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CIRCULAR DISTORTION OF CURVES AND QUASICIRCLES

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1. Introduction

Suppose that C is a Jordan curve in the extended complex plane $\overline{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. Then C is a K-*quasicircle*, $1 \le K < \infty$, if it is the image of the unit circle under a K-quasiconformal self mapping f of $\overline{\mathbf{C}}$. Thus C is a 1-quasicircle if and only if C is a circle or line [A], [LV].

Next we say that C has *circular distortion* c, $1 \le c < \infty$, if for each Möbius transformation φ, either $\varphi(C)$ separates the boundary circles of an annulus

$$A = A(z_0; r, s) = \{z \in \mathbf{C} : r \le |z - z_0| \le s\} \tag{1.1}$$

with radii ratio $s/r = c$ or $\varphi(C)$ contains the point ∞. The circular distortion is a Möbius invariant which measures how far a Jordan curve differs from being a circle or line. In particular, C has circular distortion 1 if and only if it is a circle or line.

Kühnau recently established the following relation between these two concepts [K].

1.2. Theorem. *If C is a K-quasicircle in $\overline{\mathbf{C}}$, then C has circular distortion c where c depends only on K.*

Kühnau found sharp bounds for the constant c in terms of K and asked if the converse of Theorem 1.2 is true, that is, if each curve C with circular distortion c is a K-quasicircle where K depends only on c.

In Section 2 of this paper we consider two classes of curves for which this is the case—convex curves with arbitrary circular distortion and arbitrary curves with circular distortion $c < \sqrt{2}$. Then in Section 3 we present an example to show that a curve with circular distortion $c \ge 5$ need not be a quasicircle.

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2. Two classes of quasicircles

For each $z_0 \in \mathbf{C}$ and $0 < r < \infty$ we let

$$B(z_0,r) = \{z \in \mathbf{C} : |z - z_0| < r\}, \qquad B^*(z_0,r) = \{z \in \overline{\mathbf{C}} : |z - z_0| > r\}.$$

We begin by noting that each convex curve in $\mathbf{C}$ is a K-quasicircle where K depends only on its circular distortion. (See also [C], [L].)

2.1. Theorem. *If C is a convex curve which separates the boundary circles of an annulus A with radii ratio c, then C is a K-quasicircle where*

$$K = \tfrac{1}{4}\left(\sqrt{c^2+3} + \sqrt{c^2-1}\right)^2. \tag{2.2}$$

Proof. By performing a preliminary similarity mapping, we may assume that $A = A(0;1,c)$. Then for each $\theta \in [0,2\pi]$ there exists a unique point $z = re^{i\theta} \in C$ where $r = r(\theta) \in [1,c]$. Fix θ and let E denote the double cone bounded by $\partial B(0,1)$ and the two tangent rays drawn from $\partial B(0,1)$ through $z = r(\theta)e^{i\theta}$ to ∞. Because C is convex with $B(0,1) \subset \operatorname{int}(C) \subset \overline{B}(0,c)$, $C \setminus \{z\}$ lies in $\mathbf{C} \setminus E$ and

$$\limsup_{\theta' \to \theta} \frac{|r(\theta') - r(\theta)|}{|\theta' - \theta|} \le \sqrt{c^2-1}\, r(\theta). \tag{2.3}$$

Now let

$$f(se^{i\theta}) = sr(\theta)e^{i\theta}$$

for $0 \le s < \infty$, $0 \le \theta \le 2\pi$ and $f(\infty) = \infty$. Then $f : \overline{\mathbf{C}} \to \overline{\mathbf{C}}$ is a homeomorphism which maps the unit circle onto C. Next (2.3) implies that f satisfies a local Lipschitz condition at each point of $\mathbf{C}$ and hence is differentiable almost everywhere in $\mathbf{C}$. Let $\partial_\alpha f$ denote the directional derivative of f in the direction α. Then an elementary calculation and (2.3) imply that

$$\max_\alpha |\partial_\alpha f(z)| \le K \min_\alpha |\partial_\alpha f(z)|$$

at each point where f is differentiable and hence that f is K-quasiconformal where K is as in (2.2).

If C is a Jordan curve with circular distortion c, then C is a circle or line and hence a quasicircle whenever $c = 1$. We show next that C is a quasicircle whenever $c < \sqrt{2}$. Our proof is based on elementary classical properties of the exterior mapping function

$$w = g(z) = z + \sum_0^\infty b_j z^{-j}.$$

2.4. Lemma. *If g maps $B^*(z_0, s)$ conformally into $B^*(w_0, t)$, then*

$$|b_1| \le s^2 - t^2. \tag{2.5}$$

Proof. Since the coefficient b_1 is invariant under translations in the z- and w-planes, we may assume that $z_0 = w_0 = 0$. Then

$$h(z) = \frac{1}{s}\left(g(sz) + \frac{t^2 e^{i\theta}}{g(sz)}\right) = z + \sum_0^\infty c_j z^{-j}$$

maps $B^*(0,1)$ conformally into $\overline{\mathbf{C}}$,

$$\left|b_1 + t^2 e^{i\theta}\right| s^{-2} = |c_1| \le 1$$

by the area theorem [P] and we obtain (2.5) by setting $\theta = \arg b_1$.

2.6. Lemma. *If C is a Jordan curve which separates the boundary circles of an annulus A with radii ratio c and if g maps $B^*(0,1)$ onto $\operatorname{ext}(C)$, then*

$$|b_1| \le \frac{c^2 - 1}{c^2 + 1}. \tag{2.7}$$

Proof. Suppose that $A = A(w_0; r, cr)$. Then g maps $B^*(0,1)$ conformally into $B^*(w_0, r)$ and hence

$$|b_1| \le 1 - r^2 \tag{2.8}$$

by Lemma 2.4. Next

$$z = g^{-1}(w) = w + \sum_0^\infty c_j w^{-j}$$

maps $B^*(w_0, cr)$ conformally into $B^*(0,1)$ and $c_1 = -b_1$. Hence by Lemma 2.4

$$c^{-2}|b_1| = c^{-2}|c_1| \le c^{-2}\left((cr)^2 - 1\right) = r^2 - c^{-2}, \tag{2.9}$$

and (2.7) follows directly from adding (2.8) and (2.9).

2.10. Remark. The mapping

$$g(z) = z + \frac{c-1}{c+1}\frac{1}{z}$$

shows that one cannot replace the upper bound in (2.7) by anything less than $(c-1)/(c+1)$.

2.11. Theorem. *If C is Jordan curve in $\overline{\mathbf{C}}$ with circular distortion c and if f maps $B(0,1)$ conformally onto a component of $\overline{\mathbf{C}} \setminus C$, then*

$$|S_f(z)|(1-|z|^2)^2 \le 6\frac{c^2-1}{c^2+1} \tag{2.12}$$

for each z in $B(0,1)$, where S_f denotes the Schwarzian derivative of f.

Proof. Fix $z_0 \in B(0,1)$; since the left hand side of (2.12) is continuous in z_0 we may assume that $f(z_0) \neq \infty$. Let

$$\varphi(z) = \frac{z+z_0}{1+\bar{z}_0 z}, \qquad \psi(w) = \frac{(1-|z_0|^2)f'(z_0)}{w-f(z_0)},$$

and set $g(z) = \psi \circ f \circ \varphi(1/z)$ in $B^*(0,1)$. Then by a well known computation,

$$g(z) = z + \sum_0^\infty b_j z^{-j}, \qquad b_1 = -\tfrac{1}{6} S_f(z_0)(1-|z_0|^2)^2, \tag{2.13}$$

[D], [N]. Next g maps $B^*(0,1)$ onto $\operatorname{ext}(\psi(C))$ and $\psi(C)$ does not contain ∞. Thus $\psi(C)$ separates the boundary circles of an annulus with radii ratio c and we obtain (2.12) for $z = z_0$ from (2.13) and (2.7).

2.14. Theorem. *If C is a Jordan curve in $\overline{\mathbf{C}}$ with circular distortion $c < \sqrt{2}$, then C is a K-quasicircle where K depends only on c.*

Proof. If f is a conformal mapping of $B(0,1)$ onto a component of $\overline{\mathbf{C}} \setminus C$, then

$$|S_f(z)|(1-|z|^2)^2 \le 6\frac{c^2-1}{c^2+1} = b < 2$$

for $z \in B(0,1)$ by Theorem 2.11. Hence by the Ahlfors–Weill theorem, f has a K-quasiconformal extension $\tilde{f}$ to $\overline{\mathbf{C}}$ where K depends only on b and hence on c [AW], [L].

3. A geometric interpretation for circular distortion

Finally we show that there exists a Jordan curve C with circular distortion 5 which is not a quasicircle. We shall make use of the following alternative characterization for circular distortion. For the sake of simplicity, we restrict ourselves to the case where C passes through ∞.

3.1. Theorem. *Suppose that C is a Jordan curve in $\overline{\mathbf{C}}$ which contains ∞. Then C has circular distortion c if and only if there exists a constant b, $2 \le b < \infty$, such that for each point w_1 in one component of $\overline{\mathbf{C}} \setminus C$ there exists a point w_2 in the other component with*

$$b \operatorname{dist}(w_1, C) \ge |w_1 - w_2|, \qquad b \operatorname{dist}(w_2, C) \ge |w_1 - w_2|. \tag{3.2}$$

Here $b = c+1$ in the necessity and $c = b^2 + b - 1$ in the sufficiency.

Proof. For the necessity, choose w_1 in a component of $\overline{\mathbf{C}} \setminus C$ and let φ be a Möbius transformation for which $\varphi(w_1) = \infty$. Since C has circular distortion c, $\varphi(C)$ separates the boundary circles of an annulus $A = A(z_0; r, cr)$. By a preliminary change of variables we may assume that $z_0 = 0$. Then $w_2 = \varphi^{-1}(0)$ lies in the other component of $\overline{\mathbf{C}} \setminus C$.

Let C_1 and C_2 denote the images under φ^{-1} of the outer and inner boundary circles of A, respectively. Next for $j = 1, 2$ let z_j and z_j' denote the points where C_j meets the extended line L through w_1 and w_2, labeled so that z_j lies in the segment $[w_1, w_2]$, and set $r_j = |z_j - w_j|$. Then by the Möbius invariance of the cross ratio,

$$\frac{|z - w_1|}{|z - w_2|} = \frac{|z_j - w_1|}{|z_j - w_2|} \qquad \text{for } z \in C_j,\ j = 1, 2. \tag{3.3}$$

If $\infty \notin C_1$, then C_1 is a circle which does not separate w_2 from ∞,

$$|z_1' - w_1| \le |z_1' - w_2|,$$

and we obtain

$$|z_1 - w_1| \le |z_1 - w_2| \tag{3.4}$$

from (3.3) with $j = 1$ and $z = z_1'$. If $\infty \in C_1$, then $z_1' = \infty$ and (3.4) again follows from (3.3). Interchanging the roles of C_1 and C_2 in the above discussion then shows that

$$|z_2 - w_2| \le |z_2 - w_1|. \tag{3.5}$$

Next

$$\frac{|z_1 - w_2||z_2 - w_1|}{|z_1 - w_1||z_2 - w_2|} = \frac{|\varphi(z_1)|}{|\varphi(z_2)|} = c, \tag{3.6}$$

and with (3.4) and (3.5) we obtain

$$|z_1 - w_2| \le c|z_1 - w_1| \qquad |z_2 - w_1| \le c|z_2 - w_2|$$

whence

$$|w_1 - w_2| \le |z_j - w_1| + |z_j - w_2| \le (c+1)|z_j - w_j| = (c+1)r_j \tag{3.7}$$

for j=1,2.

Finally (3.3) together with (3.4) and (3.5) implies that

$$B(z_j, r_j) \subset \operatorname{int}(C_j) \subset \overline{\mathbf{C}} \setminus C$$

and hence (3.2) follows from (3.7).

For the sufficiency, suppose that φ is any Möbius transformation for which $\varphi(C)$ does not contain ∞ and let $w_1 = \varphi^{-1}(\infty)$. Then w_1 lies in a component of $\overline{\mathbf{C}} \setminus C$. Let w_2 denote the point in the other component of $\overline{\mathbf{C}} \setminus C$ for which (3.2) holds and set

$$\psi(z) = \frac{w_2 - w_1}{z - w_1}.$$

Then $\varphi \circ \psi^{-1}$ is a euclidean similarity and in order to show that $\varphi(C)$ separates the boundary circles of an annulus of radii ratio c, it suffices to consider the case where $\varphi = \psi$.

Now let $r = |w_1 - w_2|/b$ and $s = b/(b^2 - 1)$. Then

$$\varphi\big(B(w_1, r)\big) = B^*(0, b), \qquad \varphi\big(B(w_2, r)\big) = B(bs, s)$$

while (3.2) implies that

$$C \subset \overline{\mathbf{C}} \setminus \big(B(w_1, r) \cup B(w_2, r)\big).$$

Hence

$$\varphi(C) \subset \varphi\left(\overline{\mathbf{C}} \setminus \big(B(w_1, r) \cup B(w_2, r)\big)\right) \subset A\big(bs; s, (b^2 + b - 1)s\big),$$

an annulus with radii ratio $b^2 + b - 1$.

3.8. Theorem. *There exists a Jordan curve C in $\overline{\mathbf{C}}$ with circular distortion $c = 5$ which is not a quasicircle.*

Proof. For $j = 1, 2, \ldots$ let α_j and β_j denote the upper and lower semicircles

$$\alpha_j = \big\{z : |z - 1| = 2j - 1, \operatorname{Im}(z) \geq 0\big\}, \quad \beta_j = \big\{z : |z + 1| = 2j - 1, \operatorname{Im}(z) \leq 0\big\}.$$

Then

$$(3.9) \qquad \alpha_j \cap \beta_k = \begin{cases} \{0\} & \text{if } j = k = 1, \\ \{2j\} & \text{if } j = k - 1, \\ \{-2j + 2\} & \text{if } j = k + 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Hence

$$\gamma_1 = \bigcup_1^\infty (\alpha_{2j-1} \cup \beta_{2j}) \cup \{\infty\}, \qquad \gamma_2 = \bigcup_1^\infty (\alpha_{2j} \cup \beta_{2j-1}) \cup \{\infty\}$$

are arcs which have only their endpoints 0, ∞ in common and

$$C = \gamma_1 \cup \gamma_2 = \bigcup_1^\infty (\alpha_j \cup \beta_j) \cup \{\infty\}$$

is a Jordan curve.

Fix j and let $z_1 = 4j - 2$ and $z_2 = 4j$. Then $z_1 \in \gamma_1$ and $z_2 \in \gamma_2$, 0 and ∞ are separated by z_1 and z_2 in C and

$$\min\big(\operatorname{dia}(C_1), \operatorname{dia}(C_2)\big) \geq 4j = 2j\,|z_1 - z_2|$$

where C_1 and C_2 are the components of $C \setminus \{z_1, z_2\}$. Hence C is not a quasicircle by Ahlfors' well known criterion [A].

Suppose that w_1 is a point in a component of $\overline{\mathbf{C}} \setminus C$; by replacing w_1 by $-w_1$ we may assume without loss of generality that $\operatorname{Im}(w_1) \geq 0$. Next choose $j = 1, 2, \ldots$ so that

$$2j - 2 \leq |w_1 - 1| < 2j$$

and let $w_2 = 2w_0 - w_1$, where

$$w_0 = \begin{cases} 1 + (2j-1)(w_1 - 1)/|w_1 - 1| & \text{if } w_1 \neq 1, \\ 2 & \text{if } w_1 = 1. \end{cases} \tag{3.10}$$

Then $w_0 \in \alpha_j$. If $z \in \alpha_k$, then

$$|w_1 - z| \geq \big||w_1 - 1| - (2k-1)\big| \geq \big||w_1 - 1| - (2j-1)\big| = |w_1 - w_0|. \tag{3.11}$$

Similarly if $z \in \beta_k$ with endpoints $z_k = -2k$, $z_k' = 2k - 2$, then

$$|w_1 - z| \geq \min\big(|w_1 - z_k|, |w_1 - z_k'|\big), \qquad z_k, z_k' \in \bigcup_1^\infty \alpha_l,$$

and we obtain $|w_1 - z| \geq |w_1 - w_0|$ from (3.11). Thus

$$\operatorname{dist}(w_1, C) = |w_1 - w_0|.$$

A similar argument shows that

$$\operatorname{dist}(w_2, C) = |w_2 - w_0|$$

and we conclude that

$$2 \operatorname{dist}(w_1, C) = 2 \operatorname{dist}(w_2, C) = |w_1 - w_2|.$$

Finally let $z_k = 1 + i|w_k - 1|$ for $k = 0, 1, 2$. Then $U = \overline{B}(z_0, 1)$ is a closed neighborhood of $z_0 \in \alpha_j$ and $U \setminus C$ has exactly two components, one of which contains z_1 and the other z_2. Since for $j = 1, 2$ the arc

$$\{z \,:\, |z - 1| = |w_j - 1|, \operatorname{Im}(z) \geq 0\}$$

joins z_j to w_j in $\overline{\mathbf{C}} \setminus C$, w_1 and w_2 lie in different components of $\overline{\mathbf{C}} \setminus C$. Thus C satisfies the hypotheses of Theorem 3.1 with $b = 2$ and hence has circular distortion 5.

4. Concluding remarks

Remark 4.1. Theorems 2.14 and 3.8 show that a Jordan curve with circular distortion c must be a quasicircle if $c < \sqrt{2}$ and need not be if $c \geq 5$. The bound $\sqrt{2}$ is not sharp. Indeed a slightly different argument yields the same conclusion for

$$c < \frac{\sqrt{6}(1+\sqrt{37})}{12} = 1.4457\ldots$$

Remark 4.2. One can use Theorem 3.1 to construct a Jordan curve with finite circular distortion which has positive area (or two dimensional measure) and hence is certainly not a quasicircle.

We indicate the construction of such a curve C in Figure 1 which was kindly drawn for us by U. Graeber. At the jth stage of the construction, $j = 1, 2, \ldots$, we have 4^{j-1} jth generation squares $Q_{j,k}$ of sidelength $a_j = 2^{-j}(j+1)/j$. Next in each square $Q_{j,k}$ we draw four $(j+1)$th generation squares $Q_{j+1,l}$ of sidelength a_{j+1} leaving three vertical and three horizontal corridors of width $b_j = 2^{-j}/3j(j+1)$. In these corridors we draw the jth generation arcs as in Figure 1. This figure contains two generations of squares and arcs.

The intersection E of all generations of squares has area $\frac{1}{4}$. The curve C is the union of all generations of arcs together with the set E and two halflines connecting the two endpoints in $\partial Q_{1,1}$ of first generation arcs to the point ∞. Then C is a Jordan curve with positive area and it follows from Theorem 3.1 that C has finite circular distortion.

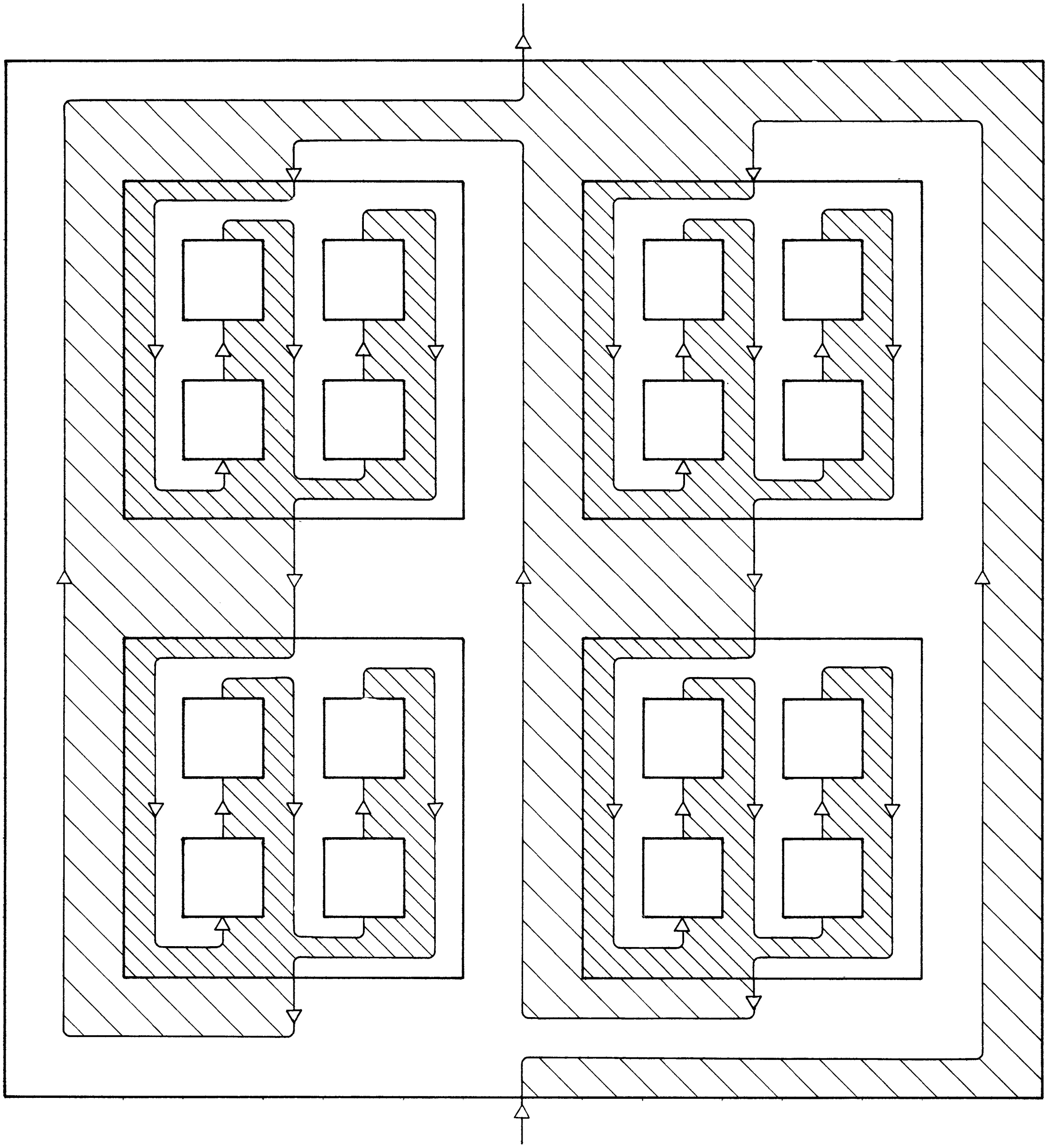

Figure 1.

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