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SOME FURTHER ARITHMETICAL IDENTITIES INVOLVING A GENERALIZATION OF RAMANUJAN'S SUM

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1. Introduction

Let G be a commutative semigroup with identity $\mathbf{1}$, with respect to a multiplication denoted by juxtaposition. Suppose there exists a finite or countable infinite set $P(\subseteq G)$ of primes such that each $\mathbf{n} \in G$ can be represented uniquely in the form

$$\mathbf{n} = \prod_{\mathbf{p} \in P} \mathbf{p}^{\mathbf{n}(\mathbf{p})},$$

where the exponents $\mathbf{n}(\mathbf{p})$ are non-negative integers of which all but a finite number are zero. (Define $\mathbf{1}(\mathbf{p}) = 0$ for all $\mathbf{p} \in P$.) Further, suppose there exists a real-valued norm $\|\cdot\|$ defined on G such that

(i) $\|\mathbf{1}\| = 1, \quad \|\mathbf{p}\| > 1 \qquad (\mathbf{p} \in P)$,
(ii) $\|\mathbf{mn}\| = \|\mathbf{m}\|\,\|\mathbf{n}\| \qquad (\mathbf{m}, \mathbf{n} \in G)$,
(iii) the set $\{\mathbf{n} \in G : \|\mathbf{n}\| \le x\}$ is finite for all real numbers x.

Then G is called [18, p. 11] an arithmetical semigroup. Throughout this paper elements in an arbitrary arithmetical semigroup are typed in boldface.

Let G be an arbitrary but fixed arithmetical semigroup. By an arithmetical function we mean a complex-valued function defined on the arithmetical semigroup G. Let A be a mapping from the set G into the set of subsets of G such that for each $\mathbf{n} \in G$, $A(\mathbf{n})$ is a subset of the set of divisors of $\mathbf{n}$. Then the A-convolution of two arithmetical functions f and g is defined by

$$(f_A g) = \sum_{\mathbf{d} \in A(\mathbf{n})} f(\mathbf{d}) g(\mathbf{n}/\mathbf{d}).$$

In this paper we confine ourselves to regular convolutions and we shall assume the reader to be familiar with this notion (see e.g. [13, Section 1.3], [22, Chapter 4], [27]). For example, the Dirichlet convolution D, where $D(\mathbf{n})$ is the set of all divisors of $\mathbf{n}$, and the unitary convolution U, where $U(\mathbf{n}) = \{\mathbf{d} : \mathbf{d}|\mathbf{n}, (\mathbf{d}, \mathbf{n}/\mathbf{d}) = \mathbf{1}\}$, are regular.

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For a positive integer k, we define

$$A_k(\mathbf{n}) = \{\mathbf{d} \in G : \mathbf{d}^k \in A(\mathbf{n}^k)\}.$$

It has been shown (see [13, p. 10], [32, p. 267]) that the A_k-convolution is regular whenever the A-convolution is regular. The symbol $(\mathbf{a}, \mathbf{b})_{A,k}$ denotes the greatest kth power divisor of $\mathbf{a}$ which belongs to $A(\mathbf{b})$.

The classical Ramanujan's sum $C(n; r)$ is defined by

$$C(n;r) = \sum_{\substack{m (\mathrm{mod}\, r) \\ (m,r)=1}} \exp(2\pi imn/r),$$

where n is a non-negative integer and r is a positive integer. Its well-known arithmetical representation is given by

$$C(n;r) = \sum_{d|(n,r)} d\mu(r/d).$$

In [14] the author together with P.J. McCarthy defined a generalized Ramanujan's sum by

$$C_{A,k}(n_1,\ldots,n_u;r) = \sum_{\substack{m_1,\ldots,m_u (\mathrm{mod}\, r^k) \\ ((m_i),r^k)_{A,k}=1}} \exp\big(2\pi i(m_1n_1 + \cdots + m_un_u)/r^k\big),$$

where $n_1, \ldots, n_u$ are non-negative integers, r is a positive integer and $(m_i) = (m_1, \ldots, m_u)$, the greatest common divisor of $m_1, \ldots, m_u$. We noted [14] that

$$C_{A,k}(n_1,\ldots,n_u;r) = \sum_{d^k \in A(((n_i),r^k)_{A,k})} d^{ku}\mu_{A_k}(r/d) = \sum_{\substack{d \in A_k(r) \\ d^k|(n_i)}} d^{ku}\mu_{A_k}(r/d),$$

where μ_{A_k} is the inverse of E, the function $\equiv 1$, with respect to the A_k-convolution. For an arithmetical semigroup this suggests we define

$$C_{A,k}(\mathbf{n}_1,\ldots,\mathbf{n}_u;\mathbf{r}) = \sum_{\mathbf{d}^k \in A(((\mathbf{n}_i),\mathbf{r}^k)_{A,k})} \|\mathbf{d}\|^{ku}\, \mu_{A_k}(\mathbf{r}/\mathbf{d}).$$

In [13] we defined a generalized Ramanujan's sum by

$$S_{A,k}^{f,g}(\mathbf{n}_1,\ldots,\mathbf{n}_u;\mathbf{r}) = \sum_{\mathbf{d}^k \in A(((\mathbf{n}_i),\mathbf{r}^k)_{A,k})} f(\mathbf{d})g(\mathbf{r}/\mathbf{d}).$$

In other words,

$$S^{f,g}_{A,k}(\mathbf{n}_1,\ldots,\mathbf{n}_u;\mathbf{r}) = \sum_{\substack{\mathbf{d}\in A_k(\mathbf{r})\\ \mathbf{d}^k|(\mathbf{n}_i)}} f(\mathbf{d})g(\mathbf{r}/\mathbf{d}) = \left(\chi((\mathbf{n}_i);(\cdot)^k)f_{A_k}g\right)(\mathbf{r}),$$

where $\chi(\mathbf{n};\mathbf{d}) = 1$ if $\mathbf{d}|\mathbf{n}$, and $= 0$ otherwise.

The purpose of [13] was to derive arithmetical identities of classical type involving that sum. The purpose of the present paper is to give more arithmetical identities for that sum. We shall also list a large number of known special cases of the given identities. At the end of this paper we shall note that two identities here can be extended to totally A-k-even functions $(\bmod\,\mathbf{r})$.

2. Preliminaries

We define an arithmetical function f to be quasi-A-multiplicative [13, p. 14] if $f(\mathbf{1}) \neq 0$ and $f(\mathbf{1})f(\mathbf{mn}) = f(\mathbf{m})f(\mathbf{n})$ whenever $\mathbf{m},\mathbf{n} \in A(\mathbf{mn})$. Quasi-$A$-multiplicative functions f with $f(\mathbf{1}) = 1$ are called A-multiplicative [45]. It is easy to see that an arithmetical function f with $f(\mathbf{1}) \neq 0$ is quasi-A-multiplicative if, and only if, $f/f(\mathbf{1})$ is A-multiplicative. Quasi-U-multiplicative functions are called quasi-multiplicative [19]. For those functions $f(\mathbf{1}) \neq 0$ and $f(\mathbf{1})f(\mathbf{mn}) = f(\mathbf{m})f(\mathbf{n})$ whenever $(\mathbf{m},\mathbf{n}) = \mathbf{1}$. All quasi-$A$-multiplicative functions are quasi-multiplicative.

Let $A^{(1)}$, $A^{(2)}$, $\ldots$, $A^{(u)}$ be regular convolutions. Then we define the $A^{(1)}A^{(2)}\cdots A^{(u)}$-convolution of arithmetical functions $f(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)$ and $g(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)$ by

$$(1)\qquad \begin{aligned} &f(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)_{A^{(1)}A^{(2)}\cdots A^{(u)}}g(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)\\ &= \sum_{\mathbf{d}_1\in A^{(1)}(\mathbf{n}_1)}\ \sum_{\mathbf{d}_2\in A^{(2)}(\mathbf{n}_2)}\cdots\sum_{\mathbf{d}_u\in A^{(u)}(\mathbf{n}_u)} f(\mathbf{d}_1,\mathbf{d}_2,\ldots,\mathbf{d}_u)\cdot\\ &\qquad\cdot g(\mathbf{n}_1/\mathbf{d}_1,\mathbf{n}_2/\mathbf{d}_2,\ldots,\mathbf{n}_u/\mathbf{d}_u). \end{aligned}$$

It is easy to see that an $A^{(1)}A^{(2)}\cdots A^{(u)}$-convolution of arithmetical functions is associative. Also, if h is an $A^{(i)}$-multiplicative function, then

$$(2)\quad \begin{aligned} &\left(f(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)_{A^{(1)}A^{(2)}\cdots A^{(u)}}g(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)\right)h(\mathbf{n}_i)\\ &= f(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)h(\mathbf{n}_i)_{A^{(1)}A^{(2)}\cdots A^{(u)}}g(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)h(\mathbf{n}_i). \end{aligned}$$

Let f be an arithmetical function of one variable and e, m, $u \in \mathbf{N}$, $u \geq 2$, $0 \leq m < u$. Then we define $P_e(f)(\mathbf{n}_1,\ldots,\mathbf{n}_m;\mathbf{n}_{m+1},\ldots,\mathbf{n}_u)$ to be the arithmetical function of u variables such that

$$(3)\quad P_e(f)(\mathbf{n}_1,\ldots,\mathbf{n}_m;\mathbf{n}_{m+1},\ldots,\mathbf{n}_u) = \begin{cases} f(\mathbf{n}_u), & \text{if } \mathbf{n}_1 = \cdots = \mathbf{n}_m = (\mathbf{n}_{m+1})^e\\ & \quad = \cdots = (\mathbf{n}_u)^e,\\ 0 & \text{otherwise.}\end{cases}$$

In particular, we denote

$$P_1(f)(\mathbf{n}_1,\ldots,\mathbf{n}_m;\mathbf{n}_{m+1},\ldots,\mathbf{n}_u)=P(f)(\mathbf{n}_1,\ldots,\mathbf{n}_u).$$

If $m=0$, then we have

$$P_e(f)(\mathbf{n}_1,\ldots,\mathbf{n}_m;\mathbf{n}_{m+1},\ldots,\mathbf{n}_u)=P(f)(\mathbf{n}_1,\ldots,\mathbf{n}_u). \tag{4}$$

We note that some special cases of the function $P_e(f)$ can be found in [39, p. 627] and [42, p. 86].

It is easy to see that if f, g are arithmetical functions of one variable and $u\geq 2$, $1\leq i\leq u$, then

$$g(\mathbf{n}_i)P(f)(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)=P(fg)(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u). \tag{5}$$

Also, if f, g are arithmetical functions of one variable, $1\leq j\leq u$ and

$$A^{(j)}(\mathbf{n})\subseteq A^{(i)}(\mathbf{n})$$

whenever $\mathbf{n}\in G$, $1\leq i\leq u$, $i\neq j$, then

$$\begin{aligned}P(f)(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)_{A^{(1)}A^{(2)}\cdots A^{(u)}}P(g)(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u)\\ =P(f\ _{A^{(j)}}\ g)(\mathbf{n}_1,\mathbf{n}_2,\ldots,\mathbf{n}_u).\end{aligned} \tag{6}$$

3. Identities

Theorem 1. *Suppose f and g are arithmetical functions and $0\leq m\leq u$. Then for $\mathbf{n}_1,\ldots,\mathbf{n}_u$, $\mathbf{r}\in G$*

$$\begin{aligned}&S_{A,k}^{f,g}(\mathbf{n}_1,\ldots,\mathbf{n}_m,(\mathbf{n}_{m+1})^k,\ldots,(\mathbf{n}_u)^k;\mathbf{r})\\ &\quad=E(\mathbf{n}_1)\cdots E(\mathbf{n}_u)g(\mathbf{r})_{D\cdots DA_k}P_k(f)(\mathbf{n}_1,\ldots,\mathbf{n}_m;\mathbf{n}_{m+1},\ldots,\mathbf{n}_u,\mathbf{r}),\end{aligned}$$

where $E(\mathbf{n})=1$ for all $\mathbf{n}\in G$.

Proof. By (1) and (3),

$$\begin{aligned}&E(\mathbf{n}_1)\cdots E(\mathbf{n}_u)g(\mathbf{r})_{D\cdots DA_k}P_k(f)(\mathbf{n}_1,\ldots,\mathbf{n}_m;\mathbf{n}_{m+1},\ldots,\mathbf{n}_u,\mathbf{r})\\ &\quad=\sum_{\mathbf{d}_1|\mathbf{n}_1}\cdots\sum_{\mathbf{d}_u|\mathbf{n}_u}\sum_{\mathbf{d}\in A_k(\mathbf{r})}E(\mathbf{n}_1/\mathbf{d}_1)\cdots E(\mathbf{n}_u/\mathbf{d}_u)g(\mathbf{r}/\mathbf{d})\\ &\qquad\cdot P_k(f)(\mathbf{d}_1,\ldots,\mathbf{d}_m;\mathbf{d}_{m+1},\ldots,\mathbf{d}_u,\mathbf{d})\\ &\quad=\sum_{\substack{\mathbf{d}\in A_k(\mathbf{r}),\mathbf{d}|\mathbf{n}_{m+1},\ldots,\mathbf{n}_u\\ \mathbf{d}^k|\mathbf{n}_1,\ldots,\mathbf{n}_m}}g(\mathbf{r}/\mathbf{d})P_k(f)(\mathbf{d}^k,\ldots,\mathbf{d}^k;\mathbf{d},\ldots,\mathbf{d},\mathbf{d})\\ &\quad=\sum_{\substack{\mathbf{d}\in A_k(\mathbf{r}),\mathbf{d}|\mathbf{n}_{m+1},\ldots,\mathbf{n}_u\\ \mathbf{d}^k|\mathbf{n}_1,\ldots,\mathbf{n}_m}}g(\mathbf{r}/\mathbf{d})f(\mathbf{d})\\ &\quad=S_{A,k}^{f,g}(\mathbf{n}_1,\ldots,\mathbf{n}_m,(\mathbf{n}_{m+1})^k,\ldots,(\mathbf{n}_u)^k;\mathbf{r}),\end{aligned}$$

which was to be proved.

Theorem 2. *Suppose $h_1, \ldots, h_u$ are quasi-D-multiplicative functions, h is a quasi-A_k-multiplicative function and f, g, H are arbitrary arithmetical functions. Then for $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r} \in G$*

$$\begin{aligned}(h_1 \cdots h_u h)(\mathbf{1}) &\sum_{\substack{\mathbf{d}\in A_k(\mathbf{r})\\ \mathbf{d}|(\mathbf{n}_i)}} S_{A,k}^{f,g}((\mathbf{n}_1/\mathbf{d})^k, \ldots, (\mathbf{n}_u/\mathbf{d})^k; \mathbf{r}/\mathbf{d}) \\ &\cdot h_1(\mathbf{n}_1/\mathbf{d}) \cdots h_u(\mathbf{n}_u/\mathbf{d}) h(\mathbf{r}/\mathbf{d}) H(\mathbf{d}) \\ &= \sum_{\substack{\mathbf{d}\in A_k(\mathbf{r})\\ \mathbf{d}|(\mathbf{n}_i)}} h_1(\mathbf{n}_1/\mathbf{d}) \cdots h_u(\mathbf{n}_u/\mathbf{d})(hg)(\mathbf{r}/\mathbf{d})\big((fh_1 \cdots h_u h)_{A_k} H\big)(\mathbf{d}).\end{aligned}$$

Proof. It suffices to consider the case $h_1(\mathbf{1}) = \cdots = h_u(\mathbf{1}) = h(\mathbf{1}) = 1$. Let L denote the left-hand side of the identity in Theorem 2. Then, by (1)–(6) and Theorem 1,

$$\begin{aligned} L &= \Big(E(\mathbf{n}_1) \cdots E(\mathbf{n}_u) g(\mathbf{r})_D \cdots {}_{DA_k} P(f)(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r})\Big) \\ &\quad \cdot h_1(\mathbf{n}_1) \cdots h_u(\mathbf{n}_u) h(\mathbf{r})_D \cdots {}_{DA_k} P(H)(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r}) \\ &= \Big(h_1(\mathbf{n}_1) \cdots h_u(\mathbf{n}_u)(hg)(\mathbf{r})_D \cdots {}_{DA_k} h_1(\mathbf{n}_1) \cdots h_u(\mathbf{n}_u) h(\mathbf{r}) \\ &\quad \cdot P(f)(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r})\Big)_D \cdots {}_{DA_k} P(H)(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r}) \\ &= \Big(h_1(\mathbf{n}_1) \cdots h_u(\mathbf{n}_u)(hg)(\mathbf{r})_D \cdots {}_{DA_k} P(fh_1 \cdots h_u h)(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r})\Big) \\ &\quad \cdot {}_D \cdots {}_{DA_k} P(H)(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r}) \\ &= h_1(\mathbf{n}_1) \cdots h_u(\mathbf{n}_u)(hg)(\mathbf{r})_D \cdots {}_{DA_k} \\ &\quad \cdot \Big(P(fh_1 \cdots h_u h)(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r})_D \cdots {}_{DA_k} P(H)(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r})\Big) \\ &= h_1(\mathbf{n}_1) \cdots h_u(\mathbf{n}_u)(hg)(\mathbf{r})_D \cdots {}_{DA_k} P\big((fh_1 \cdots h_u h)_{A_k} H\big)(\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r}). \end{aligned}$$

We thus arrive at our result.

Theorem 3. *Suppose H is a quasi-A_k-multiplicative function, H_1, H_2, $\ldots$, H_u are quasi-D-multiplicative functions and f, g, h, h_1, h_2, $\ldots$, h_u are arbitrary arithmetical functions. Then for $\mathbf{n}_1, \mathbf{n}_2, \ldots, \mathbf{n}_u, \mathbf{r} \in G$, $a_1, \ldots, a_u = 0, 1$*

$$(H_1 \cdots H_u H)(\mathbf{1}) \sum_{\mathbf{d}_1|\mathbf{n}_1} \cdots \sum_{\mathbf{d}_u|\mathbf{n}_u} \sum_{\boldsymbol{\delta}\in A_k(\mathbf{r})} S_{A,k}^{f,g}(\mathbf{d}_1^{k^{a_1}}, \ldots, \mathbf{d}_u^{k^{a_u}}; \boldsymbol{\delta})$$
$$\cdot H_1(\mathbf{d}_1) \cdots H_u(\mathbf{d}_u) H(\boldsymbol{\delta}) h_1(\mathbf{n}_1/\mathbf{d}_1) \cdots h_u(\mathbf{n}_u/\mathbf{d}_u) h(\mathbf{r}/\boldsymbol{\delta})$$
$$= \sum_{\substack{\mathbf{d}\in A_k(\mathbf{r}) \\ \mathbf{d}^k|(\mathbf{n}_i^{k^{a_i}})}} (fH)(\mathbf{d}) H_1\big(\mathbf{d}^{k^{1-a_1}}\big) \cdots H_u\big(\mathbf{d}^{k^{1-a_u}}\big)\big(h_{A_k}(gH)\big)(\mathbf{r}/\mathbf{d})$$
$$\cdot (H_1 {}_D h_1)\big(\mathbf{n}_1/\mathbf{d}^{k^{1-a_1}}\big) \cdots (H_u {}_D h_u)\big(\mathbf{n}_u/\mathbf{d}^{k^{1-a_u}}\big).$$

Proof. It suffices to consider the case $H_1(\mathbf{1}) = \cdots = H_u(\mathbf{1}) = H(\mathbf{1}) = 1$. Further, without loss of generality we may assume $a_1 = \ldots = a_m = 0$, $a_{m+1} = \cdots = a_u = 1$ $(0 \le m \le u)$. Then, by (1), the left-hand side of the identity in Theorem 3 can be written as

$$h_1(\mathbf{n}_1) \cdots h_u(\mathbf{n}_u) h(\mathbf{r})_D \ldots {}_D {}_{A_k} S_{A,k}^{f,g}(\mathbf{n}_1, \ldots, \mathbf{n}_m, (\mathbf{n}_{m+1})^k, \ldots, (\mathbf{n}_u)^k; \mathbf{r})$$
$$\cdot H_1(\mathbf{n}_1) \cdots H_u(\mathbf{n}_u) H(\mathbf{r}).$$

Thus, applying Theorem 1 and formulas (2), (3), we have the theorem.

Remark. The functions h_1, ..., h_u, h are of great importance in Theorem 3. In fact, if $h = E_0$, defined by $E_0(\mathbf{1}) = 1$ and $E_0(\mathbf{n}) = 0$ for $\mathbf{n} \ne \mathbf{1}$, then the summation over $\mathbf{d}_1$, ..., $\mathbf{d}_u$, $\boldsymbol{\delta}$ reduces to the summation over $\mathbf{d}_1$, ..., $\mathbf{d}_u$. A similar reduction is valid with respect to any subset of the set of functions h_1, ..., h_u, h.

Notation. For an arithmetical function f, denote

$$f^\wedge(A, \mathbf{e}; x) = \sum_{\substack{\|\mathbf{n}\|\le x \\ \mathbf{n}\in A(\mathbf{ne})}} f(\mathbf{n}) \qquad (x \in \mathbf{R}),$$

$$f^\wedge(x) = f^\wedge(D, \mathbf{e}; x) = \sum_{\|\mathbf{n}\|\le x} f(\mathbf{n}) \qquad (x \in \mathbf{R}).$$

Theorem 4. *Suppose f, g, h, h_1, h_2, ..., h_m $(0 \le m \le u)$ are arithmetical functions. Then for x_1, ..., x_m, $y > 0$ $(x_1, \ldots, x_m, y \in \mathbf{R})$, $\mathbf{n}_{m+1}$, ..., $\mathbf{n}_u$, $\mathbf{r} \in G$, a_1, ..., $a_m = 0, 1$*

$$(7) \qquad \sum_{\|\mathbf{i}_1\|\le x_1} \cdots \sum_{\|\mathbf{i}_m\|\le x_m} \sum_{\|\mathbf{j}\|\le y} S_{A,k}^{f,g}\Big(\mathbf{i}_1^{k^{a_1}}, \ldots, \mathbf{i}_m^{k^{a_m}}, \mathbf{n}_{m+1}, \ldots, \mathbf{n}_u; \mathbf{j}\Big)$$

$$\cdot h_1^\wedge(x_1/\|\mathbf{i}_1\|)\cdots h_m^\wedge(x_m/\|\mathbf{i}_m\|)h^\wedge(A_k,\mathbf{j};y/\|\mathbf{j}\|)$$
$$= \sum_{\substack{\|\mathbf{d}\|^k\le\min\{y^k,x_1^{k^{a_1}},\dots,x_m^{k^{a_m}}\}\\ \mathbf{d}^k|\mathbf{n}_{m+1},\dots,\mathbf{n}_u}} f(\mathbf{d})(h_1{}_DE)^\wedge(x_1/\|\mathbf{d}\|^{k^{1-a_1}})$$
$$\cdots(h_m{}_DE)^\wedge(x_m/\|\mathbf{d}\|^{k^{1-a_m}})(h\ {}_{A_k}g)^\wedge(A_k,\mathbf{d};y/\|\mathbf{d}\|),$$

$$(8)\quad \sum_{\|\mathbf{i}_1\|\le x_1}\cdots\sum_{\|\mathbf{i}_m\|\le x_m} S_{A,k}^{f,g}\left(\mathbf{i}_1^{k^{a_1}},\dots,\mathbf{i}_m^{k^{a_m}},\mathbf{n}_{m+1},\dots,\mathbf{n}_u;\mathbf{r}\right)$$
$$\cdot h_1^\wedge(x_1/\|\mathbf{i}_1\|)\cdots h_m^\wedge(x_m/\|\mathbf{i}_m\|)$$
$$= \sum_{\substack{\|\mathbf{d}\|^k\le\min\{x_1^{k^{a_1}},\dots,x_m^{k^{a_m}}\}\\ \mathbf{d}^k|\mathbf{n}_{m+1},\dots,\mathbf{n}_u;\mathbf{d}\in A_k(\mathbf{r})}} f(\mathbf{d})g(\mathbf{r}/\mathbf{d})(h_1{}_DE)^\wedge(x_1/\|\mathbf{d}\|^{k^{1-a_1}})\cdots$$
$$\cdot(h_m{}_DE)^\wedge(x_m/\|\mathbf{d}\|^{k^{1-a_m}}),\quad m\ge 1.$$

Proof. Let L denote the left-hand side of (7). Then, using the notation of χ given in the introduction, we can write

$$L=\sum_{\|\mathbf{i}_1\|\le x_1}\cdots\sum_{\|\mathbf{i}_m\|\le x_m}\sum_{\|\mathbf{j}\|\le y}\sum_{\substack{\mathbf{d}\in A_k(\mathbf{j})\\ \mathbf{d}^{k^{1-a_\nu}}|\mathbf{i}_\nu,\nu=1,\dots,m}}\chi\left((\mathbf{n}_{m+1},\dots,\mathbf{n}_u);\mathbf{d}^k\right)$$
$$\cdot f(\mathbf{d})g(\mathbf{j}/\mathbf{d})\sum_{\|\mathbf{b}_1\|\le x_1/\|\mathbf{i}_1\|}h_1(\mathbf{b}_1)\cdots\sum_{\|\mathbf{b}_m\|\le x_m/\|\mathbf{i}_m\|}h_m(\mathbf{b}_m)\sum_{\substack{\|\mathbf{a}\|\le y/\|\mathbf{j}\|\\ \mathbf{a}\in A_k(\mathbf{ja})}}h(\mathbf{a}).$$

Now we shall change the order of summation. It can be proved that the rule

$$(\mathbf{c}_1,\dots,\mathbf{c}_m,\mathbf{v},\mathbf{d},\mathbf{b}_1,\dots,\mathbf{b}_m,\mathbf{a})\to(\mathbf{c}_1/\mathbf{b}_1,\dots,\mathbf{c}_m/\mathbf{b}_m,\mathbf{v}/\mathbf{a},\mathbf{d},\mathbf{b}_1,\dots,\mathbf{b}_m,\mathbf{a})$$

defines a bijection from the set of $(2m+3)$-tuples $(\mathbf{c}_1,\dots,\mathbf{c}_m,\mathbf{v},\mathbf{d},\mathbf{b}_1,\dots,\mathbf{b}_m,\mathbf{a})$ satisfying

$$\|\mathbf{c}_1\|\le x_1,\dots,\|\mathbf{c}_m\|\le x_m,\quad \|\mathbf{v}\|\le y,$$
$$\mathbf{d}\in A_k(\mathbf{v}),\quad \mathbf{d}^{k^{1-a_1}}|\mathbf{c}_1,\dots,\mathbf{d}^{k^{1-a_m}}|\mathbf{c}_m,$$
$$\mathbf{b}_1|\mathbf{c}_1\mathbf{d}^{-k^{1-a_1}},\dots,\mathbf{b}_m|\mathbf{c}_m\mathbf{d}^{-k^{1-a_m}},\qquad \mathbf{a}\in A_k(\mathbf{v}/\mathbf{d})$$

onto the set of $(2m+3)$-tuples $(\mathbf{i}_1,\dots,\mathbf{i}_m,\mathbf{j},\mathbf{d},\mathbf{b}_1,\dots,\mathbf{b}_m,\mathbf{a})$ satisfying

$$\|\mathbf{i}_1\|\le x_1,\dots,\|\mathbf{i}_m\|\le x_m,\qquad \|\mathbf{j}\|\le y,$$

$$\mathbf{d} \in A_k(\mathbf{j}), \qquad \mathbf{d}^{k^{1-a_1}} | \mathbf{i}_1, \ldots, \mathbf{d}^{k^{1-a_m}} | \mathbf{i}_m,$$

$$\|\mathbf{b}_1\| \leq x_1/\|\mathbf{i}_1\|, \ldots, \|\mathbf{b}_m\| \leq x_m/\|\mathbf{i}_m\|, \quad \|\mathbf{a}\| \leq y/\|\mathbf{j}\|, \quad \mathbf{a} \in A_k(\mathbf{ja}).$$

Thus we obtain

$$L = \sum_{\|\mathbf{c}_1\| \leq x_1} \cdots \sum_{\|\mathbf{c}_m\| \leq x_m} \sum_{\|\mathbf{v}\| \leq y} \sum_{\substack{\mathbf{d} \in A_k(\mathbf{v}) \\ \mathbf{d}^{k^{1-a_v}} | \mathbf{c}_v, v=1,\ldots,m}} \chi\big((\mathbf{n}_{m+1}, \ldots, \mathbf{n}_u); \mathbf{d}^k\big) f(\mathbf{d})$$

$$\cdot \sum_{\mathbf{b}_1 | \mathbf{c}_1 \mathbf{d}^{-k^{1-a_1}}} h_1(\mathbf{b}_1) \ldots \sum_{\mathbf{b}_m | \mathbf{c}_m \mathbf{d}^{-k^{1-a_m}}} h_m(\mathbf{b}_m) \sum_{\mathbf{a} \in A_k(\mathbf{v}/\mathbf{d})} h(\mathbf{a}) g\big((\mathbf{v}/\mathbf{d})/\mathbf{a}\big)$$

$$= \sum_{\|\mathbf{c}_1\| \leq x_1} \cdots \sum_{\|\mathbf{c}_m\| \leq x_m} \sum_{\|\mathbf{v}\| \leq y} \sum_{\substack{\mathbf{d} \in A_k(\mathbf{v}) \\ \mathbf{d}^{k^{1-a_v}} | \mathbf{c}_v, v=1,\ldots,m}} \chi\big((\mathbf{n}_{m+1}, \ldots, \mathbf{n}_u); \mathbf{d}^k\big) f(\mathbf{d})$$

$$\cdot (h_1 {}_D E)\big(\mathbf{c}_1 \mathbf{d}^{-k^{1-a_1}}\big) \cdots (h_m {}_D E)\big(\mathbf{c}_m \mathbf{d}^{-k^{1-a_m}}\big)(h_{A_k} g)(\mathbf{v}/\mathbf{d}).$$

Further, it can be proved that the rule

$$(\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{t}, \mathbf{d}) \to (\mathbf{e}_1 \mathbf{d}^{k^{1-a_1}}, \ldots, \mathbf{e}_m \mathbf{d}^{k^{1-a_m}}, \mathbf{td}, \mathbf{d})$$

defines a bijection from the set of $(m+2)$-tuples $(\mathbf{e}_1, \ldots, \mathbf{e}_m, \mathbf{t}, \mathbf{d})$ satisfying

$$\|\mathbf{d}\|^k \leq \min\{y^k, x_1^{k^{a_1}}, \ldots, x_m^{k^{a_m}}\},$$

$$\|\mathbf{e}_1\| \leq x_1/\|\mathbf{d}\|^{k^{1-a_1}}, \ldots, \|\mathbf{e}_m\| \leq x_m/\|\mathbf{d}\|^{k^{1-a_m}}, \quad \|\mathbf{t}\| \leq y/\|\mathbf{d}\|, \quad \mathbf{t} \in A_k(\mathbf{td}),$$

onto the set of $(m+2)$-tuples $(\mathbf{c}_1, \ldots, \mathbf{c}_m, \mathbf{v}, \mathbf{d})$ satisfying

$$\|\mathbf{c}_1\| \leq x_1, \ldots, \|\mathbf{c}_m\| \leq x_m, \quad \|\mathbf{v}\| \leq y, \quad \mathbf{d} \in A_k(\mathbf{v}), \quad \mathbf{d}^{k^{1-a_1}} | \mathbf{c}_1, \ldots, \mathbf{d}^{k^{1-a_m}} | \mathbf{c}_m.$$

Thus

$$L = \sum_{\|\mathbf{d}\|^k \leq \min\{y^k, x_1^{k^{a_1}}, \ldots, x_m^{k^{a_m}}\}} \chi\big((\mathbf{n}_{m+1}, \ldots, \mathbf{n}_u); \mathbf{d}^k\big) f(\mathbf{d})$$

$$\cdot \sum_{\|\mathbf{e}_1\| \leq x_1/\|\mathbf{d}\|^{k^{1-a_1}}} (h_1 {}_D E)(\mathbf{e}_1) \cdots \sum_{\|\mathbf{e}_m\| \leq x_m/\|\mathbf{d}\|^{k^{1-a_m}}} (h_m {}_D E)(\mathbf{e}_m)$$

$$\cdot \sum_{\substack{\|\mathbf{t}\| \leq y/\|\mathbf{d}\| \\ \mathbf{t} \in A_k(\mathbf{td})}} (h_{A_k} g)(\mathbf{t})$$

$$= \sum_{\substack{\|\mathbf{d}\|^k \leq \min\{y^k, x_1^{k^{a_1}}, \ldots, x_m^{k^{a_m}}\} \\ \mathbf{d}^k | \mathbf{n}_{m+1}, \ldots, \mathbf{n}_u}} f(\mathbf{d})(h_1 {}_D E)^\wedge(x_1/\|\mathbf{d}\|^{k^{1-a_1}}) \cdots$$

$$\cdot (h_m {}_D E)^\wedge(x_m/\|\mathbf{d}\|^{k^{1-a_m}})(h_{A_k} g)^\wedge(A_k, \mathbf{d}; y/\|\mathbf{d}\|).$$

This proves (7). The proof of (8) goes through on similar lines.

Theorem 5. *Suppose h_1, h_2, ..., h_m $(0 \le m \le u)$ are quasi-D-multiplicative functions, h is a quasi-A_k-multiplicative function and f, g are arbitrary arithmetical functions. Then for $x_1, \dots, x_m, y > 0$ $(x_1, \dots, x_m, y \in \mathbf{R})$, $\mathbf{n}_{m+1}, \dots, \mathbf{n}_u$, $\mathbf{r} \in G$, $a_1, \dots, a_m = 0, 1$*

$$(9)\ (h_1 \cdots h_m h)(\mathbf{1}) \sum_{\|\mathbf{i}_1\| \le x_1} \cdots \sum_{\|\mathbf{i}_m\| \le x_m} \sum_{\|\mathbf{j}\| \le y} S_{A,k}^{f,g}(\mathbf{i}_1^{k^{a_1}}, \dots, \mathbf{i}_m^{k^{a_m}}, \mathbf{n}_{m+1}, \dots, \mathbf{n}_u; \mathbf{j}) \cdot h_1(\mathbf{i}_1) \cdots h_m(\mathbf{i}_m)\, h(\mathbf{j})$$
$$= \sum_{\substack{\|\mathbf{d}\|^k \le \min\{y^k, x_1^{k^{a_1}}, \dots, x_m^{k^{a_m}}\} \\ \mathbf{d}^k | \mathbf{n}_{m+1}, \dots, \mathbf{n}_u}} f(\mathbf{d}) h(\mathbf{d}) h_1(\mathbf{d}^{k^{1-a_1}}) \cdots h_m(\mathbf{d}^{k^{1-a_m}}) \cdot h_1^{\wedge}(x_1/\|\mathbf{d}\|^{k^{1-a_1}}) \cdots h_m^{\wedge}(x_m/\|\mathbf{d}\|^{k^{1-a_m}}) (gh)^{\wedge}(A_k, \mathbf{d}; y/\|\mathbf{d}\|),$$

$$(10)\quad (h_1 \cdots h_m)(\mathbf{1}) \sum_{\|\mathbf{i}_1\| \le x_1} \cdots \sum_{\|\mathbf{i}_m\| \le x_m} S_{A,k}^{f,g}(\mathbf{i}_1^{k^{a_1}}, \dots, \mathbf{i}_m^{k^{a_m}}, \mathbf{n}_{m+1}, \dots, \mathbf{n}_u; \mathbf{r}) \cdot h_1(\mathbf{i}_1) \cdots h_m(\mathbf{i}_m)$$
$$= \sum_{\substack{\|\mathbf{d}\|^k \le \min\{x_1^{k^{a_1}}, \dots, x_m^{k^{a_m}}\} \\ \mathbf{d}^k | \mathbf{n}_{m+1}, \dots, \mathbf{n}_u; \mathbf{d} \in A_k(\mathbf{r})}} f(\mathbf{d}) g(\mathbf{r}/\mathbf{d}) h_1(\mathbf{d}^{k^{1-a_1}}) \cdots h_m(\mathbf{d}^{k^{1-a_m}}) \cdot h_1^{\wedge}(x_1/\|\mathbf{d}\|^{k^{1-a_1}}) \cdots h_m^{\wedge}(x_m/\|\mathbf{d}\|^{k^{1-a_m}}), \quad m \ge 1.$$

Theorem 5 can be proved in a similar way to Theorem 4.

Theorem 6. *Suppose z_1, z_2, ..., $z_s \in \mathbf{C}$ $(1 \le s \le u)$ and denote $z_1 + z_2 + \cdots + z_s = z$. Let f be an arithmetical function such that $f(\mathbf{m}) \ne 0$ for all $\mathbf{m} \in G$. Then for $\mathbf{n}, \mathbf{r} \in G$*

$$S_{A,k}^{f^z, \mu_{A_k}}(\mathbf{n}_1, \dots, \mathbf{n}_u; \mathbf{r}) = \sum_{\substack{\mathbf{d}_1, \dots, \mathbf{d}_s \in A_k(\mathbf{r}) \\ [\mathbf{d}_1, \dots, \mathbf{d}_s] = \mathbf{r}}} S_{A,k}^{f^{z_1}, \mu_{A_k}}(\mathbf{n}_1, \mathbf{n}_{s+1}, \dots, \mathbf{n}_u; \mathbf{d}_1) \cdots S_{A,k}^{f^{z_s}, \mu_{A_k}}(\mathbf{n}_s, \mathbf{n}_{s+1}, \dots, \mathbf{n}_u; \mathbf{d}_s),$$

where the symbol $[\cdots]$ is used for the least common multiple.

Proof. Let $R(\mathbf{r})$ denote the right-hand side of the identity of Theorem 6.

Then

$$\sum_{\mathbf{d}\in A_k(\mathbf{r})} R(\mathbf{d}) = \sum_{\mathbf{d}\in A_k(\mathbf{r})} \sum_{\substack{\mathbf{d}_1,\ldots,\mathbf{d}_s\in A_k(\mathbf{d})\\ [\mathbf{d}_1,\ldots,\mathbf{d}_s]=\mathbf{d}}} S_{A,k}^{f^{z_1},\mu_{A_k}}(\mathbf{n}_1,\mathbf{n}_{s+1},\ldots,\mathbf{n}_u;\mathbf{d}_1)\cdots$$
$$\cdot S_{A,k}^{f^{z_s},\mu_{A_k}}(\mathbf{n}_s,\mathbf{n}_{s+1},\ldots,\mathbf{n}_u;\mathbf{d}_s)$$
$$= \prod_{j=1}^{s} \sum_{\mathbf{d}_j\in A_k(\mathbf{r})} S_{A,k}^{f^{z_j},\mu_{A_k}}(\mathbf{n}_j,\mathbf{n}_{s+1},\ldots,\mathbf{n}_u;\mathbf{d}_j)$$
$$= \prod_{j=1}^{s} \chi((\mathbf{n}_j,\mathbf{n}_{s+1},\ldots,\mathbf{n}_u);\mathbf{r}^k) f^{z_j}(\mathbf{r}) = \chi((\mathbf{n}_i);\mathbf{r}^k) f^{z}(\mathbf{r}),$$

that is,

$$(RA_k E)(\mathbf{r}) = \chi\big((\mathbf{n}_i);\mathbf{r}^k\big) f^z(\mathbf{r}),$$

where χ is the function defined in the introduction. Thus

$$R(\mathbf{r}) = \big(\chi\big((\mathbf{n}_i);(\cdot)^k\big) f^z A_k \mu_{A_k}\big)(\mathbf{r}) = S_{A,k}^{f^z,\mu_{A_k}}(\mathbf{n}_1,\ldots,\mathbf{n}_u;\mathbf{r}).$$

This completes the proof.

Remark. It is easy to see that the value of the sum

$$S_{A,k}^{f,g}(\mathbf{n}_1,\ldots,\mathbf{n}_u;\mathbf{r})$$

is independent of the order of the variables $\mathbf{n}_1,\ldots,\mathbf{n}_u$. Thus Theorems 1, 4, 5 and 6 can be further generalized by rearranging the first u variables into an arbitrary order.

Theorem 7. *Suppose g, h and H are arithmetical functions such that h, H are quasi-A_k-multiplicative and $hA_k gH = (hgH)(\mathbf{1})E_0$, where $E_0(\mathbf{1}) = 1$ and $E_0(\mathbf{n}) = 0$ for $\mathbf{n} \neq \mathbf{1}$. Let f be an arbitrary arithmetical function. Then for $\mathbf{r} \in G$, $a = 0,1$*

$$\text{(11)} \qquad \sum_{\mathbf{d}\in A_k(\mathbf{r})} S_{A,k}^{f,g}(\mathbf{d}^{k^a};\mathbf{r}/\mathbf{d}) h(\mathbf{d}) H(\mathbf{r}/\mathbf{d}) = g(\mathbf{1})(fH)(\mathbf{m}) h(\mathbf{m}^{k^{1-a}})$$

if $\mathbf{r} = \mathbf{m}^{k^{1-a}+1}, \mathbf{m} \in A_k(\mathbf{r})$, and $= 0$ otherwise.

Proof. Denote by L the left-hand side of (11). Then

$$L = \sum_{\mathbf{d}\in A_k(\mathbf{r})} \sum_{\substack{\boldsymbol{\delta}\in A_k(\mathbf{r}/\mathbf{d})\\ \boldsymbol{\delta}^k | \mathbf{d}^{k^a}}} f(\boldsymbol{\delta}) g\big(\mathbf{r}/(\mathbf{d}\boldsymbol{\delta})\big) h(\mathbf{d}) H(\mathbf{r}/\mathbf{d}).$$

It can be proved that

$$\mathbf{d} \in A_k(\mathbf{r}), \qquad \boldsymbol{\delta} \in A_k(\mathbf{r}/\mathbf{d}), \qquad \boldsymbol{\delta}^k | \mathbf{d}^{k^a}$$

if, and only if,

$$\boldsymbol{\delta} \in A_k(\mathbf{r}), \qquad \boldsymbol{\delta}^{k^{1-a}+1} \in A_k(\mathbf{r}), \qquad \mathbf{d} = \boldsymbol{\delta}^{k^{1-a}}\mathbf{e}, \quad \mathbf{e} \in A_k(\mathbf{r}/\boldsymbol{\delta}^{k^{1-a}+1}).$$

Therefore

$$\begin{aligned} L &= \sum_{\substack{\boldsymbol{\delta}\in A_k(\mathbf{r}) \\ \boldsymbol{\delta}^{k^{1-a}+1}\in A_k(\mathbf{r})}} \sum_{\mathbf{e}\in A_k(\mathbf{r}/\boldsymbol{\delta}^{k^{1-a}+1})} f(\boldsymbol{\delta})g\big((\mathbf{r}/\boldsymbol{\delta}^{k^{1-a}+1})/\mathbf{e}\big)h(\boldsymbol{\delta}^{k^{1-a}}\mathbf{e})H\big(\mathbf{r}/(\boldsymbol{\delta}^{k^{1-a}}\mathbf{e})\big) \\ &= \sum_{\substack{\boldsymbol{\delta}\in A_k(\mathbf{r}) \\ \boldsymbol{\delta}^{k^{1-a}+1}\in A_k(\mathbf{r})}} \frac{(fH)(\boldsymbol{\delta})}{H(\mathbf{1})}\frac{h(\boldsymbol{\delta}^{k^{1-a}})}{h(\mathbf{1})}(h_{A_k}\, gH)\big(\mathbf{r}/\boldsymbol{\delta}^{k^{1-a}+1}\big) \\ &= \sum_{\substack{\boldsymbol{\delta}\in A_k(\mathbf{r}) \\ \boldsymbol{\delta}^{k^{1-a}+1}\in A_k(\mathbf{r})}} (fH)(\boldsymbol{\delta})h\big(\boldsymbol{\delta}^{k^{1-a}}\big)g(\mathbf{1})E_0\big(\mathbf{r}/\boldsymbol{\delta}^{k^{1-a}+1}\big). \end{aligned}$$

We thus arrive at our result.

Theorem 8. *Suppose f is a quasi-A_k-multiplicative function and $\mathbf{a}$, $\mathbf{b}$, $\mathbf{r} \in G$ with $\mathbf{a}$, $\mathbf{b} \in A_k(\mathbf{r})$. Then*

$$\sum_{\mathbf{d}\in A_k(\mathbf{r})} S_{A,k}^{f,\mu_{A_k}}\big((\mathbf{r}/\mathbf{d})^k;\mathbf{a}\big)S_{A,k}^{f,\mu_{A_k}}\big((\mathbf{r}/\mathbf{b})^k;\mathbf{d}\big) = \begin{cases} f(\mathbf{1})f(\mathbf{r}) & \text{if } \mathbf{a}=\mathbf{b}, \\ 0 & \text{if } \mathbf{a}\neq\mathbf{b}. \end{cases}$$

Theorem 9. *Suppose f is a quasi-A_k-multiplicative function such that $f(\mathbf{r}) \neq 0$ for all $\mathbf{r} \in G$. Then for all $\mathbf{n}$, $\mathbf{r} \in G$ and integers a, b*

$$\sum_{\mathbf{d}\in A_k(\mathbf{r})} S_{A,k}^{f^a,\mu_{A_k}}(\mathbf{d}^k;\mathbf{r})S_{A,k}^{f^b,\mu_{A_k}}(\mathbf{n};\mathbf{r}/\mathbf{d}) = f^a(\mathbf{r})f(\mathbf{1})^b(f^{a-b}{}_{A_k}\mu_{A_k})(\boldsymbol{\delta})f^{b-a}(\boldsymbol{\delta}),$$

where $\boldsymbol{\delta}^k = (\mathbf{n},\mathbf{r}^k)_{A,k}$.

Theorem 10. *Suppose f is an A_k-multiplicative function with $(f_{A_k}\mu_{A_k})(\mathbf{r}) \neq 0$ for all $\mathbf{r} \in G$. Then for all $\mathbf{m}$, $\mathbf{n}$, $\mathbf{r} \in G$*

$$\sum_{\mathbf{d}\in A_k(\mathbf{r})} \frac{S_{A,k}^{f,\mu_{A_k}}(\mathbf{n};\mathbf{d})S_{A,k}^{f,\mu_{A_k}}(\mathbf{m};\mathbf{d})}{(f_{A_k}\mu_{A_k})(\mathbf{d})} = \frac{f(\mathbf{r})}{(f_{A_k}\mu_{A_k})(\mathbf{r}/\boldsymbol{\delta})}$$

if $(\mathbf{n},\mathbf{r}^k)_{A,k} = (\mathbf{m},\mathbf{r}^k)_{A,k} = \boldsymbol{\delta}^k$, and $= 0$ otherwise.

Theorem 11. *Suppose f is a quasi-A_k-multiplicative function and $\mathbf{n}$, $\mathbf{r} \in G$. Denote $\mathbf{a} = \mathbf{r}/\gamma_{A_k}(\mathbf{r})$, $\mathbf{b}^k = \left(\mathbf{n}, \gamma(\mathbf{r})^k\right)_{A,k}$, where $\gamma_{A_k}(\mathbf{1}) = \mathbf{1}$ and for $\mathbf{r} \neq \mathbf{1}$, $\gamma_{A_k}(\mathbf{r})$ is the product of distinct prime divisors of $\mathbf{r}$. Then*

$$f(\mathbf{1}) S_{A,k}^{f,\mu_{A_k}}(\mathbf{a}^k\mathbf{n}; \mathbf{r}) = f(\mathbf{a})\mu_{A_k}\left(\gamma_{A_k}(\mathbf{r})\right)\mu_{A_k}(\mathbf{b})(f_{A_k}\mu_{A_k})(\mathbf{b}).$$

By quasi-multiplicativity Theorems 8–11 can be proved by considering the case in which $\mathbf{r}$ is a prime power. We omit the details.

Theorem 12. *Suppose f, g, h and H are arithmetical functions and $\mathbf{n} \in G$. Let w denote the arithmetical function such that $w(\mathbf{1}) = 0$ and for $\mathbf{r} \neq \mathbf{1}$, $w(\mathbf{r})$ is the number of distinct prime divisors of $\mathbf{r}$. Then*

$$\sum_{\mathbf{d}_1,\ldots,\mathbf{d}_u,\mathbf{e}} S_{A,k}^{f,g}(\mathbf{d}_1, \ldots, \mathbf{d}_u; \mathbf{e})h(\mathbf{e})H(\mathbf{n}/\mathbf{e}) = f(\mathbf{1})\left((u^w H)\upsilon(gh)\right)(\mathbf{n}),$$

where the summation is over $\mathbf{d}_1, \ldots, \mathbf{d}_u, \mathbf{e} \in G$ such that $\mathbf{d}_1 \cdots \mathbf{d}_u\mathbf{e} = \mathbf{n}$ and $\mathbf{d}_1, \ldots, \mathbf{d}_u, \mathbf{e}$ are pairwise relatively prime.

Proof. The left-hand side of the identity in Theorem 12 is

$$\begin{aligned}\sum_{\mathbf{d}_1,\ldots,\mathbf{d}_u,\mathbf{e}} f(\mathbf{1})g(\mathbf{e})h(\mathbf{e})H(\mathbf{n}/\mathbf{e}) &= f(\mathbf{1}) \sum_{\mathbf{e}\in U(\mathbf{n})} u^{w(\mathbf{n}/\mathbf{e})}g(\mathbf{e})h(\mathbf{e})H(\mathbf{n}/\mathbf{e}) \\ &= f(\mathbf{1})\left((gh)\upsilon(u^w H)\right)(\mathbf{n});\end{aligned}$$

hence the theorem is valid.

Theorem 13. *Let f, g, h and H be arithmetical functions and $\mathbf{n} \in G$. Then*

$$\sum_{\substack{\mathbf{d}_1\cdots\mathbf{d}_u\mathbf{e}=\mathbf{n} \\ (\mathbf{d}_1\cdots\mathbf{d}_u,\mathbf{e})=1}} S_{A,k}^{f,g}(\mathbf{d}_1, \ldots, \mathbf{d}_u; \mathbf{e})h(\mathbf{e})H(\mathbf{n}/\mathbf{e}) = f(\mathbf{1})\left((E_u H)\upsilon(gh)\right)(\mathbf{n}),$$

where $E_u = E_D E_D \cdots_D E$ (u factors).

Proof. The left-hand side of the identity is

$$\begin{aligned}\sum_{\substack{\mathbf{d}_1\cdots\mathbf{d}_u\mathbf{e}=\mathbf{n} \\ (\mathbf{d}_1\cdots\mathbf{d}_u,\mathbf{e})=1}} f(\mathbf{1})g(\mathbf{e})h(\mathbf{e})H(\mathbf{n}/\mathbf{e}) &= f(\mathbf{1}) \sum_{\mathbf{e}\in U(\mathbf{n})} \Big(\sum_{\mathbf{d}_1\cdots\mathbf{d}_u=\mathbf{n}/\mathbf{e}} 1\Big) g(\mathbf{e})h(\mathbf{e})H(\mathbf{n}/\mathbf{e}) \\ &= f(\mathbf{1}) \sum_{\mathbf{e}\in U(\mathbf{n})} E_u(\mathbf{n}/\mathbf{e})g(\mathbf{e})h(\mathbf{e})H(\mathbf{n}/\mathbf{e}) = f(\mathbf{1})\left((E_u H)\upsilon(gh)\right)(\mathbf{n}).\end{aligned}$$

We thus arrive at our result.

Remark. A large number of special cases of our results can be found in the literature. In fact, special cases of Theorem 1 can be found in [33, Corollary (2.1.5), p. 170, Theorem (2.2.6), p. 174], [38, pp. 15 and 72] and [40, Chapter 4.1]. Special cases of Theorem 2 can be found in [2, equation (2.7)], [20, equation (6)], [33, Theorem (2.2.12), p. 176], [34, Theorem 3.1], [35, equation (4.3)], [41, Theorem 3] and [44, Theorem 5.3]. Special cases of Theorem 3 can be found in [1, Theorems 1-4], [2, equations (2.8), (2.10)], [4, Theorem 8], [5, Corollaries 2, 3, 4, 5, 10.2 and 10.4], [6, equation (5.4)], [8, equations (3.1), (5.1)], [10, p. 203], [12, equation (3.16)], [18, equation (2.3), p. 194], [20, equations (4), (5)], [23, Lemma 1], [28, Theorem 1], [29, equation (1a) and Theorem 8], [30, equation (2.6)], [31, Theorem 2.4], [33, Theorem (2.1.8), p. 172, Theorem (2.2.11), p. 176], [34, Theorems 3.2, 3.3], [36, equation (3.3)], [41, Theorems 1, 2] and [44, Theorem 5.4]. Special cases of Theorem 4 can be found in [1, Theorems 5, 6], [28, Theorem 2] and [29, Theorem (1b)]. Special cases of Theorem 5 can be found in [1, Theorems 7, 8], [2, Theorem 3.2], [3, Corollary 2.1], [17, Lemma 2.6], [29, Theorem 3], [40, Theorem 4.1.2] and [44, Theorem 5.5]. Special cases of Theorem 6 can be found in [7, Lemma 4], [14, Lemma 3] and [25, Theorem 8]. Special cases of Theorem 7 can be found in [2, Theorem 2.6], [16, Theorem 1], [20, equation (3)], [37, equations (2.10), (2.11)] and [40, equation (4.14)]. Special cases of Theorem 8 can be found in [3, Theorem 2], [8, equation (4.2)], [15, Theorem 1], [18, Lemma 2.2, p. 194], [21, Theorem 5], [24, Theorem 3], [26, equation (4.1.6)] and [32, Theorem 7.2]. A special case of Theorem 9 can be found in [37, equation (2.12)]. Special cases of Theorem 10 can be found in [4, Theorem 6], [8, equation (4.5)], [9, Theorem 3.3], [21, Theorem 4], [32, Theorem 7.4] and [34, Theorem 3.4]. Special cases of Theorem 11 can be found in [11, Theorem 3] and [25, Theorem 3]. Finally, Theorem 6 of [25] is a special case of Theorems 12 and 13.

4. Totally A-k-even functions $(\operatorname{mod}\mathbf{r})$

Let $\mathbf{r} \in G$ be fixed. Then an arithmetical function $f(\mathbf{n};\mathbf{r})$ of one variable is said to be A-k-even $(\operatorname{mod}\mathbf{r})$ if $f(\mathbf{n};\mathbf{r}) = f\big((\mathbf{n},\mathbf{r}^k)_{A,k};\mathbf{r}\big)$ for all $\mathbf{n} \in G$. An arithmetical function $f(\mathbf{n}_1,\dots,\mathbf{n}_u;\mathbf{r})$ of u variables is said to be totally A-k-even $(\operatorname{mod}\mathbf{r})$ if there exists an A-k-even function $g(\mathbf{n};\mathbf{r})$ $(\operatorname{mod}\mathbf{r})$ such that $f(\mathbf{n}_1,\dots,\mathbf{n}_u;\mathbf{r}) = g\big((\mathbf{n}_1,\dots,\mathbf{n}_u);\mathbf{r}\big)$ for all $\mathbf{n}_1,\dots,\mathbf{n}_u \in G$. The concept of a totally A-k-even function $(\operatorname{mod}\mathbf{r})$ originates from [14] in the case of the arithmetical semigroup of positive integers. It can be proved (cf. [14, Theorem 1]) that an arithmetical function $f(\mathbf{n}_1,\dots,\mathbf{n}_u;\mathbf{r})$ is totally A-k-even $(\operatorname{mod}\mathbf{r})$ if, and only if, it has a unique representation of the form

$$f(\mathbf{n}_1,\dots,\mathbf{n}_u;\mathbf{r}) = \sum_{\mathbf{d}\in A_k(\mathbf{r})} \alpha(\mathbf{d};\mathbf{r}) C_{A,k}(\mathbf{n}_1,\dots,\mathbf{n}_u;\mathbf{d}),$$

where

$$\alpha(\mathbf{d};\mathbf{r}) = \mathbf{r}^{-ku} \sum_{\boldsymbol{\delta} \in A_k(\mathbf{r})} g(\mathbf{r}^k/\boldsymbol{\delta}^k;\mathbf{r}) C_{A,k}(\mathbf{r}^k/\mathbf{d}^k,\ldots,\mathbf{r}^k/\mathbf{d}^k;\boldsymbol{\delta}).$$

The coefficients $\alpha(\mathbf{d};\mathbf{r})$ are called the Fourier coefficients of $f(\mathbf{n}_1,\ldots,\mathbf{n}_u;\mathbf{r})$. It can also be proved (cf. [14, Theorem 2]) that an arithmetical function $f(\mathbf{n}_1,\ldots,\mathbf{n}_u;\mathbf{r})$ is totally A-k-even $(\operatorname{mod}\mathbf{r})$ if, and only if, it has the form

$$f(\mathbf{n}_1,\ldots,\mathbf{n}_u;\mathbf{r}) = \sum_{\mathbf{d}^k \in A\left(((\mathbf{n}_i),\mathbf{r}^k)_{A,k}\right)} f'(\mathbf{d};\mathbf{r}). \tag{12}$$

In this case

$$\alpha(\mathbf{d};\mathbf{r}) = \mathbf{r}^{-ku} \sum_{\mathbf{e} \in A_k(\mathbf{r}/\mathbf{d})} f'(\mathbf{r}/\mathbf{e};\mathbf{r})\mathbf{e}^{ku}.$$

By (12) we find that the generalized Ramanujan's sum considered in this paper is a totally A-k-even function $(\operatorname{mod}\mathbf{r})$.

The purpose of this section is to note that the equations (8) and (10) can be extended to totally A-k-even functions $(\operatorname{mod}\mathbf{r})$. In fact, if we replace the generalized Ramanujan's sum by an arbitrary totally A-k-even function $f(\mathbf{n}_1,\ldots,\mathbf{n}_u;\mathbf{r})$ $(\operatorname{mod}\mathbf{r})$ in the left-hand sides of equations (8) and (10), we must replace the factor of $f(\mathbf{d})g(\mathbf{r}/\mathbf{d})$ by $f'(\mathbf{d};\mathbf{r})$ in the right-hand sides of the equations.

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