum_k$ and $\sum_k' = \{h|_{D^*} \mid h \in \sum_k\}$, $0 \leq k < 1$, are known to be compact in the topology of locally uniform convergence. From the representation (2.5) it follows that every $F_k^p(\zeta)$, $0 \leq k < 1$, is compact. Also, $F^p(0)$ is compact.

Let Φ be an analytic functional defined on $F^p(0)$. Then Φ is defined on every $F_k^p(\zeta)$, $0 \leq k < 1$. Because the classes $F^p(0)$ and $F_k^p(\zeta)$ are compact,

$$\max_{f \in F^p(0)} |\Phi(f)| = M(1) \qquad \text{and} \qquad \max_{f \in F_k^p(\zeta)} |\Phi(f)| = M(k)$$

exist. The class $F_0^p(\zeta)$ contains only the function $f_0 = (z - \zeta)^p$, and we write $M(0) = |\Phi(f_0)|$.

Theorem 3.1. *Let Φ be an analytic functional defined on $F^p(0)$. Then for every $f \in F_k^p(\zeta)$,*

$$M(1)\frac{M(0) - kM(1)}{M(1) - kM(0)} \leq |\Phi(f)| \leq M(1)\frac{M(0) + kM(1)}{M(1) + kM(0)}. \tag{3.1}$$

In particular, if $\Phi(f_0) = 0$,

$$|\Phi(f)| \leq kM(1).$$

Proof. In [10], inequality (3.1) was established in the case $f \in \sum$. Thanks to the simple relation (2.5) the same proof applies to $F_k^p(\zeta)$.

Corollary 3.1. *Let* $f \in F_k^p(\zeta)$. *Then*

$$|a_1| \le 2p\frac{|\zeta| + 2k}{2 + k|\zeta|}. \tag{3.2}$$

Proof. Let $\Phi(f) = a_1$. Then $M(0) = p|\zeta|$ and by Theorem XI.6.3. in [1] we have $M(1) = 2p$. Thus (3.2) follows from the right-hand inequality of (3.1).

For the class $F_k^1(\zeta)$ we obtain from (3.1)

$$|a_1| \le 2\frac{|\zeta| + 2k}{2 + k|\zeta|}.$$

Corollary 3.2. *Let* $h \in \sum_k$. *Then for* $\zeta \in \bar{D}$

$$\left|h(\zeta)\right| \le 2\frac{|\zeta| + 2k}{2 + k|\zeta|}.$$

Proof. The function $f = \left(h - h(\zeta)\right)^p$ is in $F_k^p(\zeta)$. Since $a_1 = -ph(\zeta)$, the assertion follows from (3.2). As $k \to 1$, it gives the well-known sharp estimate $\left|h(\zeta)\right| \le 2$ in $\sum$.

For $\zeta = 0$ Corollary 3.2. yields the sharp estimate $\left|h(0)\right| \le 2k$ [5].

Corollary 3.3. *Let* $f \in F_k^p(0)$. *Then*

$$|a_1| \le 2pk, \qquad |a_2| \le p(2p-1)k.$$

The first estimate is sharp.

Proof. The first estimate follows from Corollary 3.1 for $\zeta = 0$.

For the second estimate, let $\Phi(f) = a_2$. For $\zeta = 0$, $f_0(z) = z^p$ and therefore $M(0) = 0$. By Theorem XI.6.3 in [1], $M(1) = p(2p-1)$ and the assertion follows from (3.1).

For $p = 1$ we obtain $|a_1| \le 2k$, $|a_2| \le k$ for the class $F_k^1(0)$ [5, 6].

Remark 3.1. The second estimate in Corollary 3.3. is not sharp for $p > 1$. For, let $f \in F_k^p(0)$. By the representation (2.5)

$$\left(f(z)\right)^{1/p} = h(z) - h(0) = z + \frac{a_1}{p} + \sum_1^\infty c_n z^{-n}$$

and

$$c_1 = \frac{a_2}{p} - \frac{p-1}{2p^2}a_1^2.$$

Since $|c_1| \le k$, this together with the first estimate in Corollary 3.3 yields

$$|a_2| \le pk + 2p(p-1)k^2.$$

Remark 3.2. Let $\Phi(f) = a_n$, $n = 3, 4, \dots$, for $f \in F^p(0)$. Then $M(0) = 0$ and by Theorem XI.6.3 in [1], $M(1) \le C(n,p)$, where $C(n,p)$ is a constant depending on n and p only. By Theorem 3.1 we have then $|a_n| \le kC(n,p)$ in $F_k^p(0)$. We note that bounds of this kind cannot be always found for the class F_k^p.

4. Class $S_k^p(\zeta)$

We denote by $S_k^p(\zeta)$ the class of k-quasimeromorphic functions f of $\bar{\mathbf{C}}$ which are analytic in D and of the form

$$f(z) = z^p + \sum_{n=1}^{\infty} a_{p+n} z^n \tag{4.1}$$

and $f(z) = 0, \infty$ only for $z = 0, \zeta$, respectively.

$S_k(\infty)$ denotes the class of k-quasiconformal homeomorphisms f of $\bar{\mathbf{C}}$ whose restrictions to D have the form

$$f(z) = z + \sum_{n=2}^{\infty} b_n z^n$$

and which leave ∞ fixed.

The proof of the following representation theorem is similar to the proof of Theorem 2.1.

Theorem 4.1. *A function $f \in S_k^p(\zeta)$ has the unique representation*

$$f = \left(\frac{h}{1 - h/h(\zeta)}\right)^p \tag{4.2}$$

where $h \in S_k(\infty)$.

Consequently, a function $f \in S_k^p(\zeta)$ takes every value in $\bar{\mathbf{C}}$ exactly p times. In particular, a function $f \in S_k^1(\zeta)$ is a homeomorphism. We write $S_k^1(\zeta) = S_k(\zeta)$.

Theorem 4.2. *A function f of $S_k^p(\zeta)$ has the unique representation*

$$f = (\tilde{h})^p$$

where $\tilde{h} \in S_k(\zeta)$.

If $f \in S_k^p(\infty)$, the representation (4.2) takes the form

$$f = h^p \tag{4.3}$$

where $h \in S_k(\infty)$ ([2]).

It follows from Theorem 4.1 that the class $S_0^p(\zeta)$ contains only the function $f_0(z) = z^p(1 - z/\zeta)^{-p}$. We use the same notation $S_k^p(\zeta)$ for the class of the restrictions $f|_D$ of all $f \in S_k^p(\zeta)$. Then every $S_k^p(\zeta)$, $0 \le k < 1$, is contained in the class S^p of analytic functions f which take every value at most p times in D and have the normalization (4.1).

Majorant principle. Let Φ be an analytic functional defined on S^p. Again, the classes S^p and $S_k^p(\zeta)$ are compact so that

$$\max_{f \in S^p} |\Phi(f)| = M(1) \qquad \text{and} \qquad \max_{f \in S_k^p(\zeta)} |\Phi(f)| = M(k)$$

exist. For the function $f_0 = z^p(1 - z/\zeta)^{-p}$ we write $M(0) = |\Phi(f_0)|$.

Theorem 3.1 applies for the classes $S_k^p(\zeta)$ and S^p: If Φ is an analytic functional defined on S^p, then (3.1) holds for every $f \in S_k^p(\zeta)$.

Corollary 4.1. *Let $f \in S_k^p(\zeta)$. Then*

$$|a_{p+1}| \le 2p\frac{1+2k|\zeta|}{2|\zeta|+k}. \tag{4.4}$$

Proof. Consider the analytic functional $\Phi(f) = a_{p+1}$. Then $M(1) = 2p$ by Theorem XI.6.5 in [1] and $M(0) = p/|\zeta|$. Thus the assertion follows from (3.1).

Corollary 4.2. *Let $f \in S_k^p(\zeta)$. Then*

$$|a_{p+2}| \le p(2p+1)\frac{(p+1)+2k(2p+1)|\zeta|^2}{2(2p+1)|\zeta|^2+k(p+1)}. \tag{4.5}$$

Proof. Let $\Phi(f) = a_{p+2}$. Then $M(0) = p(p+1)/2|\zeta|^2$, and by Corollary 8.16 in [4] $M(1) = p(2p+1)$. Thus the assertion follows from (3.1).

We obtain from (4.4) and (4.5) the estimates

$$|a_2| \le \frac{1+2k|\zeta|}{2|\zeta|+k}, \qquad |a_3| \le 3\frac{1+3k|\zeta|^2}{3|\zeta|^2+k}$$

for the class $S_k(\zeta)$. Furthermore, $|a_{p+1}| \le 2pk$ [1] and $|a_{p+2}| \le p(2p+1)k$ in $S_k^p(\infty)$. The first estimate is sharp.

Theorem 4.3. *Let $f \in S_k^p(\infty)$. If $a_{p+n} = 0$, $n = 1, \dots, N$, $(N \ge 1)$, then*

$$|a_{p+n}| \le \frac{2pk}{n}, \qquad n = N+1, \dots, 2N+1.$$

Equality holds for the functions

$$f(z) = \begin{cases} z^p(1+ke^{i\theta}z^n)^{-2p/n} & \text{for } |z|<1, \\ (z\bar{z})^p(\bar{z}^{n/2}+ke^{i\theta}z^{n/2})^{-2p/n} & \text{for } |z| \ge 1. \end{cases}$$

Proof. The function $z \mapsto 1/f(1/z)$ is in $F_k^p(0)$. It has the expansion

$$\frac{1}{f\left(\frac{1}{z}\right)} = z^p\left(1+\sum_{N+1}^{\infty} c_n z^{-n}\right),$$

where $c_n = -a_{p+n}$ for $n = N+1, \dots, 2N+1$. Hence the assertion follows from Theorem 2.6.

---

[1] A different proof of this estimate was given in [2].

Theorem 4.4. *Let $f \in S_k^p(\infty)$. Then for $z \in \bar{D}$*

$$c^p \leq \left|f(z)\right| \leq C^p,$$

where c and C are Kühnau's constants [7]. *The estimate is sharp.*

The theorem follows from the representation (4.3) and Kühnau's distortion theorem ([7]).

Remark 4.1. Let S denote the class of conformal homeomorphisms f of D which are normalized by the conditions $f(0) = 0$, $f'(0) = 1$. It is known that the classes $S_k'(\infty) = \{h|_D \mid h \in S_k(\infty)\}$, $0 \leq k < 1$, are dense in S with respect to the topology of locally uniform convergence, as $k \to 1$. However, the classes $S_k^p(\infty)$ are not dense in S^p, nor are $\sum_k^p$ dense in $\sum^p$.

References

[1] Goluzin, G.M.: Geometric theory of functions of of a complex variable. - American Mathematical Society Translations of Mathematical Monographs 26, Providence, R.I., 1969.

[2] Göktürk, Z.: On p-valent quasimeromorphic functions. - Proceedings of the Rolf Nevanlinna Symposium on Complex Analysis, Silivri 1976. Publications of the Nazım Terzioğlu Mathematical Research Institute 7, Istanbul, 1978.

[3] Göktürk, Z.: Remarks on p-valent quasimeromorphic functions. - Proceedings of the Vth Romanian-Finnish Seminar on Complex Analysis, Bucharest, 1981. Lecture Notes in Mathematics 1013. Springer-Verlag, Berlin–Heidelberg–New York, 1983.

[4] Jenkins, J.A.: Univalent functions and conformal mappings. - Ergebnisse der Mathematik 18, Springer-Verlag, Berlin, 1958.

[5] Kühnau, R.: Wertannahmeprobleme bei quasikonformen Abbildungen mit ortsabhängiger Dilatationsbeschränkung. - Math. Nachr. 40, 1969, 1–11.

[6] Kühnau, R.: Verzerrungsätze und Koeffizientenbedingungen von Grunskyschen Typ für quasikonforme Abbildungen. - Math. Nachr. 48, 1971, 77–105.

[7] Kühnau, R.: Eine Verschärfung des Koebeschen Viertelsatzes für quasikonform fortsetzbare Abbildungen. - Ann. Acad. Sci. Fenn. Ser. A I Math. 1, 1975, 77–83.

[8] Lebedev, N.A., and V.I. Smirnov: Functions of a complex variable. - London Iliffe Book Ltd., London, 1968.

[9] Lehto, O., and K.I. Virtanen: Quasiconformal mappings in the plane. - Grundlehren der mathematischen Wissenschaften 126. Springer-Verlag, New York–Heidelberg–Berlin, 1973.

[10] Lehto, O.: Quasiconformal mappings and singular integrals. - Istituto Nazionale di Alta Matematica. Symposia Mathematica XVIII. Academic Press, Bologna, 1976.

[11] Schober, G.: Univalent functions. - Lecture Notes in Mathematics 478. Springer-Verlag, New York–Heidelberg–Berlin, 1975.

Boğaziçi University
Department of Mathematics
İstanbul
Turkey

Received 16 March 1984
Revised 31 May 1988