Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 14, 1989, 345–350

SOME CONDITIONS FOR QUASICONFORMALITY

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It is known that if D, D' are domains in $\overline{R}^n$ and $f\colon D \to D'$ is a homeomorphism such that the linear dilatation $H(x,f) < H$ for every $x \in D$, then f is quasiconformal. The linear dilatation $H(x,f)$ is defined in a point $x \in D$ such that $x \neq \infty$ and $f(x) \neq \infty$ by $\limsup_{r\to 0}\bigl(L(x,f,r)/(l(x,f,r)\bigr)$, where $L(x,f,r) = \max_{|y-x|=r}\bigl|f(y)-f(x)\bigr|$, $l(x,f,r) = \min_{|y-x|=r}\bigl|f(y)-f(x)\bigr|$. For $0 < \alpha < 1$ we define $L_\alpha(x,f,r) = \max_{|y-x|=\alpha r}\bigl|f(y)-f(x)\bigr|$ and $H_\alpha(x,f) = \limsup_{r\to 0}\bigl(L_\alpha(x,f,r)/(l(x,f,r)\bigr)$, and it is obvious that $H_\alpha(x,f) \le H(x,f)$. We shall show that if there exist $0<\alpha<1$ and $H>0$ such that $H_\alpha(x,f) < H$ for every $x \in D$, then f is K-quasiconformal, with $K(\alpha,H,n) = H^{n-1}\alpha^{1-n}$. We shall follow the classical proof of the fact that the metric definition of quasiconformality implies the analytic definition of the quasiconformality, and we shall give a detailed proof for the sake of completeness. We shall use this result to prove that if $f\colon D \to D'$ is a homeomorphism for which there exist $\varepsilon > 0$ and $\delta > 0$ such that $M(\Gamma'_A) < \delta$ for every ring A in D with $M(\Gamma_A) \le \varepsilon$, then f is K-quasiconformal with $K(\varepsilon,\delta,n) = \bigl(\exp(\omega_{n-1}\varepsilon^{-1})^{1/(n-1)}/g(\delta)\bigr)^{n-1}$, where g is a determined function. Therefore f is quasiconformal if and only if f carries rings of small modulus into rings whose modulus is dominated by a constant δ. P. Caraman [1], generalizing a result of Renggli, proved that if $f\colon D \to D'$ is a homeomorphism with f carrying path families Γ from D such that $M(\Gamma) = \infty$ into path families Γ' such that $M(\Gamma') = \infty$, then f is quasiconformal. We shall prove that if $f\colon D \to D'$ is a homeomorphism such that there exists a constant $\delta > 0$ such that $M(\Gamma') > \delta$ for every path family Γ from D with $M(\Gamma) = \infty$, then f is quasiconformal. These results (Theorem 2, Theorem 3 and Theorem 4 from our paper) can easily be deduced from Lemma 1 [1], but we give an alternative proof which permits us to evaluate the quasiconformality constant K, too.

I wish to thank the referee for his careful reading of the manuscript and his advice. The final form of the present paper was substantially improved by his suggestions.

Throughout the paper, D, D' will be domains in $\overline{R}^n$ and $f\colon D \to D'$ a homeomorphism. We shall use the notations and the results from [2]. We denote by $A(r)$ the set of all rings $A = R(C_0,C_1)$ in $\overline{R}^n$ with the following properties: 1) C_0 contains the origin and a point a such that $|a| = 1$. 2) C_1 contains ∞ and a point b such that $|b| = r$. We denote $h(r) = \inf M(\Gamma_A)$ over all rings $A \in A(r)$. Then $h\colon (0,\infty) \to (0,\infty)$ is a decreasing function such that $\lim_{r\to 0} h(r) = \infty$ and

doi:10.5186/aasfm.1989.1405

$\lim_{r\to\infty} h(r) = 0$ (see [2, Theorem 11.7, p. 34]), and we define g: $(0,\infty) \to (0,\infty)$, $g(x) = h^{-1}(x)$ for every $x \in (0,\infty)$. Also, if $A = R(C_0, C_1)$ is a ring such that $c, \infty \in C_1$, $a, b \in C_0$, then $M(\Gamma_A) \geq h\big((|c-a|)/(|b-a|)\big)$ (see [2, Theorem 11.9, p. 36]). For every set $X \subset R^n$, we denote by $m_p(X)$ the p-dimensional Hausdorff measure of X. We denote by $V_n = m_n(B^n)$ and by $\omega_{n-1} = m_{n-1}(S^{n-1})$, where B^n is the unit ball from R^n and S^{n-1} is the boundary of B^n. We also set $L(x,f) = \limsup_{h\to 0}\big|f(x+h) - f(x)\big|/|h|$.

Lemma 1. *Let f: $D \to D'$, $0 < \alpha < 1$ and $h > 0$ be such that $H_\alpha(x,f) < H$ for every $x \in D$. Then f is ACL.*

Proof. Let $Q = \{x \in R^n \,|\, a_i \leq x \leq b_i\}$ be a cube in $D \setminus \{\infty, f^{-1}(\infty)\}$ and consider the projection P: $R^n \to R^{n-1} = R_n^{n-1}$. For each Borel set $A \subset \operatorname{int} PQ$, we set $E_A = Q \cap P^{-1}(A)$. Setting $\varphi(A) = m\big(f(E_A)\big)$ we obtain a set function φ in $\operatorname{int} PQ$. By the Lebesgue theorem, φ has a finite derivative $\varphi'(y)$ for almost every $y \in \operatorname{int} PQ$. We shall prove that f is absolutely continuous on the segment $J = E_y$. Let F be a compact subset of $J \cap \operatorname{int} Q$ and let $F_k = \{x \in F \,|\, L_\alpha(x,f,r) \leq Hl(x,f,r)$ for every $0 < r < 1/k\}$; for $k \in N$. Since f is continuous, each F_k is a compact set, $F_k \subset F_{k+1}$ and $F = \bigcup_{k=1}^{\infty} F_k$. We now choose $k \in N$ such that $1 < kd(F, FrQ)$, we fix k and we choose $\varepsilon > 0$ and $t > 0$. By [2, Lemma 31.1, p. 106], there exists $\delta > 0$ such that for every $r \in (0, \delta/\alpha)$ there exists a finite covering of F_k with open intervals Δ_1, Δ_2, $\ldots$, Δ_p such that $m(\Delta_i) = 2\alpha r$, the centre of each Δ_i belongs to F_k, each point of J belongs to at most two Δ_i, $i = 1, 2, \ldots, p$ and $p\alpha r < m(F_k) + \varepsilon$. We now choose $r < \min(\delta/\alpha, 1/k)$ such that if $|x - z| \leq 2\alpha r$, $x, z \in Q$, it follows that $\big|f(x) - f(z)\big| < t$. Let A_i be the open n-balls whose diameter is Δ_i, x_i be the center of Δ_i and B_i the open n-balls centered at x_i and of radius r, $i = 1, 2, \ldots, p$. Then $B_i \subset E_B$, where $B = B^{n-1}(y, r)$, and we denote $L_i = L_\alpha(x_i, f, r)$ and $l_i = l(x_i, f, r)$, $i = 1, 2, \ldots, p$. Since $x_i \in F_k$, we have $L_i \leq Hl_i$, $i = 1, 2, \ldots, p$. Then $F_k \subset \bigcup_{i=1}^{p} A_i$, $d\big(f(A_i)\big) < t$, $i = 1, 2, \ldots, p$; hence

$$m_1^t\big(f(F_k)\big) \leq \sum_{i=1}^{p} d\big(f(A_i)\big) \leq 2\bigg(\sum_{i=1}^{p} L_i\bigg).$$

By Hölder's inequality we obtain

$$m_1^t\big(f(F_k)\big)^n \leq 2^n\bigg(\sum_{i=1}^{p} L_i\bigg)^n \leq 2^n\bigg(\sum_{i=1}^{p} 1\bigg)^{n-1}\bigg(\sum_{i=1}^{p} L_i^n\bigg)$$

$$= 2^n p^{n-1} \sum_{i=1}^{p} L_i^n \leq 2^n H^n p^{n-1} \sum_{i=1}^{p} l_i^n$$

$$\leq 2^n H^n \alpha^{1-n} r^{1-n} \big(m_1(F_k) + \varepsilon\big)^{n-1} \sum_{i=1}^{p} l_i^n$$

$$\leq 2^n H^n \alpha^{1-n} r^{1-n} \big(m_1(F_k) + \varepsilon\big)^{n-1} V_n^{-1} \sum_{i=1}^{p} m\big(f(B_i)\big).$$

Let $q = [1/\alpha] + 1$. Then $q\alpha > 1$, and every point from E_B belongs to at most $6q$ balls B_i. Indeed, let $\bar{y} \in E_B$ be such that $\bar{y} \in B(x_j, r)$. If $B(x_i, r) \cap B(x_j, r) \neq \emptyset$, then $|x_i - x_j| < 2r$; hence $\bar{y}$ may belong only to a ball $B(x_i, r)$ such that $|x_i - x_j| < 2r$. Suppose that we have more than $6q$ points x_i in $J \cap B(x_j, 2r)$. Let $I = \{i \in Z \mid x_i \in J \cap B(x_j, 2r)\}$. Then $\sum_{i \in I} m(\Delta_i) \geq 12q\alpha r > 12r$. Since $m\big(\bigcup_{i \in I} \Delta_i\big) < 6r$, it follows that there exists a point $x \in J$ such that x belongs to at least 3 sets Δ_i, which represents a contradiction. Then $\bar{y}$ may belong to at most $6q$ balls $B(x_i, r)$; hence every point of E_B may belong to at most $6q$ balls B_i. Since every point of $f(E_B)$ belongs to at most $6qf(B_i)$, we have

$$\sum_{i=1}^{p} m\big(f(B_i)\big) \leq 6qm\big(f(E_B)\big) = 6q\varphi(B).$$

Since $F_k \subset F$, we have

$$m_1^t\big(f(F_k)\big)^n \leq \Big(2^{n+3} H^n V_{n-1} q \big(m_1(F) + \varepsilon\big)^{n-1} \varphi(B)\Big) \big(V_n \alpha^{n-1} m(B)\big)^{-1}.$$

Letting first $r \to 0$, then $\varepsilon \to 0$ and then $t \to 0$, we obtain $m_1\big(f(F_k)\big)^n \leq C\varphi'(y) m_1(F)^{n-1}$, where $C = 2^{n+3} H^n V_{n-1} q \alpha^{1-n} V_n^{-1}$. Since $f(F_k) \nearrow f(F)$, we obtain $m_1\big(f(F)\big)^n \leq C\varphi'(y) m_1(F)^{n-1}$, and from [2, Theorem 30.9, p. 101], we see that f is absolutely continuous on J.

Lemma 2. *Let $f\colon D \to D'$, $0 < \alpha < 1$, $H > 0$, such that $H_\alpha(x, f) < H$ for every $x \in D$. Then f is a.e. differentiable.*

Proof. We have $\mu_f'(x) < \infty$ a.e. (see [2, Theorem 24.2, p. 83]) and, by the theorem of Rademacher and Stepanov, it suffices to show that $L(x, f) < \infty$ a.e. Let $x_0 \in D$ be such that $x_0 \neq \infty$, $f(x_0) \neq \infty$ and $\mu_f'(x_0) < \infty$. Then there exists $r_0 > 0$ such that $L_\alpha(x_0, f, r) < Hl(x_0, f, r)$ for $r \leq r_0$. For such r we have

$$V_n L_\alpha(x_0, f, r)^n \leq V_n H^n l(x_0, f, r)^n \leq H^n m\big(f(B(x_0, r))\big).$$

Hence we obtain the following inequality:

$$\frac{L(x_0, f, \alpha \cdot r)^n}{(\alpha \cdot r)^n} \leq \frac{H^n m\big(f(B(x_0, r))\big)}{\alpha^n m\big(B(x_0, r)\big)}.$$

Letting $r \to 0$, we see that $L(x_0, f)^n \leq H^n \alpha^{-n} \mu_f'(x_0)$.

Theorem 1. *Let $f\colon D \to D'$, $0 < \alpha < 1$, $H > 0$, such that $H_\alpha(x,f) < H$ for every $x \in D$. Then f is K-quasiconformal, with $K(\alpha, H, n) = H^{n-1}\alpha^{1-n}$.*

Proof. From Lemma 1 and Lemma 2, f is ACL and a.e. differentiable. Let $x \in D$ be such that f is differentiable at x and $|f'(x)| = \sup_{|h|=1}|f'(x)(h)|$, $m_x = \inf_{|h|=1}|f'(x)(h)|$, and let $\varepsilon > 0$ be fixed. Then there exists $\delta_\varepsilon > 0$ such that for $|h| < \delta_\varepsilon$;

$$|f(x+h) - f(x) - f'(x)(h)| \le \varepsilon |h|,$$

hence

$$|f'(x)(h)| - \varepsilon|h| \le |f(x+h) - f(x)| \le |f'(x)(h)| + \varepsilon|h|$$

for $|h| < \delta_\varepsilon$. Let $r = |h|$. Then for $r \le \delta_\varepsilon$, $(|f'(x)| - \varepsilon)\alpha r \le L_\alpha(x,f,r)$ and $l(x,f,r) \le (m_x + \varepsilon)r$. We have

$$\alpha \frac{|f'(x)| - \varepsilon}{m_x + \varepsilon} \le \frac{L_\alpha(x,f,r)}{l(x,f,r)}$$

for $r \le \delta_\alpha$. We see that if $m_x = 0$ and $|f'(x)| \ne 0$, then $H_\alpha(x,f)$ must be ∞; hence either $f'(x) = 0$ or we have $0 < \alpha|f'(x)|/m_x \le H_\alpha(x,f) < H$. For $m_x > 0$ we have

$$\frac{|f'(x)|^n}{|J_f(x)|} \le \frac{|f'(x)|^{n-1}}{m_x^{n-1}}$$

and

$$\frac{|J_f(x)|}{m_x^n} \le \frac{|f'(x)|^{n-1}}{m_x^{n-1}}$$

(see [2, (14.3), p. 44]). Hence if f is differentiable at x, we obtain $|f'(x)|^n \le H^{n-1}\alpha^{1-n}|J_f(x)|$ and $|J_f(x)| \le H^{n-1}\alpha^{1-n}m_x^n$. We now apply Theorem 34.6 from [2, p. 115], to conclude that f is K-quasiconformal, with $K(\alpha, H, n) = H^{n-1}\alpha^{1-n}$.

Theorem 2. *Let $f\colon D \to D'$ be such that one of the following conditions is satisfied:*

a) *There exist $\varepsilon > 0$ and $\delta > 0$ such that for every ring A in D such that $M(\Gamma_A) \le \varepsilon$ it follows that $M(\Gamma'_A) < \delta$.*

b) *There exist $\varepsilon > 0$ and $\delta > 0$ such that for every ring A in D such that $M(\Gamma'_A) \le \varepsilon$, it follows that $M(\Gamma_A) < \delta$.*

Then f is K-quasiconformal, with

$$K(\varepsilon, \delta, n) = \left(\exp\left(\omega_{n-1} \cdot \varepsilon^{-1}\right)^{1/(n-1)}\right)/g(\delta)\Big)^{n-1}.$$

Proof. Suppose that Condition b) holds. Let $\alpha \in (0,1)$ be such that $h(\alpha) = \delta$, $\lambda = \exp\bigl((\omega_{n-1}\varepsilon^{-1})^{1/(n-1)}\bigr)$, and suppose that there exists a point $x \in D \setminus \{\infty \cup f^{-1}(\infty)\}$ such that $H_\alpha(x,f) \geq \lambda$. Then we can find $r > 0$ such that $L_\alpha(x,f,r) \geq \lambda l(x,f,r)$ and $\overline{B}\bigl(f(x), L_\alpha(x,f,r)\bigr) \subset D'$. We abbreviate $L_\alpha(x,f,r)$ to L_α and $l(x,f,r)$ to l and let $B = f^{-1}\bigl(B(f(x),l)\bigr)$, $C = Cf^{-1}\bigl(B(f(x),L_\alpha)\bigr)$, $A = R(B,C)$. Then $M(\Gamma_A) \geq h(\alpha) = \delta$ and $M(\Gamma'_A) = \omega_{n-1}\bigl(\log(L_\alpha l^{-1})\bigr)^{1-n}$. Since $L_\alpha/l \geq \exp\bigl((\omega_{n-1}\varepsilon^{-1})^{1/(n-1)}\bigr)$ if and only if $\omega_{n-1}\bigl(\log(L_\alpha l^{-1})\bigr)^{1-n} \leq \varepsilon$, we obtain $M(\Gamma'_A) \leq \varepsilon$ and $M(\Gamma_A) \geq \delta$, which represents a contradiction. We proved that $H_\alpha(x,f) \leq \lambda$ for every $x \in D$ and that hence, by Theorem 1, f is K-quasiconformal, with $K(\varepsilon,\delta,n) = \bigl(\exp\bigl((\omega_{n-1}\varepsilon^{-1}\bigr)^{1/(n-1)}/g(\delta))\bigr)^{n-1}$. If Condition a) holds, we prove the same thing for f^{-1}, and by [2, Corollary 13.3, p. 42], the theorem is proved.

We remark that we have also proved the following theorem:

Theorem 3. *Let $f\colon D \to D'$ be such that one of the following conditions is satisfied:*

a) *There exist $\varepsilon > 0$ and $\delta > 0$ such that for every ring A in D such that $M(\Gamma_A) \geq \delta$ it follows that $M(\Gamma'_A) > \varepsilon$.*

b) *There exist $\varepsilon > 0$ and $\delta > 0$ such that for every ring A in D such that $M(\Gamma'_A) \geq \delta$, it follows that $M(\Gamma_A) > \varepsilon$.*

Then f is K-quasiconformal, with

$$K(\varepsilon,\delta,n) = \left(\exp\bigl(\omega_{n-1}\cdot\varepsilon^{-1})^{1/(n-1)}\bigr)/g(\delta)\right)^{n-1}.$$

Theorem 4. *Let $f\colon D \to D'$ be such that there exists $\delta > 0$ such that for every path family Γ from D such that $M(\Gamma) = \infty$, it results that $M(\Gamma') > \delta$. Then f is quasiconformal.*

Proof. Suppose that $H(\cdot,f)$ is an unbounded function on D. Then there exists $x_m \in D$ such that $x_m \neq \infty$, $f(x_m) \neq \infty$, $H(x_m,f) > \exp(2^m)$. Let $y_m = f(x_m)$ for every $m \in N$. Then we can find $r_m > 0$ such that $L(x_m,f,r_m) > \exp(2^m) l(x_m,f,r_m)$ and $\overline{B}(y_m,f,r_m) \subset D'$ for every $m \in N$. We abbreviate $L(x_m,f,r_m)$ to L_m and $l(x_m,f,r_m)$ to l_m and let $B_m = f^{-1}\bigl(B(y_m,l_m)\bigr)$, $C_m = Cf^{-1}\bigl(B(y_m,L_m)\bigr)$, $A_m = R(B_m,C_m)$ for every $m \in N$. We remark that we can choose the rings A_m such that the path families Γ_{A_m} are separate. Then $M(\Gamma_{A_m}) \geq h(1)$ and

$$M(\Gamma'_{A_m}) = \omega_{n-1}\left(\log\Bigl(\frac{L_m}{l_m}\Bigr)\right)^{n-1} \leq \omega_{n-1}\log\bigl(\exp(2^m)\bigr)^{1-n} \leq \omega_{n-1}2^{-m}$$

for every $m \in N$. Let $p \in N$ be such that $\omega_{n-1}\sum_{m\geq p} 2^{-m} < \delta$. If we take $\Gamma = \bigcup_{m\geq p}\Gamma_{A_m}$ and $\Gamma' = \bigcup_{m\geq p}\Gamma'_{A_m}$, we see that $M(\Gamma) = \sum_{m\geq p} M(\Gamma_{A_m}) = \infty$ and $M(\Gamma') = \sum_{m\geq p} M(\Gamma'_{A_m}) < \delta$, which represents a contradiction.

Remark 1. The referee pointed out to me that we can replace Condition a) of Theorem 2 with a weaker condition, i.e., "There exist $\varepsilon > 0$ and $\delta > 0$ such that $M(\Gamma'_A) < \delta$ for every ring A in D such that $M(\Gamma_A) = \varepsilon$". This condition will imply Condition a). Indeed, assume that A is a ring in D such that $M(\Gamma_A) < \varepsilon$. Then there exists a ring A_1 separating the boundary components of A such that $M(\Gamma_{A_1}) = \varepsilon$. If we assume that $M(\Gamma'_{A_1}) < \delta$, then $M(\Gamma_A) \le M(\Gamma'_{A_1}) < \delta$. Also, we can replace Condition a) of Theorem 3 with the following weaker condition: "There exists $\varepsilon > 0$ and $\delta > 0$ such that $M(\Gamma'_A) > \varepsilon$ for every ring A in D such that $M(\Gamma_A) = \delta$"; the proof is the same. From this remark we immediately get the following corollary:

Corollary 1. *Let $f\colon D \to D'$ be such that there exists $\alpha > 0$ and $\gamma\colon (0,\alpha] \to (0,\infty)$ such that $M(\Gamma'_A) \le \gamma\big(M(\Gamma_A)\big)$ for every ring A in D such that $0 < M(\Gamma_A) \le \alpha$. Then f is K-quasiconformal with $K = K(\alpha,\gamma,n)$.*

We remark that if $f\colon D \to D'$ is K-quasiconformal, then for every ring A in D, $M(\Gamma'_A) \le KM(\Gamma_A)$; hence the condition of Corollary 1 is satisfied for $\gamma\colon (0,\infty) \to (0,\infty)$, $\gamma(x) = Kx$. Our result shows that we can have quasiconformality for an arbitrary function γ.

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Received 17 October 1988