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ON THE INFINITY OF THE LONERGAN–HOSACK PRESENTATION

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Introduction

In 1984 F.D. Lonergan and J. Hosack [3] introduced the following group presentation:

$$G = \langle x, z : z^3xz^3x^{-1} = z^5x^2z^2x^2 = 1\rangle.$$

According to the authors there was good reason to believe that the presentation is that of a finite group. However, they reported that attempts to prove finiteness by using the Todd–Coxeter algorithm were unsuccessful.

By using the computer algebra system CAYLEY M. Slattery [5] managed to show in 1985 that G is infinite. To be precise, Slattery showed that G has a factor group which is infinite.

In this short note we wish to point out that a lot more can be said about the structure of G. In fact, we shall consider the presentation

$$G(n) = \langle x, z : z^nxz^nx^{-1} = z^{n+2}x^2z^2x^2 = 1\rangle$$

with $n \geq 2$. In particular, our results hold for $G = G(3)$. For the background material of this note the reader is advised to consult [7].

Main theorem

Consider the presentation $G(n)$ given in the introduction with $n \geq 1$. If $n = 1$, then $zx^2 = x^2z$ and $z^5 = z^{-5} = x^4$. Consequently $G(1)$ is a finite group of order 40. Now we shall establish the

Theorem. *Let $n \geq 2$ and consider the presentation $G(n)$. Now*

(1) *$G(n)$ is a nontrivial free product with amalgamation,*

(2) *$G(n)$ has a subgroup of finite index mapping onto a free group of rank 2 and $G(n)$ has a free subgroup of rank 2,*

(3) *$G(n)$ has a generating pair $\{u, v\}$ such that the subgroup $\langle u^k, v^k\rangle$ is free of rank 2 for a sufficiently large integer k,*

(4) *$G(n)$ is SQ-universal (i.e. every countable group is embeddable in some factor group of $G(n)$).*

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Proof. (1) If n is even, then the free product $Z_4 * Z_2 = \langle x, z : x^4 = z^2 = 1\rangle$ is an epimorphic image of $G(n)$. If n is odd (then $n \geq 3$), we have $D = \langle x, z : z^n = (z^2x^2)^2 = 1\rangle$ as an epimorphic image of $G(n)$. Now D is a nontrivial free product of H_1 and H_2 with the amalgamated subgroup H, where $H_1 = \langle x\rangle \cong Z$, $H_2 = \langle y, z : z^n = (z^2y)^2 = 1\rangle$ and $H = \langle x^2\rangle \cong \langle y\rangle$. By Lemma 3.2 of [6] we conclude that $G(n)$ is a nontrivial free product with amalgamation for every $n \geq 2$.

(2) The free product $Z_4 * Z_2$ has the triangle-group

$$T(2,4,5) = \langle a, b : a^2 = b^4 = (ab)^5 = 1\rangle$$

as an epimorphic image and clearly $G(n)$ has $T(2,4,5)$ as an epimorphic image provided n is even. Next consider D (the free product with amalgamation) introduced in the first part of the proof. Now D has the triangle-group

$$T(n,7,2) = \langle a, b : a^n = b^7 = (ab)^2 = 1\rangle$$

as an epimorphic image and naturally $G(n)$ has $T(n,7,2)$ as an epimorphic image for odd n ($n \geq 3$).

Thus $G(n)$ has as an epimorphic image a triangle-group

$$T(p,q,r) = \langle a, b : a^p = b^q = (ab)^r = 1\rangle$$

with $2 \leq p, q, r$ and $(1/p)+(1/q)+(1/r) < 1$. Now $T(p,q,r)$ has a surface group $F(g)$ of genus $g \geq 2$ as a subgroup of finite index (see [7]). Since $F(g)$ ($g \geq 2$) has a free group of rank 2 as an epimorphic image, it follows by considering the pre-image of $F(g)$ in $G(n)$ that $G(n)$ has a subgroup of finite index which maps onto a free group of rank 2 and consequently $G(n)$ has a free subgroup of rank 2.

(3) Since $T(p,q,r)$ can be regarded as a subgroup of $PSL(2,R)$ (see [7]), our assertion follows from [4].

(4) By [1] we know that a free group of rank 2 is SQ-universal. Clearly, a pre-image of an SQ-universal group is SQ-universal. Finally, by [2] it follows that a group which has an SQ-universal subgroup of finite index is also SQ-universal. The proof is complete.

Remarks. As indicated in the beginning of this paper $G(1)$ is a finite group of order 40 and consequently it has elements of finite order ≥ 2. If we consider $G(2)$, then $z^2xz^2x^{-1} = z^4x^2z^2x^2 = 1$ implies $x^2z^2 = z^2x^2$. Furthermore, $z^6 = x^{-4}$ and $z^{-6} = x^{-4}$, hence $z^{12} = 1$ and $x^8 = 1$. Now we state

Problem 1. Is it true that $G(n)$ has elements of finite order ≥ 2 for $n \geq 3$?

We also state

Problem 2. Is it true that $G(n)$ has a torsion-free normal subgroup of finite index for $n \geq 2$?

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