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# SOME PROPERTIES OF WEAK SOLUTIONS OF NONLINEAR SCALAR FIELD EQUATIONS

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## Introduction and the main results

In this paper, we study some properties of the weak solutions of the following nonlinear scalar field equations

$$-\sum_{i=1}^{N} \frac{\partial}{\partial x_i}\left(|\nabla u|^{p-2}\frac{\partial u}{\partial x_i}\right) + c(x)|u|^{q-2}u = f(x,u),$$

$$(1.1) \qquad u \in E = \left\{u \in L^q(R^N) \mid \frac{\partial u}{\partial x_i} \in L^p(R^N), \quad 1 \le i \le N\right\},$$

$$\text{where} \quad q \ge p \quad \text{if} \quad p \ge N \ge 2, \quad \text{and}$$

$$p \le q \le p^* = \frac{Np}{N-p}, \quad \text{when } N > q.$$

When $p = q = 2$, (1.1) is derived by considering the standing waves of the nonlinear Schrödinger equation

$$(1.2) \qquad i\Phi_t = \Delta\Phi + g(|\Phi|)\Phi,$$

where $\Phi \in \mathbf{C}$, $\Delta = \sum_{i=1}^{N}(\partial^2/\partial x_i^2)$, $x = (x^1, x^2, \dots, x^N) \in R^N$. A standing wave of (1.2) is a solution of (1.2) which has the form $\Phi(x,t) = e^{i\beta t}u(x)$; thus $u$ satisfies

$$\Delta u + g(|u|)u + \beta u = 0,$$

which is a special case of (1.1). For more details about scalar field equations see e.g. [BL].

Throughout this paper, we denote by $\|u\|_s$ the $L^s$-norm of the function $u$ over $R^N$ and $\|u\|_{s(|x|\ge R)}$ the $L^s$-norm of $u$ over the set $\{x \in R^N \mid |x| \ge R\}$ where $s > 1$. Let $p$, $q$, and the space $E$ be given as in (1.1). The norm in $E$ is defined by $\|u\|_E = \||\nabla u|\|_p + \|u\|_q$ for any $u \in E$. It is clear the $\langle E, \|\cdot\|_E\rangle$ is a reflexive Banach space.

By Nirenberg's inequality (see [N]), $E$ is imbedded in $L^t(R^N)$ for $t \ge q$ when $N \le p$ and for $q \le t \le p^* = Np/(N-p)$ when $N > p$. Furthermore, we have the following result which is a generalization of N. Trudinger's inequality (see [L]).

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**Lemma 1.3.** *Suppose that $0 < \gamma < N/(N-1)$, $r \geq q \geq N$ with $\gamma n_0 > r + N/(N-1)$, $b > 0$ where $n_0$ is a positive integer. Then for all $u \in E$*

$$\sum_{n=n_0}^{+\infty} \frac{b^n}{n!} \int_{R^N} |u|^{\gamma n} dx \leq C(|||\nabla u|||_N) \|u\|_r^r, \tag{1.4}$$

*and for $0 < \tau < 1$*

$$\sum_{n=n_0}^{+\infty} \frac{1}{n!} \Big( \int_{R^N} |u|^{\gamma n} dx \Big)^\tau \leq \tilde{C}(|||\nabla u|||_N) \|u\|_r^{r\tau},$$

*where $C(t)$, $\tilde{C}(t)$ are nonnegative real functions on $[0, +\infty)$. Furthermore for each $M > 0$, there is a constant $K(M) > 0$ such that*

$$C(|||\nabla u|||_N) \leq K(M), \qquad \tilde{C}(|||\nabla u|||_N) \leq K(M)$$

*whenever $|||\nabla u|||_N \leq M$ and furthermore there is a constant $K$ such that*

$$C(|||\nabla u|||_N) \leq K\, |||\nabla u|||_N^{\gamma n_0 a_{n_0}}$$

*whenever $|||\nabla u|||_N \leq 1$. Here $a_{n_0}$ is a positive constant depending only on $n_0$.*

*Proof.* This lemma was proved in [LZ, Lemma 1], but for completeness, we sketch the proof.

By the results of C. Talenti (see [T]), we know that if $s, t > 1$, $1/s = 1/t - 1/N$ and when $|\nabla h| \in L^t(R^N)$, then

$$\|h\|_s \leq K(N,t)\, |||\nabla h|||_t \tag{1.5}$$

where

$$K(N,t) = \frac{t-1}{N-t} \left[ \frac{N-t}{N(t-1)} \right]^{1/t} \left[ \frac{\Gamma(N+1)}{\Gamma(N/t)\Gamma(N+1-N/t)\omega_{N-1}} \right]^{1/N}$$

and $\omega_{N-1} = \pi^{N/2}/\Gamma(\frac{1}{2}N+1)$.

Setting $a_n = 1 - r/\gamma n$ $(n \geq n_0)$, $h = |u|^{1/a_n}$ and using (1.5) and Hölder's inequality we obtain, for each $u \in E$,

$$\int_{R^N} |u|^{\gamma n} dx \leq \left[ \frac{K(N, \mu_n)}{a_n} \right]^{\gamma n a_n} |||\nabla u|||_N^{\gamma n a_n} \|u\|_r^r, \tag{1.6}$$

where $\mu_n$ is such that $1/\gamma n a_n = 1/\mu_n - 1/N$. On the other hand, it is easy to see that

$$K(N, \mu_n) \leq C n^{(N-1)/N}$$

where $C > 0$ is a constant independent of $n$, so we have

$$\sum_{n=n_0}^{+\infty} \frac{b^n}{n!} \int_{R^N} |u|^{\gamma n} dx \leq \sum_{n=n_0}^{+\infty} \frac{b^n}{n!} C^{\gamma n a_n} n^{(N-1)\gamma n a_n/N} \, \||\nabla u|\|_N^{\gamma n a_n} \, \|u\|_r^r,$$

from which the lemma follows.

Next, we state the conditions imposed on $c(x)$ and $f(x,t)$ in (1.1).

($c_1$) The function $c(x)$ belongs to $C^0(R^N, R^1)$, and there is a constant $c > 0$ such that $c(x) \geq c$ for any $x \in R^N$.

($f_1$) $f(x,t) \in C^0(R^N \times R^1, R^1)$.

($f_2$) $\lim_{t\to 0} f(x,t)/|t|^{q-1} = 0$ uniformly in $x \in R^N$.

($f_3$) If $N < p$, then there is a $l$, $q < l < +\infty$ such that $\lim_{t\to\infty} f(x,t)/|t|^{l-1} = 0$ uniformly in $x \in R^N$.

If $N = p$, then there is a $\gamma$ with $0 < \gamma < N/(N-1)$ such that

$$\lim_{t\to\infty} f(x,t)/e^{|t|^\gamma} = 0 \qquad \text{uniformly in } x \in R^N.$$

If $N > p$, then there is a constant $b \geq 0$ such that

$$\lim_{t\to\infty} f(x,t)/|t|^{p^*-1} = b \qquad \text{uniformly in } x \in R^N.$$

Under the above conditions ($c_1$), ($f_1$)–($f_3$), we easily see that for any $\varepsilon > 0$, there is a $C_\varepsilon > 0$ such that

$$(1.7) \qquad |f(x,t)| \leq \varepsilon |t|^{q-1} + C_\varepsilon |t|^{p^*-1}, \text{ for all } (x,t) \in R^N \times R^1 \text{ if } N > p,$$

$$(1.8) \qquad |f(x,t)| \leq \varepsilon |t|^{q-1} + C_\varepsilon |t|^{l-1}, \text{ for all } (x,t) \in R^N \times R^1 \text{ if } N < p,$$

$$(1.9) \quad |f(x,t)| \leq \varepsilon |t|^{q-1} + C_\varepsilon \sum_{n=n_0}^{+\infty} \frac{|t|^{\gamma n + N - 1}}{n!}, \text{ for all } (x,t) \in R^N \times R^1 \text{ if } N = p.$$

A function $u \in E$ is called a weak solution of (1.1) if for each $v \in E$

$$(1.10) \qquad \int_{R^N} \left[ \sum_{i=1}^N |\nabla u|^{p-2} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} + c(x)|u|^{q-2}uv - f(x,u)v \right] dx = 0.$$

Note that under conditions ($c_1$), ($f_1$)–($f_3$), $u \equiv 0$ is always the trivial solution of (1.1). The existence of nontrivial weak solutions of (1.1) was studied in [BL] for $p = q = 2$ and in [Li], [LZ], and [YZ] for general $q \geq p$.

It is our aim in this paper to study some properties of the weak solutions of (1.1). The main result is the following:

**Theorem 1.11.** *Suppose that* $(c_1)$, $(f_1)$–$(f_3)$ *hold and* $u \in E$ *is a weak solution of* (1.1). *Then* $u \in L^\infty(R^N)$ *and there is a* $t > 1$, $R_0 = R_0(t) > 0$ *such that for any* $R \geq R_0$

$$\|u\|_{\infty(|x|\geq R)} \leq C\,\|u\|_{t(|x|\geq R/2)} < +\infty \tag{1.12}$$

*where* $C$ *is a positive constant independent of* $R$. *Furthermore,* $\lim_{|x|\to\infty} u(x) = 0$, *and* $u \in C^{1,\alpha}_{\mathrm{loc}}(R^N)$ *for some* $0 < \alpha < 1$.

**Remark 1.13.** $C^{1,\alpha}$ regularity is the best possible in general for weak solutions of degenerate elliptic equations like (1.1) as one can see from a simple example in [To].

There have been some results for $C^{1,\alpha}$ regularity of weak solutions in bounded domains (see e.g. [LU], [To]). For the unbounded domain $R^N$, H. Brezis and E.H. Lieb showed that weak solutions of semilinear elliptic systems, in particular of (1.1) when $N > p = 2$, are in $L^\infty(R^N) \cap C^{1,\alpha}_{\mathrm{loc}}(R^N)$ for all $0 < \alpha < 1$ and the solutions tend to zero as $|x| \to +\infty$ (see [BLi]). But their method seems not to extend to $p \neq 2$.

The main difficulty in proving Theorem 1.11 is to prove (1.12) and $u \in L^\infty(R^N)$. We overcome this difficulty by using the Nash–Moser method (see [GT]) together with careful estimates.

## 2. Proof of the main theorem

In this section, we prove the main result of this paper, Theorem 1.11. By the main result in [To], we need only to prove (1.12) and $u \in L^\infty(R^N)$ together with $\lim_{|x|\to\infty} u(x) = 0$.

Suppose that $u$ is a weak solution of (1.1). For any $R > 0$, $0 < r \leq R/2$, let $\eta \in C^\infty(R^N)$, $0 \leq \eta \leq 1$ with

$$\eta = \begin{cases} 1 & \text{if } |x| \geq R, \\ 0 & \text{if } |x| \leq R - r, \end{cases} \qquad |\nabla \eta| \leq 2/r. \tag{2.1}$$

We set $u^+ = \max(0, u)$, $u_L^+ = \min(u^+, L)$ where $L > 0$.

We first deal with the case where $N > p$. To this end, let $v = \eta^p u^+ u_L^{+p(\beta-1)}$, $W_L = \eta u^+ u_L^{+\beta-1}$ for any $\beta \geq 1$. Then by (1.7), by the definition of weak solutions and by Sobolev's imbedding, we have, for some constant $C > 0$, that, for each $\beta \geq 1$,

$$\begin{aligned} \|W_L\|_{p\gamma}^p &\leq C \int_{R^N} |\nabla W_L|^p dx \\ &\leq C\beta^p \Big( \int_{R^N} u^{+p^*} \eta^p u_L^{+p(\beta-1)} dx + \int_{R^N} |\nabla \eta|^p u^{+p} u_L^{+p(\beta-1)} dx \Big). \end{aligned} \tag{2.2}$$

We claim that

$$u \in L^{p^{*2}/p}(|x| \geq R) \tag{2.3}$$

for $R$ large enough. In fact, let $\beta = p^*/p$, from (2.2) we have

$$\begin{aligned}
&\Big(\int_{R^N} (\eta u^+ u_L^{+(p^*-p)/p})^{p^*} dx\Big)^{p/p^*} \\
&\leq C(N,p)\Big\{\Big[\int_{R^N} (\eta u^+ u_L^{+(p^*-p)/p})^{p^*} dx\Big]^{p/p^*} \Big(\int_{|x|\geq R-r} u^{+p^*} dx\Big)^{(p^*-p)/p} \\
&\quad + \int_{R^N} |\nabla\eta|^p u^{+p} u_L^{+p^*-p} dx\Big\} \\
&\leq C(N,p)\Big\{\Big[\int_{R^N} (\eta u^+ u_L^{+(p^*-p)/p})^{p^*} dx\Big]^{p/p^*} \|u^+\|^{p^*-p}_{p^*(|x|\geq \frac{1}{2}R)} \\
&\quad + \int_{R^N} |\nabla\eta|^p u^{+p} u_L^{+(p^*-p)} dx\Big\}.
\end{aligned}$$

Since $u^+ \in L^{p^*}(R^N)$, $\|u^+\|^{p^*-p}_{p^*(|x|\geq \frac{1}{2}R)} \leq 1/C(N,p)$ for $R$ large enough. Hence we obtain

$$\begin{aligned}
&\Big(\int_{|x|\geq R} (u^+ u_L^{+(p^*-p)/p})^{p^*} dx\Big)^{p/p^*} \leq \Big(\int_{R^N} (\eta u^+ u_L^{+(p^*-p)/p})^{p^*} dx\Big)^{p/p^*} \\
&\leq C(N,p)\int_{R^N} |\nabla\eta|^p u^{+p} u_L^{+(p^*-p)} dx \leq \frac{C}{r^p}\int_{R^N} u^{+p^*} dx.
\end{aligned} \tag{2.4}$$

Thus (2.2) follows.

Next, we note that if $\beta = p^*(t-1)/pt$ with $t = p^{*2}/(p^*-p)p$, then $\beta > 1$ and $pt/(t-1) < p^*$. Now suppose that $u^+ \in L^{p\beta t/(t-1)}(|x| \geq R-r)$ for some $\beta \geq 1$. Then (2.3) gives that

$$\begin{aligned}
\|W_L\|^p_{p^*} &\leq C\beta^p\Big\{\Big[\int_{|x|\geq R-r} (\eta^p u^{+p\beta})^{t/(t-1)} dx\Big]^{1-1/t} \\
&\quad \times \Big(\int_{|x|\geq R-r} u^{+(p^*-p)t} dx\Big)^{1/t} \\
&\quad + \frac{[R^N-(R-r)^N]^{1/t}}{\gamma p}\Big(\int_{|x|\geq R-r} u^{+p\beta t/(t-1)} dx\Big)^{1-1/t}\Big\} \\
&\leq C\beta^p\Big(1+\frac{R^{N/t}}{r^p}\Big)\Big(\int_{|x|\geq R-r} u^{+p\beta t/(t-1)} dx\Big)^{1-1/t}.
\end{aligned} \tag{2.5}$$

Letting $L \to +\infty$ in (2.5), we obtain

$$\left\|u^+\right\|^{p\beta}_{\beta p^*(|x|\geq R)} \leq C\beta^p\Big(1+\frac{R^{N/t}}{r^p}\Big)\left\|u^+\right\|^{p\beta}_{p\beta t/(t-1)(|x|\geq R-r)}.$$

If we set $\chi = p^*(t-1)/pt$, $s = pt/(t-1)$, then

$$\left\|u^+\right\|_{\beta\chi s(|x|\geq R)} \leq C^{1/\beta}\beta^{1/\beta}\Big(1+\frac{R^{N/t}}{r^p}\Big)^{1/p\beta}\left\|u^+\right\|_{\beta s(|x|\geq R-r)}. \tag{2.6}$$

Let $\beta = \chi^m$, $(m = 1,2,\ldots)$, then we get

$$\left\|u^+\right\|_{\chi^{m+1}s(|x|\geq R)} \leq C^{\chi^{-m}}\chi^{m\chi^{-m}}\Big(1+\frac{R^{N/t}}{r^p}\Big)^{1/p\chi^m}\left\|u^+\right\|_{\chi^m s(|x|\geq R-r)}.$$

It is clear that $p > N/t$. So if $r_m = 2^{-(m+1)}R$, then (2.6) implies

$$\begin{aligned}
&\left\|u^+\right\|_{\chi^{m+1}s(|x|\geq R)} \leq \left\|u^+\right\|_{\chi^{m+1}s(|x|\geq R-r_{m+1})}\\
&\leq C^{\sum_{i=1}^m \chi^{-i}}\chi^{\sum_{i=1}^m i\chi^{-i}}\exp\Big(\sum_{i=1}^m \ln(1+2^{p(i+1)})/p\chi^i\Big)\left\|u^+\right\|_{\chi s(|x|\geq R-r_1)}\\
&\leq C\left\|u^+\right\|_{p^*(|x|\geq\frac{1}{2}R)}.
\end{aligned}$$

Letting $m \to +\infty$ in the last inequality, we obtain

$$\left\|u^+\right\|_{\infty(|x|\geq R)} \leq C\left\|u^+\right\|_{p^*(|x|\geq\frac{1}{2}R)}.$$

Similarly, we can show

$$\left\|u^-\right\|_{\infty(|x|\geq R)} \leq C\left\|u^-\right\|_{p^*(|x|\geq\frac{1}{2}R)}$$

where $u^- = \max(-u,0)$; hence (1.12) holds for $N > p$ and $\lim_{|x|\to\infty} u(x) = 0$.

To show that $\|u\|_\infty < +\infty$ when $N > p$, we need only show that for any $x_0 \in R^N$, there is a ball $B_R(x_0) = \{x \in R^N \mid |x-x_0| \leq R\}$ such that $\|u\|_{\infty(B_R(x_0))} < +\infty$. But this was essentially done in [ZY] for weak solutions of equations similar to (1.1) in bounded domains. We just sketch the proof of this fact.

For any $x_0 \in R^N$, $R > 0$, $0 < r \leq \frac{1}{2}R$, let $\xi \in C_0(R^N)$ with $0 \leq \xi \leq 1$ and

$$\xi = \begin{cases} 1 & \text{if } |x-x_0| \leq R,\\ 0 & \text{if } |x-x_0| \geq R+r,\end{cases}$$

and $|\nabla\xi| \leq 2/r$. Write $\bar{v} = \xi^p u^+ u_L^{+p(\beta-1)}$, $\bar{W}_L = \xi u^+(u_L^+)^{-1}$, we can show that for $R_0$ small enough

$$u^+ \in L^{p^{*2}/p}(B_R(x_0))$$

and similarly for some $\bar{R}$ that $u^+ \in L^\infty(B_{\bar{R}}(x_0))$ by the method used above. Thus $\|u^+\|_\infty < +\infty$ hence $\|u\|_\infty < +\infty$ and we have completed the proof of Theorem 1.11 in the case $N > p$.

If now $N = p$, we set $v = u^+ u_L^{+N(\beta-1)}$, $(\beta \geq 1)$; then $v \in E$ and the definition of weak solutions gives that

$$
\begin{aligned}
(2.7) \qquad & \int_{R^N} |\nabla u^+|^N u_L^{+N(\beta-1)} dx + N(\beta-1)\int_{R^N} |\nabla u_L^+|^N u_L^{+N(\beta-1)} dx \\
& + c\int_{R^N} u^{+q} u_L^{+N(\beta-1)} dx \\
& \leq C \sum_{n=n_0}^{+\infty} \frac{1}{n!}\int_{R^N} u^{+\gamma n+N-1} u^+ u_L^{+N(\beta-1)} dx.
\end{aligned}
$$

If we set $W_L = u^+ u_L^{+\beta-1}$, then (2.7) implies

$$
\int_{R^N} |\nabla W_L|^N dx \leq C\beta^N \sum_{n=n_0}^{+\infty} \frac{1}{n!}\int_{R^N} u^{+\gamma n} |W_L|^N dx.
$$

Using Hölder's inequality, we get

$$
|||\nabla W_L|||_N^N \leq C\beta^N \sum_{n=n_0}^{+\infty} \frac{1}{n!}\Big(\int_{R^N} u^{+(\gamma+\varepsilon_0)n} dx\Big)^{\gamma/(\gamma+\varepsilon_0)} \|W_L^N\|_{t/N}
$$

where $\varepsilon_0$ is small enough such that $\gamma + \varepsilon_0 < N/(N-1)$, $t = N(\gamma+\varepsilon_0)/q_0 \geq q$. Thus Lemma 1.3 yields

$$
|||\nabla W_L|||_N \leq C\beta \|W_L\|_t .
$$

Hence by Nirenberg's inequality (see [N]) there is a $s > t$ with

$$
(2.8) \qquad \|W_L\|_s \leq C\big(|||\nabla W_L|||_N + \|W_L\|_t\big) \leq C\beta \|W_L\|_t
$$

where $C > 0$ is a constant from which we obtain $\|u^+\|_\infty < +\infty$ by standard Nash–Moser iteration. Similarly $\|u^-\|_\infty < +\infty$ and hence $\|u\|_\infty < +\infty$.

To show (1.12) for $N = p$, we can use the same method we used in the case where $N > p$. In fact, let $v = \eta^N u^+ u^{+N(\beta-1)}$, $W_L = \eta u^+ u_L^{+\beta-1}$ where $\eta$ was given by (2.1) for $\frac{1}{2}R \geq r > 0$, then by the definition of weak solutions and Lemma 1.3 we have, for any $\varepsilon > 0$, that

$$
(2.9) \quad \int_{R^N} |\nabla u^+|^N \eta^N u_L^{+N(\beta-1)} dx
$$

$$+ N\int_{R^N}\sum_{i=1}^{N}|\nabla u^+|^{N-2}\frac{\partial u^+}{\partial x_i}\frac{\partial \eta}{\partial x_i}\eta^{N-1}u^{+N(\beta-1)}u^+dx$$
$$+ N(\beta-1)\int_{R^N}|\nabla u_L^+|^N\eta^N u_L^{+N(\beta-1)}dx + c\int_{R^N}u^{+q}\eta^N u^{+N(\beta-1)}dx$$
$$\le \int_{R^N} f(x,u^+)u^+\eta^N u_L^{+N(\beta-1)}dx$$
$$\le \varepsilon\int_{R^N}u^{+q}\eta^N u_L^{+N(\beta-1)}dx + C_\varepsilon\|W_L\|_t^N$$

for some $C_\varepsilon > 0$ and $t > q$.

Taking $\varepsilon > 0$ small enough and using Young's inequality and $(c_1)$, we get

$$\begin{aligned}(2.10)\quad &\int_{R^N}|\nabla u^+|^N\eta^N u_L^{+N(\beta-1)}dx + N(\beta-1)\int_{R^N}|\nabla u_L^+|^N\eta^N u_L^{+N(\beta-1)}dx\\ &\quad + \bar{c}\int_{R^N}u^{+q}\eta^N u_L^{+N(\beta-1)}dx\\ &\le c\|W_L\|_t^N + N\int_{R^N}|\nabla u^+|^{N-1}|\nabla\eta|\eta^{N-1}u^+u_L^{+N(\beta-1)}dx\\ &\le \delta\int_{R^N}|\nabla u^+|^N\eta^N u_L^{+N(\beta-1)}dx\\ &\quad + C_\delta\int_{R^N}|\nabla\eta|^N u^{+N}u_L^{+N(\beta-1)}dx + C\|W_L\|_t^N\end{aligned}$$

where $\delta > 0$ is arbitrary and $\bar{c} > 0$ is a constant.

Choosing $\delta > 0$ small enough, we have

$$\begin{aligned}(2.11)\quad &\int_{R^N}|\nabla u^+|^N\eta^N u_L^{+N(\beta-1)}dx + N(\beta-1)\int_{R^N}|\nabla u_L^+|^N\eta^N u_L^{+N(\beta-1)}dx\\ &\quad + \tilde{C}\int_{R^N}u^{+q}\eta^N u_L^{+N(\beta-1)}dx\\ &\le C\Big(\|W_L\|_t^N + \int_{R^N}|\nabla\eta|^N u^+u_L^{+N(\beta-1)}dx\Big)\end{aligned}$$

where $C > 0$, $\tilde{C} > 0$ are constants. Hence

$$\begin{aligned}(2.12)\quad &\int_{R^N}|\nabla W_L|^N dx + \|W_L\|_t^N\\ &\le c\Big[\int_{R^N}|\nabla\eta|^N u^{+N}u_L^{+N(\beta-1)}dx + \int_{R^N}\eta^N u_L^{+N(\beta-1)}|\nabla u^+|^N dx\end{aligned}$$

$$
\begin{aligned}
&+ (\beta-1)^N \int_{R^N} \eta^N u_L^{+N(\beta-1)} |\nabla u_L^+|^N dx\Big] + \|W_L\|_t^N \\
&\le C\beta^N \Big(\int_{R^N} |\nabla\eta|^N u^{+N} u_L^{+N(\beta-1)} dx + \|W_L\|_t^N\Big) \\
&\le C\beta^N \Big\{1 + \frac{[R^N - (R-r)^N]^{(t-N)/t}}{\gamma N}\Big\} \Big\|u^+ u_L^{+\beta-1}\Big\|^N_{t(|x|\ge R-r)} .
\end{aligned}
$$

Again, by Nirenberg's inequality, for some $s > t$, we have

$$
\begin{aligned}
\|W_L\|_s &\le C(\||\nabla W_L|\|_N + \|W_L\|_t) \\
&\le C\beta\Big[1 + \frac{R^{N(t-N)/t}}{r^N}\Big]^{1/N} \Big\|u^+ u_L^{+\beta-1}\Big\|_{t(|x|\ge R-r)} .
\end{aligned}
$$

Letting $L \to +\infty$, we get

$$
\|u^+\|_{s\beta(|x|\ge R)} \le C^{1/\beta}\beta^{1/\beta}\Big[1 + \frac{R^{N(t-N)/t}}{\gamma N}\Big]^{1/N\beta} \|u^+\|_{t\beta(|x|\ge R-r)}
$$

where $C$ is a positive constant independent of $R$ and $r$. Let $\chi = s/t$, $\beta = \chi^m$ and $r_m = 2^{-(m+1)}R$ for $R > 1$. Now we obtain

$$
\|u^+\|_{\chi^{m+1}t(|x|\ge R-r_{m+1})} \le C^{\chi^{-m}} \chi^{m\chi^{-m}} \Big(1 + 2^{N(m+2)}\Big)^{\chi^{-m}/N} \|u^+\|_{\chi^m t(|x|\ge R)}
$$

from which (1.12) follows easily and hence $\lim_{|x|\to\infty} u(x) = 0$.

The case where $N < p$ can be dealt with in the same way; we omit the details. We have thus completed the proof of Theorem 1.11.

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