Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 15, 1990, 83–105

A GENERALIZATION OF THE M. RIESZ THEOREM ON CONJUGATE FUNCTIONS AND THE ZYGMUND $L \log L$ -THEOREM TO R^d, $d \geq 2$

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1. Introduction

Let D be a domain in $\mathbf{R}^d$, $d \geq 2$. Points in $\mathbf{R}^d$ are denoted by $x = (x_1, x_2, \dots, x_d)$. We use the Euclidean norm $|x|$. We shall prove the following two results.

Theorem 1. *Let $p \in (1, \infty)$ be given. Then $|x|^p$ has a harmonic majorant in D if and only if $|x_1|^p$ has a harmonic majorant in D.*

Corollary. *Let h^* and h_0 be the least harmonic majorants of $|x_1|^p$ and $|x|$, respectively. If $0 \in D$, we have*

$$h_0(0) \leq C_p h^*(0), \tag{0.1}$$

where C_p is defined by (0.2), (2.3) *and* (3.1) *and C_p is best possible.*

Theorem 2. (a) *If $|x_1| \log^+ |x_1|$ has a harmonic majorant in D, then $|x|$ has a harmonic majorant in D.*

(b) *Let us furthermore assume that the least harmonic majorant ψ of $|x_1|$ in D is such that $\psi(x) = \mathbf{O}(|x|)$, $x \to \infty$ in D. If $|x|$ has a harmonic majorant in D, then $|x_1| \log^+ |x_1|$ has a harmonic majorant in D.*

In the plane, Theorems 1 and 2 are related to classical results of M. Riesz and A. Zygmund. A discussion of the connection can be found in [6].

The proof of Theorem 1 consists of two parts. In Sections 2 and 3, we prove that if the constant C_p is chosen in the right way, there exists a function G_p which is superharmonic in $\mathbf{R}^d$ such that

$$|x|^p - C_p |x_1|^p \leq G_p(x), \qquad x \in \mathbf{R}^d. \tag{0.2}$$

A similar construction was used in the simple proof of the M. Riesz theorem on conjugate functions which was given in [4]. Also in the present paper, the two cases $1 < p \leq 2$ and $2 < p < \infty$ have to be separated from one another. To construct G_p, we have to make a detailed study of certain classical differential

doi:10.5186/aasfm.1990.1509

equations. Let us mention that we have not been able to find the estimate given in (2.12) in the literature.

The first part of Theorem 2 is similar but easier: we simply note that the function $\sqrt{1+x_1^2}\log(1+x_1^2) - 2d^{-1}\bigl(1+|x|^2\bigr)^{1/2}$ is subharmonic in $\mathbf{R}^d$.

The crucial idea in the second parts of the proofs of Theorems 1 and 2 is to study the connection between certain "harmonic measures" and the existence of certain harmonic majorants (cf. Sections 5 and 6). Essentially, we argue as in the proof of Theorem 5 in [5]: this result says that Theorem 2 holds if we assume also that D is contained in a half-space $\{x \in \mathbf{R}^d : x_1 > 0\}$. In [5], an important part of the theory was that x_1 could be interpreted as a minimal harmonic function in $\{x_1 > 0\}$ associated with the Martin boundary point at infinity: modulo multiplication by constants, this minimal function is unique and we could use it to construct certain other harmonic functions. In the present situation, the set of minimal harmonic functions associated with infinity is not known. An example given below tells us that there might be cases where the dimension of the set is 2. The theory in [5] depended in an essential way on the assumption that this set was one-dimensional. To avoid the use of minimal harmonic functions in our definitions in Section 4, we have chosen to apply a technique which we earlier used in [7]. However, the intuition behind our arguments is inspired by our work in [5]. An account of this adaption of previous work to the situation in the present paper is given in the Appendix.

Remark 1. In Theorem 2b, we assumed that the domain D was such that the least harmonic majorant ψ of $|x_1|$ in D was such that $\psi(x) = \mathbf{O}\bigl(|x|\bigr)$, $x \to \infty$ in D. The results of Benedicks [1] give an example of domains with this property. Let E be a closed subset of the hyperplane $\{x \in \mathbf{R}^d : x_1 = 0\}$. Let $\mathcal{P}_E$ be the cone of positive harmonic functions in $\Omega = \mathbf{R}^d \setminus E$ with vanishing boundary values on E (with the possible exception of a set of capacity zero). If E has positive capacity, each function $u \in \mathcal{P}_E$ satisfies the growth estimate $u(x) = \mathbf{O}\bigl(|x|\bigr)$, $x \to \infty$ (cf. Lemma 3 in [1]). The set $\mathcal{P}_E$ may have dimension 1 or 2. Under certain assumptions on the density of E near infinity, it is proved that $|x_1|$ has a harmonic majorant in Ω and that $\mathcal{P}_E$ is two-dimensional (cf. Theorems 3 and 4 in [1]). Furthermore, the least harmonic majorant $\psi \in \mathcal{P}_E$ and it follows that $\psi(x) = \mathbf{O}\bigl(|x|\bigr)$, $x \to \infty$. Any domain D contained in such a domain Ω satisfies the extra requirement in Theorem 2b.

Remark 2. It would be interesting to have an example of a domain D which is such that $|x|$ has a harmonic majorant in D while $|x_1|\log^+|x_1|$ does not have such a majorant. We conjecture that there is such an example. A further question is whether the extra condition in Theorem 2b is the right one.

A discussion of related questions in the plane can be found in [9].

We note that there are functions F in $H^1(U)$ which are such that $\operatorname{Re} F \notin L\log L$. Here U is the unit disc in the plane.

A general study of problems on harmonic majorization in $\mathbf{R}^d$ is given by D. Burkholder in [3]: his methods are probabilistic.

2. Construction of a superharmonic majorant of $|x|^p - C_p|x_1|^p$: the case $1 < p \le 2$

For $x_1 > 0$, we define $\theta = \arccos(x_1/|x|)$ and consider harmonic functions of the form $|x|^p F(\theta)$ where $1 < p \le 2$. Then F is a solution of the equation

$$(d/d\theta)\big((\sin\theta)^{d-2}F'\big) = -p(p+d-2)(\sin\theta)^{d-2}F, \tag{2.1}$$

or equivalently

$$F'' + (d-2)(\cot\theta)F' + p(p+d-2)F = 0. \tag{2.2}$$

Let $F = F_p$ be the solution of (2.1) on $(0,\pi/2)$ which satisfies the initial conditions $F(0) = 1$, $F'(0) = 0$. In Proposition 3, we shall prove that the smallest zero α of F in $(0,\pi/2)$ exists: we have $F'(\alpha) < 0$ (cf. (2.4)). For $x_1 > 0$, we define

$$G_p(x) = \begin{cases} |x|^p\big(F(\theta)/F'(\alpha)\big)p\tan\alpha, & 0 \le \theta < \alpha, \\ |x|^p - x_1^p(\cos\alpha)^{-p}, & \alpha \le \theta \le \pi/2. \end{cases}$$

We extend G_p to the whole of $\mathbf{R}^d$ via reflection: $G_p(x) = G_p(-x_1, x')$, $x_1 < 0$, where $x' = (x_2, x_3, \dots, x_d)$.

The function G_p is harmonic in circular cones around the positive and negative x_1-axis: the angle between a generatrix and the axis is α.

Proposition 1. *G_p is superharmonic in the complement of these cones.*

The proof of Proposition 1 will be given later.

The constants have been chosen in such a way that $G_p \in C^1(\mathbf{R}^d)$. Applying Green's formula in exactly the same way as in [4], we can use Proposition 1 to prove that G_p is superharmonic in $\mathbf{R}^d$. We omit the details. This argument gives the first part of

Proposition 2. *G_p is superharmonic in $\mathbf{R}^d$. Furthermore, we have*

$$|x|^p - |x_1|^p(\cos\alpha)^p \le G_p(x), \qquad x \in \mathbf{R}^d. \tag{2.3}$$

Let us first show that our definitions are correct.

Proposition 3. *Let F_p and α be as above. If $p > 1$, we have $0 < \alpha < \pi/2$.*

Proof. For $p = 1$, our initial value problem has the solution $F_1(\theta) = \cos\theta$ and we see that $\alpha(1) = \pi/2$. A standard argument shows that $\alpha(p)$ is strictly decreasing as a function of p when $p \geq 1$. This proves Proposition 3.

Proof of Proposition 2. We claim that

$$F'(\theta) < 0, \qquad 0 < \theta \leq \alpha. \tag{2.4}$$

To see this, we integrate (2.1) and obtain

$$F'(\theta) = -(\sin\theta)^{2-d} p(p+d-2) \int_0^\theta F(t)(\sin t)^{d-2} dt. \tag{2.5}$$

It is known that $F(\theta) > 0$ in $(0, \alpha)$. Hence (2.4) is proved.

Let L be the differential operator defined by

$$Lh = (d/d\theta)\big((\sin\theta)^{2-d}(d/d\theta)(\sin\theta)^{d-2}h\big) + p(p+d-2)h. \tag{2.6}$$

Our claim (2.3) is equivalent to

$$g(\theta) = (\cos\theta/\cos\alpha)^p - 1 + p\tan\alpha F(\theta)/F'(\alpha) \geq 0, \qquad 0 \leq \theta \leq \alpha. \tag{2.7}$$

Since $g(\alpha) = 0$, (2.7) is a consequence of

$$g'(\theta) = p\big(-(\cos\theta)^{p-1}(\sin\theta)(\cos\alpha)^{-p} + F'(\theta)\tan\alpha/F'(\alpha)\big) \leq 0, \; 0 < \theta \leq \alpha. \tag{2.8}$$

It is easy to check that $g'(0) = g'(\alpha) = 0$.

A computation shows that when $1 < p \leq 2$,

$$Lg' = -p(p-1)(p-2)(\sin\theta)(\cos\theta)^{p-3}(\cos\alpha)^{-p} \geq 0, \quad 0 < \theta < \alpha. \tag{2.9}$$

Let $K(\theta, \xi)$ be Green's function for the problem $Lh = 0$, $h(0) = h(\alpha) = 0$, $0 < \theta < \alpha$. If we can prove that K is non-positive it follows from (2.9) that

$$g'(\theta) = \int_0^\alpha K(\theta, \xi)(Lg')(\xi)\, d\xi \leq 0, \qquad 0 < \theta < \alpha,$$

which implies that (2.7) and thus also (2.3) will hold.

To prove that $K \leq 0$ in $(0, \alpha)$, we introduce H which is the solution of (2.1) on $(0, \alpha)$ satisfying the conditions $H(\alpha) = 1$, $H'(\alpha) = 0$. We claim that

$$H'(\alpha) > 0, \qquad 0 < \theta < \alpha. \tag{2.10}$$

To see this, we integrate the differential equation to obtain

$$H'(\theta) = (\sin\theta)^{2-d} p(p+d-2) \int_\theta^\alpha H(t)(\sin t)^{d-2} dt, 0 < \theta < \alpha.$$

Hence H' will be positive near α. Assuming (2.10) to be wrong, we let γ be the largest zero of H' in $(0,\alpha)$. Multiplying the differential equations for F and H by H and F, respectively, taking the difference and integrating over (γ,α), we obtain

$$H(\gamma)F'(\gamma)(\sin\gamma)^{d-2} = F'(\alpha)(\sin\alpha)^{d-2}.$$

From (2.4) we see that $H(\gamma) > 0$: hence we have $H''(\gamma) < 0$ (cf. (2.2)). This means that H has a maximum at γ which contradicts our assumption that $H'(\theta) > 0$ in (γ,α). Our assumption that γ exists must be wrong and we have proved (2.10).

So far, we know that $LF' = LH' = 0$ and that $F'(0) = H'(\alpha) = 0$. Thus we can write K in the following way:

$$K(\theta,\xi) = \begin{cases} AF'(\theta), & 0 < \theta < \xi, \\ BH'(\theta), & \xi \le \theta < \alpha. \end{cases}$$

The constants A and B are determinded by the system

$$\begin{cases} AF'(\xi) - BH'(\xi) = 0, \\ -AF''(\xi) + BH''(\xi) = 1. \end{cases}$$

To compute the determinant, we re-write the operator L as

$$Lh = h'' + (d-2)(\cot\theta)h' + h\big(-(d-2)(\sin\theta)^{-2} + p(p+d-2)\big). \tag{2.11}$$

Multiplying the differential equations for F' and H' by H' and F', respectively, taking the difference and integrating, we obtain

$$F'(\xi)H''(\xi) - H'(\xi)F''(\xi) = F'(\alpha)H''(\alpha)(\sin\alpha/\sin\xi)^{d-2}.$$

From (2.2) we see that $H''(\alpha) < 0$ and that (cf. (2.4))

$$F'(\xi)H''(\xi) - H'(\xi)F''(\xi) = c(\sin\xi)^{2-d},$$

where $c = c(\alpha)$ is a positive constant. It follows that

$$K(\theta,\xi) = \begin{cases} c^{-1}F'(\theta)H'(\xi)(\sin\xi)^{d-2}, & 0 < \theta < \xi, \\ c^{-1}F'(\xi)H'(\theta)(\sin\xi)^{d-2}, & \xi \le \theta < \alpha. \end{cases}$$

The non-positivity of K is now a consequence of (2.4) and (2.10). This completes the proof of Proposition 2.

Proof of Proposition 1. A computation shows that when $\alpha < \theta < \pi/2$, we have

$$\Delta G_p(x) = p|x|^{p-2}\big((p+d-2) - (p-1)(\cos\theta)^{p-2}(\cos\alpha)^{-p}\big).$$

Since $G_p \in C^1(\mathbf{R}^d)$, the singularities in the second derivatives of G_p at $\theta = \pi/2$ will not give any contribution to the Riesz mass.

Using the lower bound for θ, we obtain the estimate

$$\Delta G_p \le p|x|^{p-2}\big((p+d-2) - (p-1)(\cos\alpha)^{-2}\big),$$

for x in the complement of the cones where G_p is harmonic. The right hand member will be non-positive if and only if

$$\cos^2\big(\alpha(p)\big) \le (p-1)/(p+d-2), \qquad 1 < p \le 2. \tag{2.12}$$

If (2.12) holds, ΔG_p will be non-positive for $\theta \in (\alpha, \frac{1}{2}\pi)$ and Proposition 1 will be proved.

To prove (2.12), we make a change of variable in (2.1) and consider $f(t) = f_p(t) = F_p(\arcsin t)$ which solves the problem

$$\begin{cases} (1-t^2)f'' + t^{-1}\big(d-2-(d-1)t^2\big)f' + p(p+d-2)f = 0, \\ f(0) = 1, \quad f'(0) = 0, \quad 0 < t < 1. \end{cases} \tag{2.13}$$

Let a be the first zero of f in $(0,1)$. Then $a = \sin\alpha$ and (2.12) is equivalent to

$$a \ge \big((d-1)/(p+d-2)\big)^{1/2} = t_0. \tag{2.14}$$

Problem (2.13) has a power series solution $1 + \sum_1^\infty a_n t^{2n}$ where

$$\begin{aligned} a_1 &= -p(p+d-2)/\big(2(d-1)\big), \\ a_{n+1} &= a_n\big(2n(2n+d-2) - p(p+d-2)\big)/\big(2(n+1)(2n+d-1)\big), \\ &\qquad n = 2, 3, \ldots. \end{aligned}$$

It is clear that all these coefficients in the power series are negative and that f is strictly decreasing on $(0,1)$. Thus (2.14) will follow if we can prove that $f_p(t_0) \ge 0$. A computation shows that

$$f_p(t_0) = 1 - (p/2) - (p/2)\sum_2^\infty \prod_{n=1}^{k-1} b_n,$$

where $b_n = \big(2n(2n+d-2) - p(p+d-2)\big)(d-1)/\big(2(n+1)(2n+d-1)(p+d-2)\big)$.

For each factor in the product we have the estimate

$$b_n \leq (2n-p)/\big(2(n+1)\big). \tag{2.15}$$

In fact, b_n is increasing as a function of d and maximal when $d=\infty$. We also note that (2.15) is equivalent to

$$0 \leq (p-1)(2n-p)2n, \qquad n=1,2,\ldots,$$

which holds when $1<p\leq 2$. We conclude that

$$f_p(t_0) \geq 1-(p/2)-(p/2)\sum_2^\infty c_{k-1}, \tag{2.16}$$

where $c_k = \prod_{n=1}^k (n-p/2)/(n+1)$, $k=1,2,\ldots$.

We shall prove that

$$\sum_1^\infty c_k = (2-p)/p, \tag{2.17}$$

which implies that $f_p(t_0)\geq 0$, which is what we wanted to prove.

It remains to show that (2.17) holds. Let us consider the formulas

$$(k+2)c_{k+1} = (k+1-p/2)c_k, \qquad k=1,2,\ldots.$$

Multiplying these equations by z^k and adding them, we find that the function $v(z)=\sum_1^\infty c_k z^k$ satisfies the differential equation

$$v'(z)+v(z)\big(z^{-1}+(p/2)(1-z)^{-1}\big) = (1-p/2)(1-z)^{-1},$$

and is such that $v(0)=0$. The solution is

$$v(z) = (1-p/2)\Big((2/p)-2(pz)^{-1}(1-p/2)^{-1}\big((1-z)^{p/2}-(1-z)\big)\Big).$$

A classical Tauberian theorem tells us that if $c_k = \mathbf{O}(1/k)$, $k\to\infty$, then we have

$$\sum_1^\infty c_k = \lim_{x\uparrow 1} v(x) = (2-p)/p.$$

It is easy to prove that $c_k = \mathbf{O}\big(k^{-(1+p/2)}\big)$, $k\to\infty$. Hence the Tauberian condition on the coefficients holds. This concludes the proof of (2.17) and of Proposition 1.

3. Construction of a superharmonic majorant of $|x|^p - C_p|x_1|^p$: the case $p > 2$

Assuming that $p > 2$, we let $F = F_p$ be the solution of (2.1) on $(0, \frac{1}{2}\pi)$ which satisfies the conditions $F(\frac{1}{2}\pi) = 1$, $F'(\frac{1}{2}\pi) = 0$. In Proposition 3′ we shall prove that F has a largest zero β in $(0, \frac{1}{2}\pi)$: we have $F'(\beta) > 0$ (cf. (3.2)). For $x_1 > 0$ we define

$$G_p(x) = \begin{cases} |x|^p - |x_1|^p(\cos\beta)^{-p}, & 0 \le \theta < \beta, \\ |x|^p\big(F(\theta)/F'(\beta)\big)p\tan\beta, & \beta \le \theta \le \frac{1}{2}\pi. \end{cases}$$

We extend G_p to the whole of $\mathbf{R}^d$ via reflection: $G_p(x) = G_p(-x_1, x')$, $x_1 < 0$. The extended function will be harmonic across $\{x_1 = 0\}$ except at the origin. This is clear since our condition $F'(\frac{1}{2}\pi) = 0$ implies that the normal derivative of G_p on $\{x_1 = 0\}$ vanishes. Thus G_p is harmonic in the complement of circular cones around the positive and the negative x_1-axis: the angle between a generatrix and the axis is β.

Proposition 1′. *G_p is superharmonic in these cones.*

The proof of Proposition 1′ will be given later.

The constants have been chosen in such a way that $G_p \in C^1(\mathbf{R}^d)$. In exactly the same way as in the case $1 < p \le 2$, we prove the first part of

Proposition 2′. *G_p is superharmonic in $\mathbf{R}^d$. Furthermore, we have*

$$|x|^p - |x_1|^p(\cos\beta)^{-p} \le G_p(x), \qquad x \in \mathbf{R}^d. \tag{3.1}$$

Proposition 3′. *Let F_p and β be as above. If $p > 2$, we have $0 < \beta < \frac{1}{2}\pi$.*

Proof. It is easy to check that $F_2(\theta) = 1 - d\cos^2\theta$ and that $\cos\beta(2) = 1/\sqrt{d}$. A standard argument shows that $\beta(p)$ is strictly increasing as a function of p in $(2, \infty)$. We have proved Proposition 3′.

Proof of Proposition 2′. We claim that

$$F'(\theta) > 0, \qquad \beta \le \theta < \tfrac{1}{2}\pi. \tag{3.2}$$

Integrating (2.1), we obtain

$$F'(\theta) = (\sin\theta)^{2-d}p(p+d-2)\int_\theta^{\pi/2} F(t)(\sin t)^{d-2}\,dt, \quad 0 < \theta < \tfrac{1}{2}\pi.$$

Since F is positive in $(\beta, \frac{1}{2}\pi)$, it is clear that (3.2) is true.

Let L be the operator defined by (2.6). Our claim (3.1) is equivalent to

$$g(\theta) = (\cos\theta/\cos\beta)^p - 1 + p\tan\beta F(\theta)/F'(\beta) \ge 0, \qquad \beta \le \theta \le \tfrac{1}{2}\pi. \tag{3.3}$$

Since $g(\beta) = 0$, it suffices to prove

$$g'(\theta) = p\big(-(\cos\theta)^{p-1}(\sin\theta)(\cos\beta)^{-p} + \tan\beta F'(\theta)/F'(\beta)\big) \geq 0, \quad \beta \leq \theta \leq \tfrac{1}{2}\pi. \tag{3.4}$$

It is easy to check that $g'(\beta) = g'(\frac{1}{2}\pi) = 0$.

A computation shows that when $p > 2$,

$$Lg' = -p(p-1)(p-2)(\sin\theta)(\cos\theta)^{p-3}(\cos\beta)^{-p} < 0, \ \beta \leq \theta < \tfrac{1}{2}\pi. \tag{3.5}$$

We note that Lg' is integrable in $(\beta, \frac{1}{2}\pi)$.

Let $K(\theta,\xi)$ be Green's function for the problem $Lh = 0$, $h(\beta) = h(\frac{1}{2}\pi) = 0$, $\beta < \theta < \frac{1}{2}\pi$. If we can prove that K is non-positive, it follows from (3.5) that

$$g'(\theta) = \int_\beta^{\pi/2} K(\theta,\xi)(Lg')(\xi)\,d\xi \geq 0, \quad \beta < \theta < \tfrac{1}{2}\pi,$$

which implies that (3.3) and thus also (3.1) will hold.

To prove that $K \leq 0$ in $(\beta, \frac{1}{2}\pi)$, we introduce H which is the solution of (2.1) on $(\beta, \frac{1}{2}\pi)$ satisfying the conditions $H(\beta) = 1$, $H'(\beta) = 0$. We claim that

$$H'(\theta) < 0, \qquad \beta < \theta < \tfrac{1}{2}\pi. \tag{3.6}$$

To see this, we integrate the differential equation to obtain

$$H'(\theta) = -(\sin\theta)^{2-d}p(p+d-2)\int_\beta^\theta H(t)(\sin t)^{d-2}dt, \quad \beta < \theta < \tfrac{1}{2}\pi.$$

Clearly, H' is negative near β. Assuming (3.6) to be wrong, we let γ be the smallest zero of H' in $(\beta, \frac{1}{2}\pi)$. Multiplying the differential equations for F and H by H and F, respectively, taking the difference and integrating over (β,γ), we obtain

$$H(\gamma)F'(\gamma)(\sin\gamma)^{d-2} = F'(\beta)(\sin\beta)^{d-2}.$$

From (3.2) we see that $H(\gamma) > 0$. Thus, we have $H''(\gamma) < 0$ (cf. (2.2)). This means that H has a maximum at γ which contradicts our assumption that $H'(\theta)$ is negative in (β,γ). Thus there can be no zero of H' in $(\beta, \frac{1}{2}\pi)$ and we have proved (3.6).

So far we know that $LF' = LH' = 0$ and that $F'(\frac{1}{2}\pi) = H'(\beta) = 0$. Thus we can write K in the following way:

$$K(\theta,\xi) = \begin{cases} AH'(\theta), & \beta < \theta < \xi, \\ BF'(\theta), & \xi < \theta < \frac{1}{2}\pi. \end{cases}$$

The constants A and B are determined by the system

$$AH'(\xi) - BF'(\xi) = 0,$$
$$-AH''(\xi) + BF''(\xi) = 1.$$

The same kind of computation as in the case $1 < p \leq 2$ shows that there is a positive constant $c = c(\beta)$ such that

$$K(\theta,\xi) = \begin{cases} cH'(\theta)F'(\xi)(\sin\xi)^{d-2}, & \beta < \theta < \xi, \\ cH'(\xi)F'(\theta)(\sin\xi)^{d-2}, & \xi < \theta < \pi. \end{cases}$$

The non-positivity of K is now a consequence of (3.2) and (3.6). This completes the proof of Proposition 2$'$.

Proof of Proposition 1$'$. A computation shows that when $0 < \theta < \beta$, we have

$$\Delta G_p(x) = p|x|^{p-2}\big((p+d-2) - (p-1)(\cos\theta)^{p-2}(\cos\beta)^{-p}\big).$$

Using the upper bound for θ and the assumption $p > 2$, we obtain the estimate

$$\Delta G_p(x) \leq p|x|^{p-2}\big((p+d-2) - (p-1)(\cos\beta)^{-2}\big),$$

for x in the cones where G_p is not a harmonic function. The right hand member will be non-positive if and only if

$$\cos^2\beta(p) \leq (p-1)/(p+d-2). \tag{3.7}$$

Fortunately, the proof of (3.7) is much easier than the proof of the corresponding inequality (2.12) in the case $1 < p \leq 2$. In the proof of Proposition 3$'$, we saw that $\cos\beta(2) = 1/\sqrt{d}$ and that $\beta(p)$ is strictly increasing as a function of p in $(2,\infty)$. When $p = 2$, there is equality in (3.7). When p increases in the interval $(2,\infty)$, $\cos\beta(p)$ is strictly decreasing and the right hand member in (3.7) is strictly increasing. Hence the inequality in (3.7) is correct and we have proved Proposition 1$'$.

4. Definitions and estimates of some harmonic measures

Without loss of generality we can assume that $0 \in D$. Let ω_t be harmonic in the component of $D \cap \{x \in \mathbf{R}^d : |x_1| < t\} = D \cap A(t)$ which contains 0 with boundary values 0 on $\partial D \cap A(t)$ and 1 on $\partial A(t) \cap \bar{D}$. Let $\omega_R(\cdot, \mathbf{R}^d, D)$ be harmonic in that component of $D \cap \{x \in \mathbf{R}^d : |x| < R\}$ which contains 0 with boundary values 0 on $\partial D \cap \{|x| < R\}$ and 1 on $\bar{D} \cap \{|x| = R\}$. In components which do not contain 0, these harmonic functions are defined to be 0. From the maximum principle we see that

$$\omega_R(x) \leq \omega_R(x, \mathbf{R}^d, D), \qquad x \in D \cap \{|x| < R\}. \tag{4.1}$$

Let us also assume that $|x_1|$ has a harmonic majorant Ψ in D. We introduce two functions v_t and w_t which are harmonic in D with the following boundary values:

$$v_t(x) = \begin{cases} 1, & x \in \partial D \setminus A(t), \\ 0, & x \in \partial D \cap A(t), \end{cases}$$

$$w_t(x) = \begin{cases} |x_1|, & x \in \partial D \setminus A(t), \\ 0, & x \in \partial D \cap A(t). \end{cases}$$

We define w_t to be the difference between $\Psi(x)$ and the harmonic function in D which has boundary values $|x_1|$ on $\partial D \cap A(t)$ and 0 on $\partial D \setminus A(t)$. We note that $D \cap A(t)$ is not necessarily bounded. According to Lemma A0, CD is not thin at infinity. Applying Lemma A1, we see that w_t is positive in D.

Remark 1. One of the main new ideas in [5] was to introduce a new kind of "harmonic measure" which depended on the (essentially unique) minimal harmonic function in $\Omega \supset D$ associated with the Martin boundary point at infinity. If Ω is a half-space $\{x_1 > 0\}$, this minimal function is x_1 and one of the examples of this "harmonic measure" coincides with the function w_t given above. If D is not contained in a half-space, we avoid the minimal harmonic functions and study the function w_t directly (cf. Theorem A1 and its corollaries!).

Remark 2. We warn the reader that the symbols Ψ, v_t, w_t and ω_t have also been used in [5]. Even if the definitions are related, they are not identical to the ones used in the present paper. However, if the same symbol is used, the two concepts play similar roles in the two papers.

If $x' = (x_2, x_3, \dots, x_d)$, we let $\omega_T^0(\cdot, s)$ be the harmonic measure of the set $\{x \in \mathbf{R}^d : |x_1| = T, |x'| > s\}$ in the strip $\{x \in \mathbf{R}^d : |x_1| < T\}$. From Lemma 6 in [2a], we deduce that

$$\omega_T^0(0, s) \le C_d T^{2-d} s^{d-2} e^{-s/T}, \tag{4.2}$$

where the constant C_d depends on the dimension only.

Let K_T^0 be a harmonic function in $\{|x_1| < T\}$ with boundary values $|x|$ on $\{x \in \mathbf{R}^d : |x_1| = T, |x'| > cT \log T\}$ and 0 on the rest of the boundary. From (4.2) we see that

$$K_T^0(0) = \mathbf{O}\big(T^{1-c} (\log T)^{d-2}\big), \qquad T \to \infty. \tag{4.3}$$

If $p > 1$ and ε is a number in $(0, p-1)$, we consider also the harmonic function K_T^1 in $\{|x_1| < T\}$ which has boundary values $|x|^{p-\varepsilon}$ on $\{x \in \mathbf{R}^d : |x_1| = T, |x'| > T^{p/(p-\varepsilon)}\}$ and 0 on the rest of the boundary. From (4.2) we see that

$$K_T^1(0) = \mathbf{O}\big(T^a \exp(-T^{\varepsilon/(p-\varepsilon)})\big), \qquad T \to \infty, \tag{4.4}$$

where a is a function of p, d and ε.

It is clear that the right hand member of (4.4) tends to 0 when $T \to \infty$. It is also clear that if $c > 1$, the right hand member (4.3) will tend to 0 when $T \to \infty$.

The following Phragmén–Lindelöf theorem is a simplified version of Theorem 2 in [2b].

Let u be subharmonic in the strip $\{x \in \mathbf{R}^d : |x_1| < \frac{1}{2}\}$, let u be non-positive on the boundary of the strip and assume that when $x \to \infty$ in the strip, we have

$$\max\big((u(x), 0\big) = o\big(|x|^{(2-d)/2} \exp(\pi |x|)\big). \tag{4.5}$$

Then u is non-positive in $\{|x_1| < \frac{1}{2}\}$.

5. Proof of Theorem 1

For $1 < p < \infty$ we know that there exists a function G_p which is superharmonic in D such that (0.2) holds. With α and β as in Sections 2 and 3, we have

$$C_p = \begin{cases} (\cos\alpha)^{-p}, & 1 < p \le 2, \\ (\cos\beta)^{-p}, & 2 < p < \infty. \end{cases}$$

(cf. (2.3) and (3.1)). The function C_p is continuous in the interval $(1, \infty)$, decreasing on $(1, 2]$ and increasing on $(2, \infty)$.

We shall use the Riesz representation theorem (cf. Theorem 6.18 in [10]).

Theorem R. *Let D be a domain in $\mathbf{R}^d$ having a Green's function G and let u be superharmonic on D. If $u \ge 0$, there is a unique measure μ on D such that $u = G\mu + h$ where $G\mu$ is the Green potential of μ and h is the greatest harmonic minorant of u on D.*

Let us assume that h^* is the least harmonic majorant of $|x_1|^p$ in D. If ε is a small positive number, it is easy to see that

$$|x_1|^{p-\varepsilon} \le h^*(x) + C(p, \varepsilon), \qquad x \in D,$$

where $C(p, \varepsilon) = \max(t^{p-\varepsilon} - t^p)$, $t \in [0, 1]$. According to (0.2) with p replaced by $p - \varepsilon$, we have

$$|x|^{p-\varepsilon} \le C_{p-\varepsilon} |x_1|^{p-\varepsilon} + G_{p-\varepsilon}(x) \le C_{p-\varepsilon} h^*(x) + G_{p-\varepsilon}(x) + Q(p, \varepsilon) \tag{5.1}$$

where $Q(p, \varepsilon) = C_{p-\varepsilon} C(p, \varepsilon) \to 0$, $\varepsilon \to 0_+$.

Since the right hand member is non-negative and superharmonic in D, we can apply Theorem R and see that there exists a measure μ_ε such that

$$C_{p-\varepsilon} h^*(x) + G_{p-\varepsilon}(x) + Q(p, \varepsilon) = h_\varepsilon(x) + G\mu_\varepsilon(x), \qquad x \in D,$$

where h_ε is the greatest harmonic minorant of $C_{p-\varepsilon}h^* + G_{p-\varepsilon} + Q(p,\varepsilon)$ in D. In particular, we have

$$|x|^{p-\varepsilon} \le (h_\varepsilon + G\mu_\varepsilon)(x), \qquad x \in D, \tag{5.2}$$

Lemma 5.1. *If* $|x_1|^p$ *has a harmonic majorant in* D *and if* (5.2) *holds, we have*

$$|x|^{p-\varepsilon} \le h_\varepsilon(x), \qquad x \in D. \tag{5.3}$$

The proof of Lemma 5.1 will be given later.

To complete the proof of Theorem 1, we use (5.1) and the definition of $G_{p-\varepsilon}$ to deduce the estimate

$$0 \le h_\varepsilon(x) \le C_{p-\varepsilon}h^*(x) + B_p(\varepsilon)|x|^{p-\varepsilon} + Q(p,\varepsilon), \qquad x \in D, \tag{5.4}$$

where

$$B_p(\varepsilon) = \begin{cases} 1, & 1 < p \le 2, \\ (p-\varepsilon)\tan\beta\big(F'(\beta)\big)^{-1}, & 2 < p-\varepsilon < \infty, \end{cases}$$

with $\beta = \beta(p-\varepsilon)$.

When $\varepsilon \to 0_+$, the constants tend to finite limits and we see that there is a constant C_0 which does not depend on ε such that for ε small, we have

$$0 \le h_\varepsilon(x) \le C_0\big(h^*(x) + |x|^{p-\varepsilon}\big) + Q(p,\varepsilon), \qquad x \in D.$$

In a standard way we can now prove that $\operatorname{grad} h_\varepsilon$ is locally uniformly bounded in D as $\varepsilon \to 0_+$. From the Arzelà–Ascoli theorem, it follows that there exists a sequence $\{\varepsilon_n\}$ tending to zero such that $\{h_{\varepsilon_n}\}_{n=1}^\infty$ converges pointwise to a function h_0 in D as $n \to \infty$: the convergence is locally uniform. The function h_0 is harmonic in D and a majorant of $|x|^p$.

To prove the Corollary, we first note that (0.1) is an immediate consequence of the identity

$$C_{p-\varepsilon}h^*(0) = h_\varepsilon(0) + G\mu_\varepsilon(0) - Q(p,\varepsilon).$$

We give the details of the proof that C_p is best possible in the case $1 < p \le 2$. We recall that $\alpha = \alpha(p)$ and $F = F_p$ were defined in Section 2. If $e_1 = (1,0,\dots,0)$ and $s \in (1,p)$, we consider the cones

$$\Gamma_p = \{x \in \mathbf{R}^d : 0 \le \theta < \alpha(p)\}, \quad \Gamma'_p = \{x \in \mathbf{R}^d : x + e_1 \in \Gamma_p\},$$

and the inequalities

$$|x_1|^s \le |x_1 + 1|^s + 1, \qquad x \in \Gamma'_s, \tag{5.5}$$

(5.6) $$|x+e_1|^s \le |x|^s + 2|x| + 1, \qquad x \in \Gamma'_s.$$

We define $\varphi = \arccos\big((1+x_1)/|e_1+x|\big)$. Let U_s^* and U_s be the least harmonic majorants of $|1+x_1|^s$ and $|x+e_1|^s$ in Γ'_p, respectively. It is easy to see that

$$U_s^*(x) = |x+e_1|^s \big(\cos\alpha(p)\big)^s F_s(\varphi)/F_s\big(\alpha(p)\big),$$
$$U_s(x) = |x+e_1|^s F_s(\varphi)/F_s\big(\alpha(p)\big).$$

Let h_s^* and h_s be the least harmonic majorants of $|x_1|^s$ and $|x|^s$ in Γ'_p, respectively. Let V be the least harmonic majorant of $2|x|+1$ in Γ'_q, where $q \in (1,p)$ is fixed. If $q < s < p$, it follows from (5.5) and (5.6) that

$$h_s^*(x) \le U_s^*(x) + 1, \qquad U_s(x) \le h_s(x) + V(x), \qquad x \in \Gamma'_p.$$

Since $\lim_{s\to p} F_s\big(\alpha(p)\big) = 0$, we conclude that

$$\liminf_{s\to p} h_s(0)/h_s^*(0) \ge \big(\cos(\alpha(p))\big)^{-p} = C_p,$$

which proves that C_p is best possible.

Proof of Lemma 5.1. From Corollary A2 with $\Phi(t) = t^p$, we see that

(5.7) $$T^p \omega_T(0) \to 0, \qquad T \to \infty.$$

From now on, we assume that T is large. Let g_T be harmonic in $D \cap \{|x_1| < T\}$ with boundary values $|x|^{p-\varepsilon}$ on $\bar{D} \cap \{|x_1| = T\}$ and zero on $\partial D \cap \{|x_1| < T\}$. We claim that

$$g_T \le 2T^p \omega_T + K_T^1,$$

in $D \cap \{|x_1| < T\}$. (The definition of K_T^1 is given in Section 4.)

This follows from Lemma A1 since

$$|x|^{p-\varepsilon} \le 2T^p, \qquad |x_1| = T, \qquad |x'| \le T^{p/(p-\varepsilon)}.$$

We recall that $x' = (x_2, \ldots, x_d)$. Using (4.4) and (5.7), we deduce that

(5.8) $$g_T(0) \to 0, \qquad T \to \infty.$$

If $|x|^{p-\varepsilon} - h_\varepsilon(x)$ has a positive lower bound on a set of positive harmonic measure on ∂D, then it would follow from (5.2) that $G\mu_\varepsilon$ would have a positive harmonic minorant in D which is impossible (cf. Corollary 6.19 in [10]). Hence $|x|^{p-\varepsilon} - h_\varepsilon(x)$ has non-positive boundary values on ∂D. Let us now in $D \cap \{|x_1| < T\}$ consider the function $|x|^{p-\varepsilon} - (h_\varepsilon + g_T)(x)$ which is non-positive on $\partial\{D \cap \{|x_1| < T\}\}$ and dominated from above by $|x|^{p-\varepsilon}$. This growth is far below the bound below which a Phragmén–Lindelöf theorem holds in the strip (cf. (4.5)). Hence we see that

$$|x|^{p-\varepsilon} \le (h_\varepsilon + g_T)(x), \qquad x \in D \cap \{|x_1| < T\}.$$

Finally, letting $T \to \infty$ and using (5.8), we obtain (5.3). We have proved Lemma 5.1 and completed the proof of Theorem 1.

6. Proof of Theorem 2

We first note that $|x_1|\log^+|x_1|$ has a harmonic majorant in D if and only if $f(x_1) = \sqrt{1+x_1^2}\log(1+x_1^2)$ has a harmonic majorant in D. Assuming that f has a harmonic majorant in D, we shall prove that $g(x) = C_d\big(1+|x|^2\big)^{1/2}$ has a harmonic majorant in D. Here the constant C_d is chosen in such a way that $u = f - g$ is subharmonic in $\mathbf{R}^d$ (we can choose $C_d = 2/d$).

We can now argue as in the proof of Theorem 4 in [5]. Let h_0 be the least harmonic majorant of f in D. Since we have $u \le h_0 - g$, $h_0 - u$ is a nonnegative superharmonic function in D. According to Theorem R, we have $h_0 - u = G\mu + h$, where h is the greatest harmonic minorant of $h_0 - u$ in D and $G\mu$ is the Green potential of a nonnegative measure μ in D.

We wish to prove that h is a harmonic majorant of g in D. So far, we know that $g \le G\mu + h$. Just as in the proof of Theorem 1, it is clear that $g - h$ is non-positive on ∂D. Applying Corollary A1 with $\varphi(t) = t\log^+ t$, we see that

$$T\log T\omega_t(0) \to 0, \qquad T \to \infty. \tag{6.1}$$

From now on, we assume that T is large. Let g_T be harmonic in $D\cap \big\{|x_1| < T\big\}$ with boundary values $|x|$ on $\bar{D}\cap\big\{|x_1| = T\big\}$ and zero on $\partial D\cap \big\{|x_1| < T\big\}$. We claim that

$$g_T \le 3T\log T\omega_T + K_T^0 \qquad \text{in } D\cap\big\{|x_1 < T\big\}.$$

(The definition of K_T^0 can be found in Section 4: let us choose the constant c in the definition to be 2.)

This follows from Lemma A1 since

$$|x| \le 3T\log T, \qquad |x_1| = T, \qquad |x'| < 2T\log T.$$

Using (4.3) and (6.1), we deduce that

$$g_T(0) \to 0, \qquad T \to \infty. \tag{6.2}$$

Let us now consider the function $g - h - 2C_d g_T$ in $D\cap\big\{|x_1| < T\big\}$: it is non-positive on the boundary of this set and dominated from above by $|x|$. The same Phragmén–Lindelöf argument as in the proof of Theorem 1 tells us that

$$g \le h + 2C_d g_T \qquad \text{in } D\cap\big\{|x_1| < T\big\}.$$

Letting $T \to \infty$ and using (6.2) we see that h is a harmonic majorant of g. This concludes the proof of Theorem 2a.

In the proof of Theorem 2b, we assume that $|x|$ has a harmonic majorant in D and that the least harmonic majorant Ψ of $|x_1|$ in D is such that $\Psi(x) \le C|x|$ when $x \in D$ is large. It is known that $w_t \le \Psi$ in D. We define $\Gamma_t = \{x \in D : \Psi(x) = t\}$. It is easy to see that $|x| > t/C$ on Γ_t and thus that

$$w_t(x) \le t\omega_{t/C}(x, \mathbf{R}^d, D), \qquad x \in D \cap \{|x| \le t/C\}. \tag{6.3}$$

In the last step we used the maximum principle.

Our assumption that $|x|$ has a harmonic majorant in D implies that

$$\int_0^\infty \omega_t(0, \mathbf{R}^d, D)\, dt < \infty, \tag{6.4}$$

(cf. Theorem 2 in [8] with $\Phi(t) = t$!).

Another consequence of the maximum principle is the inequality

$$\omega_t(0) \le \omega_t(0, \mathbf{R}^d, D). \tag{6.5}$$

From (6.4) and (6.5) we deduce that $t\omega_t(0) \to 0$ as $t \to \infty$: this is (A4) in Lemma A5 and Theorem A1. Combining (6.3) and (6.4) we obtain

$$\int_1^\infty w_t(0) t^{-1} dt < \infty,$$

This is condition (A 7′) with $\Phi(t) = t \log^+ t$. Since (A7) is equivalent to (A 7′) we can apply Theorem A1 and conclude that $|x_1| \log^+ |x_1|$ has a harmonic majorant in D. We have proved Theorem 2.

Remark. Let h_g be the least harmonic majorant of g in D. It follows from the proof that we have

$$h_g(0) \le h_0(0) + 2d^{-1}.$$

(We recall that h_0 is the least harmonic majorant of f in D.)

Appendix

Definitions of the harmonic measures used in the Appendix are given in Section 4.

Let D be a domain in $\mathbf{R}^d$, $d \ge 2$. We shall many times need the maximum principle in the form given below in Lemma A1. In order to show that all domains D which are studied in the present paper are such that the complement CD is not thin at infinity, we need a simple observation which we state in Lemma A0. For a discussion of thinness and its relation to harmonic measure, we refer to [8].

Lemma A0. *Assume that $|x_1|$ has a harmonic majorant Ψ in D. Then CD is not thin at infinity.*

Proof. Let $f : [0,\infty) \to [1,\infty)$ be strictly increasing, continuous and unbounded. Furthermore, we assume that as $t \to \infty$, we have

(i) $$f(t)/t \to \infty,$$

(ii) $$t^{-1}\bigl(\log f(t)\bigr) \to 0,$$

(iii) $$t\bigl(f(t)\bigr)^{-1}\bigl(\log f(t)\bigr) \to 0.$$

Necessary and sufficient for CD not to be thin at infinity is that

$$\omega_R(0,\mathbf{R}^d,D) \to 0, \qquad R\to\infty, \quad \text{when } d\ge 3,$$

$$(\log R)\omega_R(0,\mathbf{R}^d,D) \to 0, \qquad R\to\infty, \quad \text{when } d=2,$$

(cf. the proof of Lemma 4 and Lemma 6 in [8]).

We write $\omega_R(\cdot,\mathbf{R}^d,D)$ as $\omega_R^{(1)}+\omega_R^{(2)}$, where $\omega_R^{(1)}$ is the harmonic measure of $D\cap\{x\in\mathbf{R}^d : |x|=R, |x|<f(|x_1|)\}$ in D and $\omega_R^{(2)}$ is the harmonic measure of the rest of $D\cap\{|x|=R\}$ in D. Since $|x_1|>f^{-1}(R)$ on $\{x\in\mathbf{R}^d : |x|=R<f(|x_1|)\}$, it follows from the classical maximum principle for harmonic functions in a bounded domain that

$$f^{-1}(R)\omega_R^{(1)}(x,\mathbf{R}^d,D)\le\Psi(x), \qquad x\in D\cap\{x\in\mathbf{R}^d : |x|<R\}.$$

To estimate $\omega_R^{(2)}$ we note that this function is majorized by the harmonic function in $\{|x|<\mathbf{R}\}$ which is 1 on $\{x\in\mathbf{R}^d : |x|=R>f(|x_1|)\}$ and 0 on the rest of $\{|x|=R\}$. We conclude that

$$\omega_R^{(2)}(0,\mathbf{R}^d,D)\le \text{Const.}\,\bigl(f^{-1}(R)/R\bigr)^{d-1}.$$

Combining these two estimates we obtain

$$\omega_R(0,\mathbf{R}^d,D)\le\Psi(0)/\bigl(f^{-1}(R)\bigr)+\text{Const.}\,\bigl(f^{-1}(R)/R\bigr)^{d-1}.$$

Letting $R\to\infty$, it is easy to check that the criteria for CD not to be thin at infinity are fulfilled in the two cases $d\ge 3$ and $d=2$. This concludes the proof of Lemma A0.

Lemma A1. *Let D be a domain in $\mathbf{R}^d$, $d \geq 2$, such that CD is not thin at infinity. If h is harmonic and bounded from above in D and if h has non-positive boundary values on ∂D, then $h \leq 0$ in D.* (Cf. Lemma 1 in [5].)

In [7] we considered the question whether certain radial functions had harmonic majorants and it was natural to work in subdomains of the type $D \cap \{|x| < R\}$. In the present paper we apply the same technique to subdomains of the type $D \cap \{|x_1| < t\} = D \cap A(t)$.

Let $\mathcal{L}$ be the class of convex increasing functions on $[0, \infty)$ which are such that if $\Phi \in \mathcal{L}$, we have $\Phi(0) = 0$, $\Phi'(0) = 0$ and $\lim_{t\to\infty} \Phi(t)/t > 0$. Let $g(x, \cdot)$ be Green's function for D with pole at x.

Lemma A2. *Let $\Phi \in \mathcal{L}$. If $\Phi(|x_1|)$ has a harmonic majorant in D, then $g(x, \cdot)$ exists and we have*

$$\int_0^\infty \Phi'(t) v_t(0)\, dt + \int_D g(0, y) \Phi''(|y_1|)\, dy < \infty. \tag{A1}$$

Proof. Let h be the least harmonic majorant of $\Phi(|x_1|)$ in D. In $D_t = A(t) \cap D$ we have $\Phi(|x_1|) = (h_t - P_t)(x)$, where h_t is the least harmonic majorant of $\Phi(|x_1|)$ in $A(t) \cap D$ and

$$P_t(x) = \text{Const.} \int_{D_t} g_t(x, y) \Phi''(|y_1|)\, dy.$$

Here $g_t(x, \cdot)$ is Green's function for D_t with pole at x. Since $h_t \leq h$ for all t, the sequence $\{h_n\}_1^\infty$ converges to h_0 which is harmonic in D and a majorant of $\Phi(|x_1|)$. It follows that $h_0 = h$.

Thus the sequence $\{P_n\}_1^\infty$ converges to a function P in D defined by

$$P(x) = \text{Const.} \int_D g(x, y) \Phi''(|y_1|)\, dy.$$

We conclude that the second integral in (A1) is finite.

Using Lemma A1 we see that $h \geq \Phi(t) v_t$ in D for all $t > 0$. It follows that the first integral in (A1) is at most $2h(0)$. This completes the proof of Lemma A2.

Outside their domains of definition we define the Green functions to be zero. Let $d\sigma(y) = dy_2 \cdots dy_d$. We introduce

$$\gamma(t) = \int_{\{y_1 = t\}} g(0, y)\, d\sigma(y).$$

Lemma A3. *Let $\Phi \in \mathcal{L}$, and let $\Phi(|x_1|)$ have a harmonic majorant in D and assume furthermore that $\Phi(t)/t \to \infty$, $t \to \infty$. Then $\gamma(t) + \gamma(-t) \to 0$, $t \to \infty$.*

Proof. Let us first prove that if $\gamma(t_0) < \infty$ for some $t_0 > 0$, then $\gamma(t)$ is decreasing on $[t_0, \infty)$. This is a consequence of the proof of Lemma A2. The subharmonic function $g_n(0,\cdot)$ is nonnegative and bounded in the strip $\{y \in \mathbf{R}^d : 0 < t_0 < y_1 < n\}$: it is integrable over $\{y_1 = t_0\}$ and vanishes on $\{y_1 = n\}$. Hence $g_n(0,\cdot)$ is dominated by the Poisson integral in the strip with boundary values $g_n(0,\cdot)$ on $\{y_1 = t_0\}$ and 0 on $\{y_1 = n\}$. If $\gamma_n(t)$ is the integral of $g_n(0,\cdot)$ over the hyperplane $\{y_1 = t\}$, we deduce that

$$\gamma_n(t) \le \gamma_n(t_0), \qquad t_0 < t < n.$$

Letting $n \to \infty$, we see that $\gamma(t) \le \gamma(t_0)$, $t > t_0$. The same argment shows that $\gamma(t)$ is decreasing on $[t_0, \infty)$ and that $\gamma(t)$ is increasing on $(-\infty, -t_0]$ if $\gamma(-t_0) < \infty$.

From the convergence of the second integral in (A1) and the monotonicity of $\gamma(t) + \gamma(-t)$ on $(0, \infty)$, we see that

$$\big(\gamma(T) + \gamma(-T)\big)\Phi'(T) \le \int_D g(0,y)\Phi''\big(|y_1|\big)\,dy < \infty.$$

Since $\Phi'(T) \to \infty$ as $T \to \infty$ we obtain the conclusion of Lemma A3.

Let $\tilde\omega_t$ be harmonic in $A(2t) \setminus \big(\overline{A(t)} \setminus D\big)$ with boundary values 0 on $\overline{A(t)} \setminus D$ and 1 on $\partial A(2t)$. We extend $\tilde\omega_t$ to a function superharmonic in $\mathbf{R}^d \setminus \big(\overline{A(t)} \setminus D\big)$ by defining $\tilde\omega_t$ to be 1 on $\{x \in \mathbf{R}^d : |x_1| \ge 2t\}$.

Comparing boundary values of the harmonic functions in $D \cap A(2t)$ and in $D \cap A(t)$, we see that

$$\omega_t(0) \ge \tilde\omega_t(0) \ge \omega_{2t}(0). \tag{A2}$$

In D, we have $\tilde\omega_t = H_t + Q_t$, where H_t is the greatest harmonic minorant of $\tilde\omega_t$ in D and Q_t is a Green potential. It is easy to see that $H_t(0) \le v_t(0)$.

Let μ be the Riesz mass of Q_t restricted to $\mathbf{R}^d \setminus \big(\overline{A(t)} \setminus D\big)$: it is known that $\operatorname{supp}\mu \subset \partial A(2t)$. We define $K(x) = \big(|x_1| - t\big)^+/t$, $t < |x_1| < 2t$. Then we have

$$d\mu(x)/d\sigma = \text{Const.}\,\big|(\partial\tilde\omega_t/\partial n)(x)\big| \le \text{Const.}\,\big|(\partial K/\partial n)(x)\big| = \text{Const.}\,t^{-1}, \ |x_1| = 2t,$$

$$Q_t(0) = \int_{\partial A(2t)} g(0,y)\,d\mu(y) \le \text{Const.}\,t^{-1}\big(\gamma(2t) + \gamma(-2t)\big). \tag{A3}$$

Lemma A4. *Under the assumptions of Lemma* A3, *we have* $t\omega_t(0) \to 0$, $t \to \infty$.

Proof. The convergence of the first integral in (A1) implies that we have $\Phi(t)v_t(0) \to 0$, $t \to \infty$ and thus that $tv_t(0) \to 0$, $t \to \infty$. From the discussion above we see that

$$t\tilde{\omega}_t(0) \le t(H_t + Q_t)(0) \le tv_t(0) + \text{ Const.}\big(\gamma(2t) + \gamma(-2t)\big).$$

From Lemma A3 it is clear that the right hand member tends to 0 as $t \to \infty$. Using (A2) we see that $t\omega_t(0) \to 0$, $t \to \infty$. We have proved Lemma A4.

Lemma A5. *Let the domain D be such that*

$$\text{(A4)} \qquad t\omega_t(0) \to 0, \qquad t \to \infty,$$

$$\text{(A5)} \qquad \int_0^\infty t\, d\big(-v_t(0)\big) < \infty.$$

Then we have

$$\text{(A6)} \qquad \omega_{2t}(x) \le w_t(x)/t, \qquad x \in A(2t) \cap D.$$

Remark. This is Lemma 6b in [5], adapted to the situation in the present paper. The proof is similar to the proof in [5]. We include the details for completeness.

Proof. From our assumptions we see that there exists a function $L_0\colon [0,\infty) \to [0,\infty)$ such that $L_0(t)/t \to \infty$, $t \to \infty$, which is such that

$$\int_0^\infty L_0(t)\, d\big(-v_t(0)\big) < \infty, \qquad L_0(t)\omega_t(0) \to 0, t \to \infty.$$

Let L be the greatest convex minorant of L_0. Clearly, we have $L(t)/t \to \infty$, $t \to \infty$. Since $L\big(|x_1|\big)$ is subharmonic in D, we can use Lemma A1 in $A(t) \cap D$ to deduce that

$$L\big(|x_1|\big) \le \int_0^\infty L(t)\, d\big(-v_t(x)\big) + L(t)\omega_t(x), \qquad x \in A(t) \cap D.$$

Letting $t \to \infty$, we see that $L\big(|x_1|\big)$ has a harmonic majorant h in D. We define $m(t) = \inf h(x)$, $x \in D \cap \big\{|x_1| = t\big\}$. Once more using Lemma A1 we deduce that if $t < t_1$ we have

$$|x_1| - w_t(x) \le t + t_1 h(x)/m(t_1), \qquad x \in D \cap \big(A(t_1) \setminus A(t)\big).$$

Letting $t_1 \to \infty$ we see that

$$w_t(x) \ge |x_1| - t, \qquad x \in D \cap A(t); \quad w_t(x) \ge t, \quad x \in D \cap \big\{|x_1| = 2t\big\}.$$

Applying Lemma A1 we obtain (A6). We have proved Lemma A5.

Theorem A1. *Let $\Phi \in \mathcal{L}$ and assume furthermore that $\Phi(t)/t \to \infty$, $t \to \infty$. Then $\Phi(|x_1|)$ has a harmonic majorant in D if and only if* (A4) *and* (A7) *hold:*

(A7)
$$\int_0^\infty v_t(0)\, d\Phi(t) < \infty.$$

Remark. Condition (A7) is equivalent to

(A7′)
$$\int_0^\infty w_t(0)\, d\big(\Phi(t)/t\big) < \infty.$$

Remark. Theorem A1 is an analogue of Theorem 1 in [5].

Proof. If $\Phi(|x_1|)$ has a harmonic majorant in D, (A4) and (A7) are consequences of Lemmas A2 and A4.

Conversely, assume that (A4) and (A7′) hold. It follows that we have $\Phi(t)w_t(x)/t \to 0$, $t \to \infty$. Let $c > 1$ be given. From the proof of Lemma A5 we see that $\omega_{tc}(x) \le (c-1)^{-1} w_t(x)/t$, $x \in A(tc) \cap D$. We conclude that $\Phi(t/c)\omega : t(x) \to 0$, $t \to \infty$. It follows that

$$h(x, c) = \int_0^\infty \Phi(t/c)\, d\big(-v_t(x)\big)$$

is a harmonic majorant of $\Phi(|x_1|/c)$ in D. Letting $c \downarrow 1$, we obtain an increasing sequence of harmonic functions which is bounded at 0 since (A7) holds. This proves Theorem 1.

For functions Φ in $\mathcal{L}$ we define $\Lambda(t) = \int_0^t s\, d\big(\Phi(s)/s\big)$.

Corollary A1. *Let Φ be as in Theorem* A1. *If $\Phi(|x_1|)$ has a harmonic majorant in D then*

(A8)
$$\int_0^\infty \Lambda'(t)\omega_{2t}(0)\, dt < \infty,$$

(A9)
$$\Phi(t)\omega_{2t}(0) \to 0, \qquad t \to \infty.$$

Remark. In Theorem 2 in [5] it is proved that a condition which formally looks almost exactly like (A8) is equivalent to the existence of a certain harmonic majorant. In the context of the present paper we can only prove Corollary A1. We note that the definitions of ω_t in [5] and here are different.

Proof. If $\Phi(|x_1|)$ has a harmonic majorant in D, it follows as above that (A4) and (A7) hold. From Lemma A5 we see that (A6) holds. Our conclusion (A8) is an immediate consequence of (A7′) and (A6). Since (A7′) holds we know that $\Phi(t)w_t(x)/t \to 0$, $t \to \infty$. Using this fact and (A6) we obtain (A9).

Corollary A2. *Let Φ be as in Theorem* A1. *Furthermore, we assume that Φ satisfies a doubling condition when t is large and that*

$$\Phi(t) \sim \Lambda(t) \qquad \text{for all large } t. \tag{A10}$$

Then $\Phi(|x_1|)$ has a harmonic majorant in D if and only if

$$\int_0^\infty \Phi'(t)\omega_t(0)\,dt < \infty. \tag{A11}$$

Consequently we have

$$\Phi(t)\omega_t(0) \to 0, \qquad t \to \infty. \tag{A12}$$

Proof. Under our assumptions conditions (A8) and (A11) are equivalent. Corollary A2 is now an immediate consequence of Theorem A1 and Corollary A1.

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Received 28 March 1989