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ON THE TYPE OF SEWING FUNCTIONS WITH A SINGULARITY

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When a Riemann surface is defined by identifying boundary arcs of plane regions, the process is called a conformal sewing. A permissible identifying homeomorphism can be termed a sewing function, or a function with the sewing property (the terms are from [5]). In the case of sewing along real intervals, the question arises: when does a function with the sewing property on two adjacent intervals possess the global sewing property. This is a removability problem, with the common endpoint taking the role of a singularity.

Our assumptions are as follows: Suppose that φ is an increasing homeomorphism between two bounded open intervals; let the situation be normalized by the condition $\varphi(0)=0$. Suppose further that the function φ is locally quasisymmetric off the point zero. The latter assumption guarantees that 1) the restrictions of φ to both sides of zero admit sewing (of the lower and upper half-planes), and 2) the conformal structure of the resulting (doubly connected) Riemann surface R is essentially unique.

The surface R has a parabolic or hyperbolic end at zero. Accordingly, the function φ is called parabolic or hyperbolic. The former case occurs exactly when φ is a global sewing function. The type is determined by the values of φ in an arbitrarily small neighborhood of zero. In this paper, we will study the type of φ mainly in the special cases where φ is locally bilipschitz or analytic off the point zero, by applying and modifying some results of [5].

Let us first cite a few earlier results. There exist hyperbolic functions φ which, for $x\to 0$, approach an arbitrary increasing zero-preserving homeomorphism arbitrarily rapidly ([1]), as well as symmetric ($\varphi(x)\equiv-\varphi(-x)$) hyperbolic functions φ with analyticity and positive derivative off the point zero (Example 2.9 of [5]). A sufficient parabolicity condition is

$$(1)\qquad \int_0 \left(x+\frac{\varphi(x)}{\varphi'(x)}+\frac{|\varphi(-x)|}{\varphi'(-x)}\right)^{-1}\left(1+\ln^2\frac{|\varphi(-x)|}{\varphi(x)}\right)^{-1}dx=\infty,$$

which was obtained in [5] (p. 13) as a re-formulation of a condition of [4]. The function φ defined in a neighborhood of 0 by the expressions x, $x\exp\bigl(-|\ln x|^p\bigr)$

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for $x \leq 0$, $x > 0$, respectively, is parabolic for $p \leq 1/2$, fulfilling (1); for $p > 1/2$ the function is hyperbolic ([5] p. 18). However, there also exist parabolic functions φ "arbitrarily asymmetric", that is, vanishing arbitrarily rapidly for $x \to 0_+$, arbitrarily slowly for $x \to 0_-$ (Example 2.10 of [5]; the situation implies the local quasisymmetry).

The latter remark implies that there exist parabolic functions φ satisfying, for instance, $\varphi(x) < x$ for $x < 0$, $\varphi(x) < x^2$ for $x > 0$. However, such functions neither satisfy the condition (1) (the asymmetry being too great) nor, it seems, the other parabolicity conditions of [5] (Theorems 2.1, 3.8). In order to obtain an explicit parabolic example where the asymmetry of the three values $\varphi(x)$, 0, $\varphi(-x)$ is, at least on separate intervals, essentially greater than in the above example in the case $p = 1/2$, we will introduce the following function.

Example. Let $f\colon]0, a[\to \mathbf{R}$ be a function satisfying $0 < f(x) < x$, and define a function φ as follows. Let φ coincide with the identity for $x \leq 0$. Then let us take positive points a_n, b_n, c_n, $n \in \mathbf{N}$, with $a_n > b_n > c_n$, $a_{n+1} < f(c_n) < c_n$, converging to 0, and such that the series $\sum \ln(a_n/b_n)$ diverges. Set $\varphi(x) = x$ for $x \in [b_n, a_n]$, $\varphi(c_n) = f(c_n)$, and let φ be linear elsewhere. As the convergence $f(x) \to 0$ can be arbitrarily rapid, the definition permits a large oscillation between symmetry and asymmetry. The function φ clearly satisfies the condition (1), and it is thus parabolic.

As a further application of the same criterion, we can easily prove a proposition which is of course a classical result if $\varphi'(0) \neq 0$.

Theorem 1. *An analytic φ is parabolic.*

Proof. For a function φ analytic at zero, the Taylor development gives

$$\varphi(x) = ax^p\big(1 + \psi_1(x)\big), \qquad \varphi'(x) = apx^{p-1}\big(1 + \psi_2(x)\big),$$

with $a > 0$, $p \geq 1$, $\psi_1(0) = \psi_2(0) = 0$, where p is the order (odd) of the zero of φ. Clearly φ satisfies (1). □

Our next theorem is a slightly generalized version of Theorem 2.8 of [5] (by which the above-mentioned hyperbolicity for $p > 1/2$ was established). The convexity assumptions have been weakened to the bilipschitz property, which made it necessary to add the separate condition (needed in the proof) for the derivative φ_1'; in the condition for the derivative φ_2', the equality sign is allowed. As the purely analytic proof remains essentially the same, we will only state the result.

Theorem 2. *If φ is locally bilipschitz off zero, then it is hyperbolic if the functions φ_1, φ_2 defined by the formulae*

$$\varphi_1(x) = -\varphi^{-1}(-x), \qquad \varphi_2(x) = \varphi\big(\varphi_1(x)\big)$$

satisfy the following three conditions (the latter two a.e.) in an interval $]0,a]$:

$$\varphi_2(x) < x,$$

$$\varphi_1'(x) \geq c\frac{\varphi_1(x)}{x}, \qquad c > 0,$$

$$\varphi_2'(x) \geq \frac{\varphi_2(x)}{x}\left(\frac{\ln \varphi_2(x)}{\ln x}\right)^k, \qquad k > 1/2.$$

A function with $\varphi(x) = x^p$ for $x \geq 0$, $\varphi(x) = -|x|^q$ for $x < 0$ (p, $q \in \mathbf{R}_+$) is hyperbolic for $p \neq q$ ([5], originally [4]). For $p = q$, the function is parabolic, by (1). A similar correspondence holds even in a more general case, shown below by means of Theorem 2 and (1).

Theorem 3. *If φ is locally bilipschitz off zero, and there exist finite positive one-sided limits*

$$\lim_{x\to 0_+} \frac{\varphi'(x)}{x^r}, \qquad \lim_{x\to 0_-} \frac{\varphi'(x)}{|x|^s}$$

for some r, $s > -1$, then φ is parabolic for $r = s$, hyperbolic for $r \neq s$.

Corollary. *If the restrictions of φ to non-negative and non-positive values are analytic at 0, then φ is parabolic if and only if the zeros at 0 are of the same order.*

Proof of Theorem 3. We use the notation $f(x) \sim g(x)$, meaning that $f(x)/g(x) \to 1$ for $x \to 0$. The assumptions on the limits in the theorem can be written in the form

$$\varphi'(x) \sim ax^r \quad \text{for } x > 0, \qquad \varphi'(x) \sim b|x|^s \quad \text{for } x < 0,$$

with a, $b > 0$. In our situation, the integral of $\varphi'(x)$ from 0 to x equals $\varphi(x)$. It follows that

$$\varphi(x) \sim cx^p \quad \text{for } x > 0, \qquad \varphi(x) \sim -d|x|^q \quad \text{for } x < 0,$$

$$p = r + 1 > 0, \qquad c = a/p > 0, \qquad q = s + 1 > 0, \qquad d = b/q > 0.$$

For $r = s$, the above formulae for φ and φ' imply that the parabolicity condition (1) is fulfilled. Suppose $r > s$, implying $p > q$. (If $r < s$, consider the function $x \mapsto -\varphi(-x)$). The functions of Theorem 2 now satisfy for $x > 0$

$$\varphi_1(x) \sim \left(\frac{x}{d}\right)^{1/q}, \qquad \varphi_2(x) \sim c\left(\frac{x}{d}\right)^{p/q} = Cx^{p/q},$$

$$\varphi_1'(x) = 1/\varphi'\left(\varphi^{-1}(-x)\right) = 1/\varphi'\left(-\varphi_1(x)\right),$$

$$\varphi_1(x)/\varphi_1'(x) \sim \left(\frac{x}{d}\right)^{1/q} b \left(\frac{x}{d}\right)^{s/q} = qx,$$

$$\varphi_2'(x) = \varphi'(\varphi_1(x))\,\varphi_1'(x) \sim a \left(\frac{x}{d}\right)^{r/q} b^{-1} \left(\frac{x}{d}\right)^{-s/q} = \frac{pc}{qx}\left(\frac{x}{d}\right)^{p/q},$$

$$\varphi_2(x)\bigl|\ln \varphi_2(x)\bigr|^k \sim c\left(\frac{x}{d}\right)^{p/q}\left(\frac{p}{q}\right)^k |\ln x|^k,$$

which shows that the assumptions of the theorem are fulfilled (with $k < 1$). Hence φ is hyperbolic. □

The rather complicated Theorem 3.10 of [5] gives a condition necessary for the parabolicity of φ; by converting the condition, one can obtain a hyperbolicity criterion. Our bilipschitz assumption allows us to simplify the original expression of the auxiliary function ξ.

Theorem 4. *If φ is locally bilipschitz off zero, then it is hyperbolic if there exist functions τ, η such that the condition*

$$\int_0 \frac{\xi(x)}{x^2|\ln x|^{2k}}\,dx < \infty$$

is fulfilled, where the function ξ is defined by the formulae

$$\xi(x) = \psi\bigl(\eta^{-1}(x)\bigr) + \bigl|\psi(-x)\bigr| + \frac{\tau(x)}{\tau'(x)} + \frac{\eta(x)}{\psi'(x)} + \frac{x}{\psi'(-x)},$$

$$\psi = \varphi \circ \tau,$$

and the functions τ, η are zero-preserving increasing homeomorphisms of intervals, locally bilipschitz off zero, satisfying for $x > 0$ the conditions

$$\tau(x) = -\tau(-x), \qquad \eta(x) < x,$$

$$\eta'(x) \ge \frac{\eta(x)}{x}\left(\frac{\ln \eta(x)}{\ln x}\right)^k, \qquad k > \tfrac{1}{2}.$$

Proof. (We will merely outline the proof, since the original one applies to the present situation, with the exception of a few details.)

Let φ be locally bilipschitz off zero, and τ, η as in the latter part of the theorem. As in [5], the real functions τ, $\psi = \varphi \circ \tau$ are extended into domains in the lower and upper half-planes, respectively, denoting the locally quasiconformal extensions by w_1, w_2. The original definition $w_1(\varrho e^{i\theta}) = \tau(\varrho)e^{i\theta}$ is retained. In the general case of [5], the expression of w_2 contains an integral, whereas our case, with φ, ψ bilipschitz, allows us to define a locally quasiconformal map w_2 as follows: set $w_2(x+iy) = u(x,y) + iy$, where the function u has the value $\psi(x)$

for both $x > 0$, $y < \eta(x)$ and $x < 0$, $y < |x|$, and is linear in x for each y in the region $x \in [-y, \eta^{-1}(y)]$.

Assume that φ is parabolic, i.e., a sewing function. There then exists a homeomorphism f of a neighborhood of the origin, whose complex dilatation coincides with that of w_1 for $y < 0$, with that of w_2 for $y > 0$; the map f is now locally quasiconformal off 0. Using the function η, a homeomorphism h is defined as in [5], to be locally quasiconformal off 0 in a neighborhood of 0. Also the map $f \circ h$ is then locally quasiconformal off 0. Its complex dilatation therefore satisfies an asymptotic integral condition, derived in [2] and applied in [5]. The new definition of the function u simplifies some expressions. At the very end of the proof, use is made of the condition for the derivative η'; the equality sign can be added (even in the general case of [5]). The integral condition turns out to be equivalent to the divergence of the integral in our theorem. As the parabolicity assumption thus leads to a contradiction with the convergence assumption, the assertion thus follows. □

Theorem 2 is a corollary of Theorem 4: the assumptions of the former imply those of the latter, with $\tau = \varphi_1$, $\eta = \varphi_2$ for $x > 0$ (one obtains $\xi(x) = O(x)$). Let us state a further special result:

Corollary. *If φ is locally bilipschitz off zero, and the functions φ_1, φ_2 of Theorem 2 satisfy for a value $p > 1$ the condition*

$$\int_0 x^{-2}|\ln x|^{-2}\left(x + \frac{\varphi_1(x)}{\varphi_1'(x)} + \varphi_2(x^{1/p}) + \frac{x^p}{\varphi_2'(x)}\right) dx < \infty,$$

then φ is hyperbolic.

Namely, the condition implies that the assumptions of Theorem 4 are fulfilled, with $\tau = \varphi_1$, $\eta = x^p$ for $x > 0$ ($k = 1$).

As a conclusion of this paper, we will present a parabolic example function which is very asymmetric in spite of being analytic off zero. The idea is similar to that of Example 2.10 of [5]. Our example suggests that there exist "arbitrarily asymmetric" parabolic functions which are analytic or locally bilipschitz off zero; for the classes of functions with either property, there would then exist no hyperbolicity criterion regarding only the amount of the asymmetry.

Example. *There exists a parabolic φ which is analytic off zero, with positive derivative, and satisfies the conditions*

$$\varphi(x) > |\ln x|^{-1} \quad \text{for } x > 0, \qquad |\varphi(x)| < e^{1/x} \quad \text{for } x < 0.$$

We define a sewing function φ as follows. Let D_1, D_2 be the lower and upper half-planes, G_1, G_2 the domains below and above the Jordan curve $C = C_+ \cup C_-$, where the arc C_+ is defined by

$$y = f(x)(1 + \sin 1/x),$$

$$f(x) = \left(\ln|\ln x|\right)^{-p}, \qquad p \in]0, \tfrac{1}{2}[$$

for $x \in [0, x_0]$, and continued to ∞ along the positive real axis from a zero $x_0 < 1/e$ of the function $1 + \sin 1/x$, and the arc C_- is obtained by reflecting C_+ with respect to the origin. Let $f_i\colon D_i \to G_i$ $(i = 1, 2)$ be two conformal maps, with $f_i(0) = 0$, $f_i(\infty) = \infty$, $f_1(a) = f_2(b) = c \in C_+$, $a, b > 0$, $\operatorname{Re} c \le x_0$. The boundary function $\varphi = f_2^{-1} \circ f_1$ sews D_1 to D_2. It is analytic, with $\varphi'(x) \ne 0$, in a neighborhood of 0, for $x \ne 0$; this is implied by the local analyticity of C.

Denoting the module of a quadrilateral by M, we have, for $0 < x < a$,

$$\begin{aligned} M\left(G_1(c, f_1(x), 0, \infty)\right) &= M\left(D_1(a, x, 0, \infty)\right) = M\left(D_2(-a, -x, 0, \infty)\right) \\ &= \frac{2}{\pi}\mu\left(\sqrt{\frac{x}{a}}\right) < \frac{2}{\pi}\ln\frac{4}{\sqrt{x/a}} < |\ln x| \end{aligned}$$

for x small (cf. [3], p. 61). On the other hand (by [3], p. 23),

$$M\left(G_1(c, f_1(x), 0, \infty)\right) = 1/M\left(G_1(f_1(x), 0, \infty, c)\right) \ge 1/F(s_1/s_2),$$

where

$$F(s_1/s_2) = \pi\left(1 + 2\ln(1 + 2s_1/s_2)\right)\left(\ln(1 + 2s_1/s_2)\right)^{-2},$$

and s_1, s_2 are the distances of the opposite sides. For the quadrilateral $G_1(f_1(x), 0, \infty, c)$, we have

$$s_1 > d_1, \qquad s_2 < k \operatorname{Re} f_1(x)$$

for x small, where d_1, k are positive constants. For x small, the ratio s_1/s_2 is large, and

$$F(s_1/s_2) < K/\ln s_1/s_2$$

(K a constant). We thus obtain

$$|\ln x| > 1/F(s_1/s_2) > K^{-1}\ln s_1/s_2,$$

implying

$$x^{-K} > s_1/s_2 > d_1/\left(k \operatorname{Re} f_1(x)\right),$$

that is,

$$\operatorname{Re} f_1(x) > (d_1/k)x^K.$$

For $G_2\left(\infty, 0, f_1(x), c\right)$, we now have

$$s_1 > f\left((1-\varepsilon)\operatorname{Re} f_1(x)\right) > (1-\varepsilon_1)f(x),$$

where ε, ε_1 are small for x small, and further

$$s_2 < d_2, \qquad s_1/s_2 > K_1 f(x),$$

$$M\left(G_2(\infty,0,f_1(x),c)\right) \le F(s_1/s_2) < F\big(K_1 f(x)\big) < K_2 f(x)^{-2}$$

for x small. On the other hand,

$$M\left(G_2(\infty,0,f_1(x),c)\right) = M\left(D_2(\infty,0,\varphi(x),b)\right) = 1/M\left(D_2(0,\varphi(x),b,\infty)\right),$$

$$\begin{aligned} M\left(D_2(0,\varphi(x),b,\infty)\right) &= \frac{2}{\pi}\mu\left(\sqrt{\frac{b-\varphi(x)}{b}}\right) = \frac{2}{\pi}\mu\left(\sqrt{1-\sqrt{\varphi(x)/b}^{\,2}}\right) \\ &= \frac{2}{\pi}\frac{\pi^2}{4}\mu\left(\sqrt{\varphi(x)/b}\right)^{-1} < K_3\big|\ln\varphi(x)\big|^{-1} \end{aligned}$$

(by [3], p. 61). We thus obtain

$$\big|\ln\varphi(x)\big| < K_4 f(x)^{-2} < \ln|\ln x|,$$

implying the required inequality $\varphi(x) > |\ln x|^{-1}$ in a neighborhood of 0, for $x > 0$.

The inequality required for $x < 0$ is obtained by similar arguments applied to φ^{-1}, starting with the domains D_2, G_2.

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