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AREA THEOREMS AND FREDHOLM EIGENVALUES

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1. Introduction

In this paper we shall derive an area theorem for conformal mappings onto a domain whose Fredholm eigenvalue is bounded from below. Furthermore, we will prove the following extremum property of circular rings: The smallest non-trivial Fredholm eigenvalue of a doubly connected domain having a fixed conformal modulus attains its maximal value if, and only if, the boundary of the domain consists of two circles.

Let G be an unbounded plane domain bounded by n closed analytic Jordan curves such that the complement G^C of G consists of simply connected closed regions. We denote by $\sum(G)$ the class of analytic and univalent functions $f(z)$ $(z \in G \setminus \{\infty\})$ having the Laurent series expansion

$$f(z) = z + \alpha_0 + \frac{\alpha_1}{z} + \frac{\alpha_2}{z^2} + \cdots \tag{1}$$

at infinity. Let $\sum\langle G, \kappa\rangle$ be the subclass of $\sum(G)$ containing all functions $f(z)$ for which the domain $G^* = f(G)$ has the smallest non-trivial Fredholm eigenvalue λ_2 satisfying

$$\lambda_2 \geq \frac{1}{\kappa}, \tag{2}$$

where the positive number $\kappa < 1$ is fixed.

A classical definition of the Fredholm eigenvalue λ_2 is contained in [4; p. 3 ff.]. For our purposes we need the generalized definition of λ_2 given by G. Springer. We denote by G^* an unbounded domain of finite connectivity bounded by closed pairwise disjoint Jordan curves. Then the Fredholm eigenvalue λ_2 of G^* is the greatest number $\lambda > 1$ satisfying

$$\frac{\lambda+1}{\lambda-1} \geq \frac{\int\int_{G^*}[\nabla H(w)]^2\,du\,dv}{\int\int_{G^{*C}}[\nabla H(w)]^2\,du\,dv} \geq \frac{\lambda-1}{\lambda+1}, \qquad (w = u + iv), \tag{3}$$

for all continuous functions $H(w)$ which are harmonic in G^* and in the interior of G^{*C} and have a single-valued harmonic conjugate (see [21] and cf. [1], [3], [10]

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and [19]). Note that $H(w)$ may be a real or a complex harmonic function. In the latter case the expression $[\nabla H(w)]^2$ means $|H_u(w)|^2 + |H_v(w)|^2$.

Let $\{\varphi_\nu(z), \Phi_\nu(z)\}$, $(\nu = 1, 2, \ldots)$ be the orthonormal system of functions fulfilling

$$\overline{d\varphi_\nu(z)} = d\Phi_\nu(z) \tag{4}$$

on the boundary ∂G of G. The functions $\varphi_\nu(z)$ are analytic in the closure of G. The functions $\Phi_\nu(z)$ have a pole at infinity and are analytic for all finite z in the closure of G. This system of functions is introduced in [16]. The functions of this orthonormal system are closely related to mappings onto parallel slit domains. For example, $\Phi_1(z) + \varphi_1(z)$ and $\Phi_1(z) - \varphi_1(z)$ map G univalently onto parallel slit domains.

Besides (4), we need another main property of this system. If $T(w)$ is an analytic function in the complement of $f(G)$ for all mappings $f(z) \in \sum(G)$, then

$$\Big\{T[f(z)]\Big\}' = \sum_{\nu=1}^{\infty} \Gamma_\nu \Phi'_\nu(z) + \sum_{\nu=1}^{\infty} \gamma_\nu \varphi'_\nu(z) \tag{5}$$

holds for all $z \in G$ in a sufficiently small neighbourhood of points on ∂G and for all mappings $f(z) \in \sum(G)$. A simple example of such a function $T(w)$ is $T(w) = w$. In this case we have

$$f'(z) = a_{11}\Phi'_1(z) + \sum_{\nu=1}^{\infty} \gamma_\nu \varphi'_\nu(z),$$

where $a_{11} > 0$ is a fixed number depending only on the domain G. Besides, a_{11}^2 represents the radius of that circular disk which is the exact range of the coefficient α_1 in (1) when $f(z)$ belongs to the class $\sum(G)$.

Finally, it should be noted that

$$\iint_{[f(G)]^C} |T'(w)|^2 du\, dv = \pi\left(\sum_{\nu=1}^{\infty} |\Gamma_\nu|^2 - \sum_{\nu=1}^{\infty} |\gamma_\nu|^2\right) \qquad (w = u + iv). \tag{6}$$

This follows from (5) for all mappings $f(z) \in \sum(G)$ (see [16; Theorem 4.13]).

2. Derivation of an area theorem

The basic idea of the following investigations is taken from [13]. We consider a continuous function $H(w)$ defined by

(7)
$$\nabla H(w) = \begin{cases} \nabla \operatorname{Re}(e^{-i\theta}T(w)) & \text{if } w \in (f(G))^C, \\ \nabla \operatorname{Re}(e^{-i\theta}(T(w) - \sum \Gamma_\nu \Phi_\nu(z)) + e^{i\theta} \sum \overline{\Gamma}_\nu \varphi_\nu(z)) & \text{if } w = f(z) \in f(G), \end{cases}$$

where θ is an arbitrary given number satisfying $0 \leq \theta < \pi$ and Γ_ν are the coefficients in (5). Since $f(z) \in \sum\langle G, \kappa\rangle$, we write (3) in the form

$$\iint_{[f(G)]^C} (\nabla H(w))^2 du\, dv \geq \frac{1-\kappa}{1+\kappa} \iint_{f(G)} (\nabla H(w))^2 du\, dv. \tag{8}$$

The first integral in (8) is the same as

$$\iint_{[f(G)]^C} |T'(w)|^2 du\, dv.$$

By (6) the value of this integral is equal to

$$\pi\left(\sum_{\nu=1}^{\infty} |\Gamma_\nu|^2 - \sum_{\nu=1}^{\infty} |\gamma_\nu|^2\right).$$

After a transformation into the z-plane we find for the other integral in (8) that

$$\iint_{f(G)} (\nabla H(w))^2 du\, dv = \pi \sum_{\nu=1}^{\infty} |\gamma_\nu - e^{2i\theta}\overline{\Gamma}_\nu|^2. \tag{10}$$

Now we state our first result.

Theorem 1. *Let θ and κ be arbitrary numbers satisfying $0 \leq \theta < \pi$ and $0 < \kappa < 1$, respectively. If $f(z) \in \sum\langle G, \kappa\rangle$ then*

$$\sum_{\nu=1}^{\infty} \left|\gamma_\nu + \tfrac{1}{2}(1-\kappa)\overline{\Gamma}_\nu e^{2i\theta}\right|^2 \leq \tfrac{1}{4}(1+\kappa)^2 \sum_{\nu=1}^{\infty} |\Gamma_\nu|^2 \tag{11}$$

holds for the coefficients Γ_ν, γ_ν ($\nu = 1, 2, \dots$) in (5).

In the next section we shall give some examples for Theorem 1.

3. Examples

Let G be the exterior of the unit circle and $T(w) = w$. In this case we write $\sum\langle\kappa\rangle$ instead of $\sum\langle G, \kappa\rangle$ (see also [13] and [17; p. 287 ff.]). Then (11) has the form

$$\left(|\alpha_1| + \tfrac{1}{2}(1-\kappa)\right)^2 + \sum_{\nu=2}^{\infty} \nu|\alpha_\nu|^2 \leq \tfrac{1}{4}(1+\kappa)^2. \tag{12}$$

The coefficients α_ν ($\nu = 1, 2, \dots$) are given in (1). The same inequality was proved by L.V. Ahlfors in [2] for the mappings of the known class $\sum(Q)$ where κ and Q satisfy

$$\kappa = \frac{Q-1}{Q+1}. \tag{13}$$

But our class $\sum\langle\kappa\rangle$ is much wider than $\sum(Q)$ (see [13]). A better inequality for the class $\sum\langle\kappa\rangle$ than (12), namely

$$\sum_{\nu=1}^{\infty} \nu|\alpha_\nu|^2 \le \kappa^2, \tag{14}$$

has been derived in [13]. (14) and also (12) imply that

$$|\alpha_1| \le \kappa. \tag{15}$$

Equality in (12), (14) and (15) holds for the mappings

$$f^*(z) = \begin{cases} z + \kappa e^{2i\alpha}/z + \text{const} & \text{if } |z| > 1, \\ z + \kappa e^{2i\alpha}\bar{z} + \text{const} & \text{if } |z| \le 1, \end{cases}$$

where α is an arbitrary number satisfying $0 \le \alpha < \pi$. It should be noted that $f^*(z) \in \sum\langle\kappa\rangle$ because the Fredholm eigenvalue λ_2 (in the classical sense) of an ellipse

$$E = \left\{(x,y) : \frac{x^2}{(1+\kappa)^2} + \frac{y^2}{(1-\kappa)^2} = 1\right\}$$

is exactly $1/\kappa$ (see [19]).

Now let G be a doubly connected domain. In this case it is more difficult to consider an explicit example because the system $\{\varphi_\nu(z), \Phi_\nu(z)\}$ $(\nu = 1, 2, \ldots)$ is, in general, not available. Therefore, we shall investigate conformal mappings of an annulus $\{z : R < |z| < 1/R\}$ $(0 < R < 1)$ onto an unbounded domain having the Fredholm eigenvalue $\lambda_2 \ge 1/\kappa$. For the sake of simplicity we propose $f(1) = \infty$ and

$$f(z) - \frac{1}{z-1} = \sum_{\nu=-\infty}^{\infty} \delta_\nu z^\nu \qquad (R < |z| < 1/R). \tag{16}$$

Furthermore, we need a modification of (7). Letting $T(w) = w$, it is necessary to find a harmonic function in the annulus with the boundary values $1/(\bar{z}-1)$. Starting from the following expression with some unknown numbers $s_\nu^{(1)}$ and $s_\nu^{(2)}$ $(\nu \in \mathbf{Z})$

$$\frac{1}{\bar{z}-1} = \sum_{\substack{\nu=-\infty \\ \nu\ne 0}}^{\infty} \left(s_\nu^{(1)} z^\nu + s_\nu^{(2)} \bar{z}^\nu\right) + s_0^{(1)} \ln|z| + s_0^{(2)}, \qquad |z| = R, 1/R,$$

and making use of $z \cdot \bar{z} = R^2$ or $1/R^2$ on the boundary of the annnulus we get

$$s_\nu^{(1)} = -\frac{R^{2\nu}}{R^{4\nu}-1},$$

$$s_\nu^{(2)} = \begin{cases} (R^{-4\nu}-1)^{-1}, & \text{when } \nu > 0, \\ (1-R^{4\nu})^{-1}, & \text{when } \nu < 0, \end{cases},$$

with $\nu \in \mathbf{Z} \setminus \{0\}$. Then we define

$$\nabla H(w) = \nabla \operatorname{Re}\Bigl(e^{-i\theta}\bigl(f(z) - \frac{1}{z-1}\bigr) + e^{i\theta} \sum_{\substack{\nu=-\infty \\ \nu\neq 0}}^{\infty} \bigl(s_\nu^{(1)} z^\nu + s_\nu^{(2)} \bar{z}^\nu\bigr)\Bigr)$$

when $w = f(z) \in f(G)$. Thus we obtain

$$\begin{aligned} \sum_{\nu=1}^{\infty} \nu \frac{1-R^{4\nu}}{R^{2\nu}} \Bigl(\bigl|\delta_\nu + \frac{R^{4\nu}}{1-R^{4\nu}} + \tfrac{1}{2}(1-\kappa)\frac{R^{2\nu}}{1-R^{4\nu}} e^{2i\theta}\bigr|^2 \\ + \bigl|\delta_{-\nu} - \frac{R^{4\nu}}{1-R^{4\nu}} - \tfrac{1}{2}(1-\kappa)\frac{R^{2\nu}}{1-R^{4\nu}} e^{2i\theta}\bigr|^2 \Bigr) \\ \le \tfrac{1}{2}(1+\kappa)^2 \sum_{\nu=1}^{\infty} \nu \frac{R^{2\nu}}{1-R^{4\nu}}, \end{aligned} \tag{17}$$

where the number θ may be arbitrarily chosen as before.

Multiply connected domains provide another difficulty. We will show in Section 4 that $\sum\langle G,\kappa\rangle$ is empty for too small numbers $\kappa > 0$. To ensure that $\sum\langle G,\kappa\rangle \neq \emptyset$, we choose $\kappa \ge R^2$ because $1/R^2$ is the Fredholm eigenvalue λ_2 of the given annulus G (see [20]). This difficulty illustrates a remarkable property of a circle, namely $\lambda_2 = \infty$. Hence, the identical mapping is always a member of $\sum\langle\kappa\rangle$ for all $\kappa \in (0,1)$, consequently $\sum\langle\kappa\rangle \neq \emptyset$.

In what follows we will investigate for which mappings $f(z) \in \sum\langle G,\kappa\rangle$ equality holds in (11).

We propose that for all mappings $f(z) \in \sum$ the function $T\bigl(f(z)\bigr) - T(z)$ is analytic in $\{z : |z| > 1\}$. In other words, the singularities of $T\bigl(f(z)\bigr)$ do not depend on $f(z)$. Consequently, we have

$$T\bigl(f(z)\bigr) = \sum_{\nu=1}^{\infty} \Omega_\nu z^\nu + \text{ const } + \sum_{\nu=1}^{\infty} \omega_\nu z^{-\nu}, \qquad 1 < |z| < r, \tag{18}$$

for r sufficiently close to 1. The numbers Ω_ν, $\nu = 1, 2, \ldots$, depend only on our choice of $T(w)$, and (18) is valid for all functions $f(z) \in \sum$. In particular, the area theorem for the class $\sum\langle\kappa\rangle$ may be written in the form

$$\sum_{\nu=1}^{\infty} \nu \bigl|\omega_\nu + \tfrac{1}{2}(1-\kappa)e^{2i\theta}\overline{\Omega}_\nu\bigr|^2 \le \bigl(\tfrac{1}{2}(1+\kappa)\bigr)^2 \sum_{\nu=1}^{\infty} \nu |\Omega_\nu|^2. \tag{19}$$

Firstly, we give an example for a mapping $f^*(z)$ for which equality holds in (19). In the following step it is shown that $f^*(z)$ belongs to $\sum\langle\kappa\rangle$. Defining a quasiconformal mapping $f^*(z)$ of the whole plane by

$$f^*_{\bar{z}}(z) = \kappa\frac{\overline{T'(z)}}{T'(z)}f^*_z(z), \qquad |z| < 1, \tag{20}$$

and $f^*(z) \in \sum$, we conclude that $T\big(f^*(z)\big)$ also satisfies (20) for $|z| < 1$. Consequently, the function

$$F(z) = \begin{cases} T\big(f^*(z)\big) - T(z) - \kappa\overline{T(1/\bar{z})}, & \text{if } |z| > 1, \\ T\big(f^*(z)\big) - T(z) - \kappa\overline{T(z)}, & \text{if } |z| \le 1, \end{cases} \tag{21}$$

is a solution of (20) which is analytic in the exterior of the unit circle. Hence, $F(z)$ must be a constant. Since $T(z)$ is an analytic function in the interior of the unit circle, (18) implies that

$$T'(z) = \sum_{\nu=1}^{\infty} \nu\Omega_\nu z^{\nu-1}, \qquad |z| < 1.$$

Because of (21) we obtain

$$T\big(f^*(z)\big) = \sum_{\nu=1}^{\infty} \Omega_\nu z^\nu + \text{const} + \kappa\sum_{\nu=1}^{\infty} \overline{\Omega}_\nu z^{-\nu} \qquad 1 < |z| < r. \tag{22}$$

By choosing $\theta = 0$ it is easy to see that equality in (19) holds for $f^*(z)$. Because of (20) the inequality $\lambda_2 \ge 1/\kappa$ follows from a known result of L.V. Ahlfors (see [1]); consequently $f^*(z) \in \sum\langle\kappa\rangle$.

Suppose now that equality in (19) holds for any mapping $f(z) \in \sum\langle\kappa\rangle$. Then the function $H(w)$ in (7) must be a real eigenfunction belonging to $1/\kappa$. This means that $e^{-i\theta}\kappa T(w) - e^{i\theta}\overline{T(w)}$ can be extended to an analytic function in the complement of $f\big(\{z : |z| > 1\}\big)$ (see [12; Theorem 5]). Consequently, it follows from (18) that

$$\omega_\nu = \kappa e^{2i\theta}\overline{\Omega}_\nu, \qquad \nu = 1, 2, \ldots .$$

It can be proved as above that a suitable continuation of $f(z)$ satisfies

$$f_{\bar{z}}(z) = \kappa e^{2i\theta}\frac{\overline{T'(z)}}{T'(z)}f_z(z).$$

Summarizing these facts, we can state our next result.

Theorem 2. *If the analytic function* $T(w)$, $w \in \big(f(\{z : |z| > 1\})\big)^C$, *satisfies*

$$T\big(f(z)\big) = \sum_{\nu=1}^{\infty} \Omega_\nu z^\nu + \text{const} + \sum_{\nu=1}^{\infty} \omega_\nu z^{-\nu}$$

for all mappings $f(z) \in \sum\langle\kappa\rangle$, *where the numbers* Ω_ν *are fixed, then equality in*

$$\sum_{\nu=1}^{\infty} \nu\big|\omega_\nu + \tfrac{1}{2}(1-\kappa)e^{2i\theta}\overline{\Omega}_\nu\big|^2 \leq \big(\tfrac{1}{2}(1+\kappa)\big)^2 \sum_{\nu=1}^{\infty} \nu|\Omega_\nu|^2 \tag{23}$$

holds only for the functions $f^*(z) \in \sum\langle\kappa\rangle$ *which can be extended to solutions of*

$$f^*_{\bar z}(z) = \kappa e^{2i\theta} \frac{\overline{T'(z)}}{T'(z)} f^*_z(z), \qquad |z| < 1.$$

4. Considerations of multiply connected domains

It will be shown that the inequality (11) is, in general, unsharp for multiply connected domains. For the sake of simplicity, we investigate only the case $T(w) = w$. Suppose that equality (11) holds for any mapping $f^*(z) \in \sum\langle\kappa\rangle$. As before, we conclude from (8) that $\kappa e^{-i\theta} f^*(z) - e^{i\theta}\overline{f^*(z)} +$ const can be extended to an analytic function in G. This provides

$$f^*(z) = a_{11}\Phi_1(z) + \kappa a_{11} e^{2i\theta}\varphi_1(z) + \text{ const}, \qquad z \in G, \tag{24}$$

where a_{11} is a positive constant depending only on the domain G (see [16; Chapter 5, Section 2]).

Considering the example of an annulus (see Section 3), we get by using the harmonic function with the boundary values $1/(\bar z - 1)$

$$f^*(z) - \frac{1}{z-1} = \sum_{\nu=1}^{\infty} \frac{R^{4\nu}}{1-R^{4\nu}}(z^{-\nu} - z^\nu) + \kappa e^{2i\theta} \sum_{\nu=1}^{\infty} \frac{R^{2\nu}}{1-R^{4\nu}}(z^\nu - z^{-\nu}) + \text{const}. \tag{25}$$

It is easy to see that equality in (17) holds for $f^*(z)$. But the question whether $f^*(z)$ belongs to the class $\sum\langle G,\kappa\rangle$ remains still open. To find an answer for this question we shall prove an extremal property of circular domains.

Theorem 3. *If the doubly connected domain* G^* *bounded by two Jordan curves is conformally equivalent to the annulus* $\{w : R < |w| < 1/R\}$, $0 < R < 1$, *then the Fredholm eigenvalue* λ_2 *of* G *fulfills*

$$\lambda_2 \leq \frac{1}{R^2}. \tag{26}$$

Equality can occur in (26) *if, and only if,* G^* *is a circular domain.*

Proof. Denoting by $g(z)$ a homeomorphic and conformal mapping of G^* onto $\{w : R < |w| < 1/R\}$, we shall estimate the quotient of Dirichlet's integrals in (3) for the function

$$h(z) = \begin{cases} g(z) + 1/\overline{g(z)}, & \text{if } z \in G^*, \\ h_1(z) & \text{if } z \in B_1, \\ h_2(z) & \text{if } z \in B_2, \end{cases} \tag{27}$$

where B_1 is the component of G^{*C} bounded by the inverse image of $\{w : |w| = R\}$ and B_2 is the other component of G^{*C}. It follows from the definition of $g(z)$ that

$$\iint_{G^*} \left(\left| h_z(z) \right|^2 + \left| h_{\bar{z}}(z) \right|^2 \right) dx\, dy = 2\pi \left(\frac{1}{R^2} - R^2 \right). \tag{28}$$

Let $h_1(z)$ be the harmonic function in $B_1 \setminus \partial B_1$ which satisfies

$$h_1(z) = g(z) + 1/\overline{g(z)} = g(z) \cdot (1 + 1/R^2)$$

on the boundary of B_1. This implies that

$$\begin{aligned} &\iint_{B_1} \left(\left| h_{1z}(z) \right|^2 + \left| h_{1\bar{z}}(z) \right|^2 \right) dx\, dy \\ &\qquad \geq \iint_{B_1} \left(\left| h_{1z}(z) \right|^2 - \left| h_{1\bar{z}}(z) \right|^2 \right) dx\, dy \\ &\qquad = \pi \left(1 + \frac{1}{R^2} \right)^2 R^2. \end{aligned} \tag{29}$$

If $h_2(z)$ is the harmonic function in the interior of B_2 which fulfills

$$h_2(z) = g(z) + \frac{1}{\overline{g(z)}} = \left(1 + \frac{1}{R^2} \right) \cdot \frac{1}{\overline{g(z)}}$$

on the boundary of B_2, then

$$\begin{aligned} &\iint_{B_2} \left(\left| h_{2z}(z) \right|^2 + \left| h_{2\bar{z}}(z) \right|^2 \right) dx\, dy \\ &\qquad \geq \iint_{B_2} \left(\left| h_{2\bar{z}}(z) \right|^2 - \left| h_{2z}(z) \right|^2 \right) dx\, dy \\ &\qquad = \pi \left(1 + \frac{1}{R^2} \right)^2 R^2. \end{aligned} \tag{30}$$

Because of (28), (29) and (30) an application of the second inequality in (3) leads to

$$\frac{2\pi(1-R^4)/R^2}{\pi R^2(1+1/R^2)^2+\pi R^2(1+1/R^2)^2}\geq\frac{\lambda_2-1}{\lambda_2+1}.$$

This proves (26). Equality in (26) can occur if, and only if, $h_{1\bar{z}}(z)=0$ and $h_{2z}(z)=0$ holds in B_1 and B_2, respectively. Consequently, $h_1(z)$ and $\overline{h_2(z)}$ are analytic functions. Since $h_2(z)=\left(1+1/R^2\right)/\overline{g(z)}$ on the boundary of B_2, it follows from the argument principle that $h_2(z)$ has only one simple zero in B_2. Hence $g(z)$ is a linear fractional transformation. This completes the proof.

Now we consider again our example of an annulus. By choosing $\kappa=R^2$ we conclude from Theorem 3 that $\sum\langle G,\kappa\rangle$ consists only of the mappings

$$f(z)=\frac{1}{z-1}+\text{const.}$$

On the other hand, equality in (17) can occur only for the mappings $f^*(z)$ in (25). This contradiction shows that (17) provides an unsharp estimate for all κ $(1>\kappa\geq R^2)$ sufficiently close to R^2. Furthermore, the condition $\kappa\geq R^2$ is also necessary for $\sum\langle G,\kappa\rangle\neq\emptyset$ when G is a doubly connected domain having the conformal modulus $1/R^2$.

5. Remarks

1. From Theorem 1 some estimations of functionals defined on $\sum\langle G,\kappa\rangle$ may be derived. Suppose that the numbers Γ_ν, $\nu=1,2,\ldots$, in (5) do not depend on the mapping $f(z)\in\sum\langle G,\kappa\rangle$. Then the expression

$$\sum_{\nu=1}^{\infty}\Gamma_\nu\gamma_\nu \tag{31}$$

defines a functional for all mappings $f(z)\in\sum\langle G,\kappa\rangle$. Now Theorem 1 leads to

$$\left|\sum_{\nu=1}^{\infty}\Gamma_\nu\gamma_\nu\right|\leq\kappa\sum_{\nu=1}^{\infty}\left|\Gamma_\nu\right|^2; \tag{32}$$

in fact, Schwarz's inequality yields

$$\begin{aligned}\sum_{\nu=1}^{\infty}&\left|\gamma_\nu+\tfrac{1}{2}(1-\kappa)e^{2i\theta}\overline{\Gamma}_\nu\right|^2\sum_{\nu=1}^{\infty}\left|\overline{\Gamma}_\nu\right|^2\\&\geq\left|\sum_{\nu=1}^{\infty}\gamma_\nu\Gamma_\nu+\tfrac{1}{2}(1-\kappa)e^{2i\theta}\sum_{\nu=1}^{\infty}|\Gamma_\nu|^2\right|^2\\&\geq\left(\operatorname{Re}\left(e^{-2i\theta}\sum_{\nu=1}^{\infty}\gamma_\nu\Gamma_\nu+\tfrac{1}{2}(1-\kappa)\sum_{\nu=1}^{\infty}|\Gamma_\nu|^2\right)\right)^2\end{aligned}$$

and thus we get from (11)

$$\mathrm{Re}\left(e^{-2i\theta}\sum_{\nu=1}^{\infty}\gamma_\nu\Gamma_\nu\right) \le \kappa\sum_{\nu=1}^{\infty}|\Gamma_\nu|^2.$$

In general, (32) is an unsharp inequality. For simply connected domains G, however, this estimate becomes a sharp one. The inequality (32) generalizes the inequalities (4), (6) and (7) in [13] which were proved in a similar way. It should be noted that the representation (31) is valid for some known functionals, for instance for the coefficient α_1 in (1), for the Schwarzian derivative and for Golusin's functional (see [6], [7] and [8]). The estimate of the range of α_1 has the form

$$|\alpha_1 - m| \le \kappa a_{11}^2, \tag{33}$$

where m, a_{11}^2 are the centre and the radius of the circular disk which belongs to the class $\sum(G)$. Note that the extremal mapping $f^*(z)$ in Theorem 2 also fulfills

$$\Big(T\big[f^*(z)\big]\Big)_{\bar z} = \kappa e^{2i\theta}\overline{\Big(T\big[f^*(z)\big]\Big)_{z}}, \qquad |z| < 1.$$

2. In Section 3 we have considered the example of an annulus. We point out that the series on the right-hand side of (17) satisfies

$$\sum_{\nu=1}^{\infty}\frac{\nu R^{2\nu}}{1-R^{4\nu}} = \frac{1}{2\pi^2}\mathbf{K}(k)\big(\mathbf{K}(k)-\mathbf{E}(k)\big),$$

where $\mathbf{E}(k)$ and $\mathbf{K}(k)$ are the known complete elliptic integrals (see [18; p. 693, 5.1.29.3]) and k fulfills

$$R^2 = \exp\left[-\pi\frac{\mathbf{K}\big(\sqrt{1-k^2}\big)}{\mathbf{K}(k)}\right].$$

The last term is closely related to the conformal modulus of Grötzsch's domain (see [5] and [15, Chapter II, Sections 1 and 2]).

3. In this paper the derived inequalities do not contain coefficients of some unknown (extremal) mappings unlike the area theorems in [6] and [7]. On the other hand, the area theorems in [6] and [7] are also sharp for multiply connected domains G. This illustrates the shortcoming of the area theorem in Section 2. A common property of these area theorems in [6], [7] and in this paper is, however, that the area theorem for mappings $f(z) \in \sum(G)$ (see [16; Theorem 5.1]) can be interpreted as a special case of them.

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