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THE LOWER BOUND OF THE MAXIMAL DILATATION OF THE BEURLING–AHLFORS EXTENSION

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1. Introduction

An increasing continuous function h defined on an interval $I \subset \mathbf{R}^1$ is ϱ-quasisymmetric on I if

$$\varrho^{-1} \le \frac{h(x+t)-h(x)}{h(x)-h(x-t)} \le \varrho \tag{1}$$

for all x and $t>0$ such that $[x-t,x+t] \subset I$.

A well-known result due to Beurling and Ahlfors [1] states that the map $f_{h,r}$ defined by

$$f_{h,r}(z) = \tfrac{1}{2}\big[\alpha(z)+\beta(z)+ir\big(\alpha(z)-\beta(z)\big)\big], \tag{2}$$

where $r>0$,

$$\alpha(z) = \int_0^1 h(x+yt)\,dt, \quad \beta(z) = \int_{-1}^0 h(x-yt)\,dt, \quad z = x+iy, \tag{3}$$

is a quasiconformal extension of h to the upper half-plane H if h is ϱ-quasisymmetric on $\mathbf{R}^1$. Such a map $f_{h,r}\colon H \to H$ is called a Beurling–Ahlfors extension of h. Beurling and Ahlfors proved that if h is ϱ-quasisymmetric on $\mathbf{R}^1$, there is a number $r>0$ such that the maximal dilatation $K[f_{h,r}] \le \varrho^2$. This estimation has been replaced by

$$K[f_{h,1}] \le 8\varrho, \quad K[f_{h,1}] \le 4.2\varrho, \quad \text{and} \quad K[f_{h,1}] \le 2\varrho$$

due to T. Reed [8], Li Zhong [7] and M. Lehtinen [3], respectively. M. Lehtinen [4] even proved that $K[f_{h,r}] \le 2\varrho - 1$ for some $r>0$.

In this paper, the lower bound of $K[f_{h,r}]$ will be examined. We denote

$$K_\varrho := \sup_{h\in S_\varrho} \big\{ \inf_{r>0} K[f_{h,r}] \big\},$$

where S_ϱ is the set of all ϱ-quasisymmetric functions on $\mathbf{R}^1$. We shall give an example of a ϱ-quasisymmetric function h such that

$$K[f_{h,r}] \ge (2\varrho+1)\Big(1-\frac{1}{\sqrt{\varrho}}\Big)$$

for every $r>0$. This implies the following theorem.

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Theorem. *We have* $K_\varrho \geq (2\varrho+1)(1-1/\sqrt{\varrho})$ *for every* $\varrho \geq 1$.

This result tells us that the coefficient of ϱ in a linear upper bound of $K[f_{h,r}]$ generally cannot be smaller than 2. This means that the above results by Lehtinen are sharp in a certain sense.

By this theorem and the results of Lehtinen, we have

Corollary. *We have* $\lim_{\varrho\to\infty} K_\varrho/\varrho = 2$.

2. Piecewise linear quasisymmetric functions

To give a special ϱ-quasisymmetric function, we need some lemmas on piecewise linear quasisymmetric functions.

Lemma 1. *Let* $E \subset [0,1]$, $\{0,1\} \subset E$, *be a set of finite points and* h: $[0,1] \to [0,1]$, $h(0)=0$, $h(1)=1$, *be increasing and continuous on* $[0,1]$ *and linear on each interval in* $[0,1]\setminus E$. *If (1) is true for all* x *and* $t>0$ *such that* $\{x-t,x,x+t\}\cap E$ *has at least two points, then* h *is* ϱ*-quasisymmetric on* $[0,1]$.

This lemma is proved by Hayman and Hinkkanen ([2]).

Noting that h is ϱ-quasisymmetric if and only if $f\circ h\circ g$ is ϱ-quasisymmetric when f and g are increasing linear functions, Lemma 1 can easily be generalized to the following statement:

Lemma 1′. *Let* $E \subset [a,b]$, $\{a,b\} \subset E$, *be a set of finite points and* h: $[a,b] \to [c,d]$, $h(a)=c$, $h(b)=d$, *be increasing and continuous on* $[a,b]$ *and linear on each interval in* $[a,b]\setminus E$. *If (1) is true for all* x *and* $t>0$ *such that* $\{x-t,x,x+t\}\cap E$ *has at least two points, then* h *is* ϱ*-quasisymmetric on* $[a,b]$.

Lemma 2. *Let* $E\subset \mathbf{R}^1$ *be a set of* n *points and* h: $\mathbf{R}^1\to\mathbf{R}^1$ *be increasing and continuous on* $\mathbf{R}^1$ *and linear on each interval in* $\mathbf{R}^1\setminus E$. *Suppose that (1) is true for all* x *and* $t>0$ *such that* $\{x-t,x,x+t\}\cap E$ *has at least two points and*

$$(4)\qquad \varrho^{-1} \leq \lim_{x\to+\infty} \frac{h(x)}{-h(-x)} \leq \varrho.$$

Then h *is* ϱ*-quasisymmetric on* $\mathbf{R}^1$.

Proof. Without any loss of generality, we may assume that $n\geq 2$. For if $n=1$, the condition (4) implies that h is ϱ-quasisymmetric on $\mathbf{R}^1$. Suppose that $E=\{x_1,x_2,\dots,x_n\}$ with $x_1<x_2<\cdots<x_n$. Let A be a sufficiently large number and $E'=E\cup\{-A,A+2x_1\}$.

To prove that h is ϱ-quasisymmetric on $\mathbf{R}^1$, it is sufficient to show that h is ϱ-quasisymmetric on $[-A,A+2x_1]$ for any sufficiently large A. By Lemma 1′, we should only check whether (1) is true for all x and $t>0$ such that $\{x-t,x,x+t\}\cap E'$ has at least two points. But we have supposed that (1) is true for all x

and $t > 0$ such that $\{x-t, x, x+t\} \cap E$ has at least two points. So it is sufficient to show that

$$\varrho^{-1} \le \frac{h(A+2x_1) - h(x_1)}{h(x_1) - h(-A)} \le \varrho, \tag{5}$$

$$\varrho^{-1} \le \frac{h(A+2x_1) - h(x_j)}{h(x_j) - h(2x_j - A - 2x_1)} \le \varrho, \tag{6}$$

and

$$\varrho^{-1} \le \frac{h(A+2x_j) - h(x_j)}{h(x_j) - h(-A)} \le \varrho \tag{7}$$

for all $j = 2, \dots, n$.

For any given $x_j \in E$, we look at the function $\varphi_j(t) = \big[h(x_j + t) - h(x_j)\big] / \big[h(x_j) - h(x_j - t)\big]$. Obviously, when $t > \tau_j = \max\big\{|x_l - x_j| \,\big|\, l = 1, \dots, n\big\}$, $\varphi_j'(t)$ keeps its sign. Hence if $\varphi_j'(t) > 0$ as $t > \tau_j$,

$$\varphi_j(\tau_j) \le \frac{h(x_j + t) - h(x_j)}{h(x_j) - h(x_j - t)} \le \lim_{n \to +\infty} \varphi_j(\eta), \tag{8}$$

and if $\varphi_j'(t) < 0$ as $t > \tau_j$,

$$\lim_{\eta \to +\infty} \varphi_j(\eta) \le \frac{h(x_j + t) - h(x_j)}{h(x_j) - h(x_j - t)} \le \varphi_j(\tau_j), \tag{9}$$

for $j = 1, 2, \dots, n$. Since h is increasing and linear on (x_n, ∞) and $(-\infty, x_1)$, $h(x) \to +\infty$ as $x \to +\infty$ and $h(x) \to -\infty$ as $x \to -\infty$. Hence

$$\lim_{\eta \to +\infty} \varphi_j(\eta) = \lim_{x \to +\infty} \frac{h(x)}{-h(-x)}$$

for $j = 1, 2 \dots, n$. By (4) we have

$$\varrho^{-1} \le \lim_{\eta \to +\infty} \varphi_j(\eta) \le \varrho, \qquad j = 1, 2, \dots, n. \tag{10}$$

From (8), (9) and (10) we see that if

$$\varrho^{-1} \le \varphi_j(\tau_j) \le \varrho, \qquad j = 1, 2, \dots, n, \tag{11}$$

then

$$\varrho^{-1} \le \frac{h(x_j + t) - h(x_j)}{h(x_j) - h(x_j - t)} \le \varrho, \qquad j = 1, 2, \dots, n,$$

for $t > \max\{\tau_1, \tau_2, \ldots, \tau_n\}$, and hence (5), (6), and (7) hold. It remains to prove (11).

Since $n > 1$, τ_j is positive for $j = 1, 2, \ldots, n$. Then we see that $\{x_j - \tau_j, x_j, x_j + \tau_j\} \cap E$ has at least two points. By the assumption of the lemma, (11) is true. The lemma is proved.

For any $s \geq 1$, we define a function h_s as follows:

(12)
$$h_s(x) := \begin{cases} 1 + s(x-1) & \text{as } x \geq 1, \\ s(1+s)^{-1} + s^{-1}\big(x - (1+s)^{-1}\big) & \text{as } (1+s)^{-1} \leq x \leq 1, \\ sx & \text{as } -s(1+s)^{-1} \leq x \leq (s+1)^{-1}, \\ -s^2(1+s)^{-1} + s^3\big(x + s(1+s)^{-1}\big) & \text{as } -1 \leq x \leq -s(1+s)^{-1}, \\ -s^2 + s(x+1) & \text{as } x \leq -1. \end{cases}$$

We are now going to show that h_s is an s^2-quasisymmetric function. This quasisymmetric function will be used to prove the main theorem in the next paragraph.

Let $E = \{-1, -s(1+s)^{-1}, (1+s)^{-1}, 1\}$. Obviously, there are $3 \times C_4^2 = 18$ cases in each of which $\{x - t, x, x + t\} \cap E$ has at least two points. We omit three cases $\{x - t, x, x + t\}$ that are on the same interval in $\mathbf{R}^1 \setminus E$. For all remaining cases, one may check (1) directly by simple computation. By Δ we denote $[h_s(x+t) - h_s(x)]/[h_s(x) - h_s(x-t)]$. Then we have

Case 1: $x = (1+s)^{-1}$, $x + t = 1$. Then $\Delta = s^{-2}$.

Case 2: $x - t = (1+s)^{-1}$, $x = 1$. Then $\Delta = s$.

Case 3: $x = -s(1+s)^{-1}$, $x + t = 1$. Then $\Delta = (s^2 + s + 1)/(s^3 + 2s)$ and $s^{-2} \leq \Delta \leq 1$.

Case 4: $x - t = -s(1+s)^{-1}$, $x + t = 1$. Then $\Delta = (s+2)/(2s^2 + s)$ and $s^{-2} \leq \Delta \leq 1$.

Case 5: $x - t = -s(1+s)^{-1}$, $x = 1$. Then $\Delta = (2s+1)/(s^2 + s + 1)$ and $s^{-1} \leq \Delta \leq 1$.

Case 6: $x = -1$, $x + t = 1$. Then $\Delta = (s^2+1)/2s$ and $1 \leq \Delta \leq s$.

Case 7: $x - t = -1$, $x + t = 1$. Then $\Delta = s^{-2}$.

Case 8: $x - t = -1$, $x = 1$. Then $\Delta = 2s/(s^2+1)$ and $s^{-1} \leq \Delta \leq 1$.

Case 9: $x = -s(1+s)^{-1}$, $x + t = (1+s)^{-1}$. Then $\Delta = s^{-1}$.

Case 10: $x - t = -s(1+s)^{-1}$, $x = (1+s)^{-1}$. Then $\Delta = s^{-2}$.

Case 11: $x = -1$, $x + t = (1+s)^{-1}$. Then $\Delta = (s^2 + s + 1)/(s+2)$ and $1 \leq \Delta \leq s$.

Case 12: $x - t = -1$, $x + t = (1-s)^{-1}$. Then $\Delta = (s+2)/(2s^2 + s)$ and $s^{-2} \leq \Delta \leq 1$.

Case 13: $x - t = -1$, $x = (1+s)^{-1}$. Then $\Delta = s^2$.

Case 14: $x = -1$, $x + t = -s(1+s)^{-1}$. Then $\Delta = s^2$.

Case 15: $x - t = -1$, $x = -s(1+s)^{-1}$. Then $\Delta = s^{-2}$.

Therefore we have

$$s^{-2} \le \frac{h_s(x+t) - h_s(x)}{h_s(x) - h_s(x-t)} \le s^2 \qquad \text{for cases } 1) - 15). \tag{13}$$

Moreover, we easily see that

$$\lim_{x \to +\infty} \frac{h_s(x)}{-h_s(-x)} = 1. \tag{14}$$

From (13) and (14), we can conclude by Lemma 2 that h_s is s^2-quasisymmetric on $\mathbf{R}^1$.

3. The proof of the main result

The quasisymmetric function h_s constructed in the previous paragraph has some special properties. Obviously,

$$h_s(0) = 0, \qquad h_s(1) = 1, \qquad h_s(-1) = -s^2. \tag{15}$$

By a simple computation, we get

$$\int_0^1 h_s(t)\,dt = \frac{s}{s+1}, \qquad \int_{-1}^0 h_s(t)\,dt = -\frac{s^2}{1+s}. \tag{16}$$

Using these properties one obtains a lower estimate of K_ϱ.

We denote h_s by h and s^2 by ϱ. Then h is a ϱ-quasisymemtric function on $\mathbf{R}^1$. Let $f_{r,h}$ be the Beurling–Ahlfors extension of h. The dilatation of $f_{h,r}$ at i is denoted by D_r. Setting $\xi = \alpha_y(i)/\alpha_x(i)$, $\eta = -\beta_y(i)/\beta_x(i)$, $\zeta = \alpha_x(i)/\beta_x(i)$, we get

$$D_r + D_r^{-1} = a(\xi,\eta,\zeta)r + b(\xi,\eta,\zeta)/r, \tag{17}$$

where

$$a(\xi,\eta,\zeta) = \left[(\zeta-1)^2 + (\zeta\xi+\eta)^2\right]/\left[2\zeta(\xi+\eta)\right]$$
$$b(\xi,\eta,\zeta) = \left[(\zeta+1)^2 + (\zeta\xi-\eta)^2\right]/\left[2\zeta(\xi+\eta)\right].$$

From (15) and (16) one obtains

$$\zeta = -1/h(-1) = s^{-2}, \tag{18}$$

$$\xi = 1 - \int_0^1 h(t)\,dt = (1+s)^{-1}, \tag{19}$$

$$\eta = 1 + \zeta \int_{-1}^0 h(t)\,dt = \frac{s}{1+s}. \tag{20}$$

Hence we have

$$D_r + D_r^{-1} \ge \Big(a(\xi,\eta,\zeta)\cdot b(\xi,\eta,\zeta)\Big)^{1/2} \tag{21}$$

$$= \frac{1}{\zeta(\xi+\eta)}\Big(\big((\zeta-1)^2 + (\zeta\xi+\eta)^2\big)\big((\zeta+1)^2 + (\zeta\xi-\eta)^2\big)\Big)^{1/2}$$

$$= s^2\bigg(\Big(\Big(\frac{1}{s^2}-1\Big)^2 + \Big(\frac{1}{s^2(s+1)} + \frac{s}{s+1}\Big)^2\Big)\Big(\Big(\frac{1}{s^2}+1\Big)^2 + \Big(\frac{1}{s^2(1+s)} - \frac{s}{1+s}\Big)^2\Big)\bigg)^{1/2}$$

$$= \frac{1}{s^2}\left(\left((s^2-1)^2 + \left(\frac{1+s^3}{1+s}\right)^2\right)\left((s^2+1)^2 + \left(\frac{s^3-1}{1+s}\right)^2\right)\right)^{1/2}.$$

Noting that

$$\frac{s^3-1}{s+1} \ge s^2 - s \quad \text{and} \quad \frac{s^3+1}{s+1} = s^2 - s + 1$$

one obtains

$$(s^2-1)^2 + \left(\frac{s^3+1}{s+1}\right)^2 = 2s^4 - 2s^3 + s^2 - 2s + 2,$$

$$(s^2+1)^2 + \left(\frac{s^3-1}{s+1}\right)^2 \ge 2s^4 - 2s^3 + 3s^2 + 1,$$

and hence

$$D + D^{-1} \ge \frac{1}{s^2}\Big((2s^4 - 2s^3 + s^2 - 2s + 2)(2s^4 - 2s^3 + 3s^2 + 1)\Big)^{1/2} \tag{22}$$

$$= \frac{1}{s^2}\Big(4s^8 - 8s^7 + 12s^6 - 12s^5 + 13s^4 - 12s^3 + 7s^2 - 2s + 2\Big)^{1/2}$$

$$= \frac{1}{s^2}\Big(\big(2s^4 - 2s^3 + 2s^2 - s\big)^2 + 5s^4 - 8s^3 + 6s^2 - 2s + 2\Big)^{1/2}.$$

Setting $P(s) = 5s^4 - 8s^3 + 6s^2 - 2s + 2$, one computes

$$P(1) = 3 > 0, \quad P'(1) = 6 > 0, \quad P''(s) = 60s^3 - 48s + 12 \ge 0 \text{ as } s \ge 1,$$

and hence $P(s) > 0$ as $s \ge 1$. Then we have

$$D + D^{-1} > \frac{1}{s^2}(2s^4 - 2s^3 + 2s^2 - s) = 2s^2 - 2s + 2 - \frac{1}{s}. \tag{23}$$

Since $D^{-1} \le 1$, we immediately obtain

$$D > 2s^2 - 2s + 1 - s^{-1} = (2s^2+1)(1-s^{-1}).$$

Replacing s by $\sqrt{\varrho}$, we get

$$D > (2\varrho + 1)(1 - 1/\sqrt{\varrho}) \tag{24}$$

and $K_\varrho > (2\varrho+1)(1-1/\sqrt{\varrho})$. The main theorem is proved.

Some remarks: 1. The author suggested another piecewise linear quasisymmetric function on $\mathbf{R}^1$ which is similar to h_s in this paper but more complicated. Li Wei and Liu Yong computed the maximal dilatation of its Beurling–Ahlfors extension ([6]) and got an asymptotic estimate.

2. There are some other results on the lower bound of K_ϱ. For instance, $K_\varrho \geq 1.587\varrho$ for large ϱ ([7]); $K_\varrho \geq 3\varrho/2$ for every $\varrho \geq 1$ and $\lim_{\varrho\to+\infty} K_\varrho/\varrho \geq 1.5625$ ([4]); $K_\varrho > 8\varrho/5$ for $\varrho > 7$ and $\lim_{\varrho\to+\infty} K_\varrho/\varrho \geq 1.765625$ ([5]).

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