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CONNECTEDNESS IN FINE TOPOLOGIES

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1. Introduction

If E is an arbitrary subset of Euclidean n-space $\mathbf{R}^n$, let $B_{\alpha,p}(E)$ denote the Bessel capacity of E, $0<\alpha<\infty$, $1<p\le n/\alpha$, that is

$$B_{\alpha,p}(E)=\inf\bigl\{\|f\|_p^p : f\in L_+^p(\mathbf{R}^n),\ G_\alpha * f\ge 1 \quad \text{in } E\bigr\}.$$

Here $L^p(\mathbf{R}^n)$ is the usual Lebesgue space of p-th power summable functions, $L_+^p(\mathbf{R}^n)$ the nonnegative elements, $\|f\|_p$ the usual norm of f in L_p, and $G_\alpha * f$ the convolution over $\mathbf{R}^n$ of f with the Bessel kernel G_α, best defined by its Fourier transform $\hat{G}_\alpha(\xi)=\bigl(1+|\xi|^2\bigr)^{-\alpha/2}$, see e.g. [St]. The reader should note that the Bessel capacity is a Choquet capacity.

As usually in nonlinear potential theories, we say that the set E is (α,p)-*thin* at x in $\mathbf{R}^n$ if the Wiener integral converges,

$$\int_0^1 \bigl(r^{\alpha p-n}B_{\alpha,p}(E\cap B(x,r))\bigr)^{1/(p-1)}\,\frac{dr}{r}<\infty. \tag{1.1}$$

Here $B(x,r)$ is the open ball $\bigl\{y\in\mathbf{R}^n : |x-y|<r\bigr\}$. If E is not (α,p)-thin at x, then we say that E is (α,p)-*fat* at x. The set $b(E)$ of points at which E is (α,p)-fat is called the (α,p)-*base* of E.

We define the (α,p)-*fine topology*, $\tau_{\alpha,p}$, to be the collection of all sets $V\subset\mathbf{R}^n$ such that V^c, the complement of V, is (α,p)-thin at each $x\in V$. Thus V is an (α,p)-fine neighborhood of $x\in V$ if and only if V^c is (α,p)-thin at x, cf. [M2, p. 162]. Topological concepts in (α,p)-fine topology are equipped with the phrase "(α,p)-fine", for example (α,p)-finely open, (α,p)-finely connected, or if no confusion arises, finely open, finely compact, etc.

The particular case of (α,p)-fine topologies when $\alpha=1$ and $1<p\le n$ is related to second order elliptic equations. As well known, the $(1,2)$-fine topology coincides with the classical fine topology of H. Cartan, the coarsest topology on

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$\mathbf{R}^n$ making all superharmonic functions continuous. A similar result holds for all p not greater than n: the $(1,p)$-fine topology is the coarsest topology in which all supersolutions of the p-Laplace equation

$$\operatorname{div}\big(|\nabla u|^{p-2}\nabla u\big) = 0 \tag{1.2}$$

are continuous. In effect, the equation (1.2) can be replaced by a more general degenerate elliptic equation

$$\operatorname{div}\mathcal{A}(x,\nabla u) = 0 \tag{1.3}$$

where $\mathcal{A}(x,h)\cdot h \approx |h|^p$. See [HKM] and Section 6 below.

There are several sources for the various properties of the Bessel capacities, the associated nonlinear potentials, and the use of the (α,p)-fine topologies in analysis. We refer the reader e.g. to [AH], [AL], [AM], [Hed], [HW], [M1–2], [MK], and [R2]. See also [F1]. However, topological properties of (α,p)-fine topologies are not yet thoroughly investigated. In [AL] D.R. Adams and J.L. Lewis showed that for $\alpha p > 1$ each (α,p)-finely open and (α,p)-finely connected set is arcwise connected; the result is false if $\alpha p \le 1$. Our main results in this paper assert that the (α,p)-fine topology is locally connected (provided $\alpha p > 1$) and it obeys Doob's quasi-Lindelöf principle: any collection of (α,p)-finely open sets has a countable subcollection whose union differs from the union of the whole family only by a set of (α,p)-capacity zero. See Sections 2 and 3. In classical potential theory these two properties are proved using the balayage of measures, a tool which is not available in this nonlinear setting. In the linear situation these results are found in [F2] and [D].

We will also show that if $\alpha p > n-1$, then an (α,p)-finely open set is finely connected, arcwise connected and euclidean connected at the same time, the assertion being false if $\alpha p \le n-1$. See Section 5. The case $\alpha p > n-1$ thus resembles the classical plane case where this result is known [F3], [GL].

In the final section, Section 6, we apply the aforementioned arcwise connectedness result for asymptotic paths of $\mathcal{A}$-subharmonic functions.

2. The quasi-Lindelöf property

In this section we show that the (α,p)-fine topologies obey Doob's quasi-Lindelöf principle.

First we prove an auxiliary result, a modified Wiener criterion.

2.1. Lemma. *Let $x_0 \in \mathbf{R}^n$. Suppose that x_k is a sequence of points with $|x_k - x_0| < 2^{-k-2}$, $k = 1,2,\ldots$. If $B_k(x_k) = B(x_k, 3\cdot 2^{-k-2})$, then a set $E \subset \mathbf{R}^n$ is (α,p)-thin at x_0 if and only if*

$$\sum_{k=1}^{\infty}\Big(2^{-k(\alpha p-n)}B_{\alpha,p}\big(E\cap B_k(x_k)\big)\Big)^{1/(p-1)} < \infty.$$

Proof. Since

$$B(x_0, 2^{-k-1}) \subset B_k(x_k) \subset B(x_0, 2^{-k}),$$

the claim follows easily.

The Kellogg property was proved in [HW, Theorem 2]:

2.2. The Kellogg property. *Let E be any set in $\mathbf{R}^n$. If $e_{\alpha,p}(E)$ is the set of all points at which E is (α,p)-thin, then*

$$B_{\alpha,p}\big(e_{\alpha,p}(E) \cap E\big) = 0.$$

To state our main result in this section recall that $\tau_{\alpha,p}$ has the *quasi-Lindelöf property* if for each family $\{U_\lambda\}$, $\lambda \in \Lambda$, of (α,p)-finely open sets there is a countable set $\Gamma \subset \Lambda$ such that

$$B_{\alpha,p}\Big(\bigcup_{\lambda\in\Lambda} U_\lambda \setminus \bigcup_{\lambda\in\Gamma} U_\lambda\Big) = 0.$$

We prove

2.3. Theorem. *The (α,p)-fine topology $\tau_{\alpha,p}$ has the quasi-Lindelöf property.*

Proof. We make use of the following local capacity: Let $\{B_k\}$, $k = 1,2,\ldots$, be the collection of all balls $B \subset \mathbf{R}^n$ with rational centers and radii. Write

$$\operatorname{cap}(E) = \sum_{k=1}^{\infty} 2^{-k} \frac{B_{\alpha,p}(E \cap B_k)}{B_{\alpha,p}(B_k)},$$

for $E \subset \mathbf{R}^n$. Then, clearly, $\operatorname{cap}(\cdot)$ is a subadditive set function and $B_{\alpha,p}(E) = 0$ if and only if $\operatorname{cap}(E) = 0$.

Suppose then that the sets U_λ, $\lambda \in \Lambda$, are (α,p)-finely open and that

$$U = \bigcup_{\lambda\in\Lambda} U_\lambda .$$

Let

$$\delta = \inf\Big\{\operatorname{cap}\big(U \setminus \bigcup_{\lambda\in\Gamma} U_\lambda\big) : \Gamma \subset \Lambda \text{ countable}\Big\}.$$

Then choosing countable sets $\Gamma_j \subset \Lambda$, $j = 1,2,\ldots$, with

$$\operatorname{cap}\big(U \setminus \bigcup_{\lambda\in\Gamma_j} U_\lambda\big) \le \delta + 1/j$$

and putting

$$\Gamma_0 = \bigcup_{j=1}^{\infty} \Gamma_j$$

we obtain

$$\delta = \operatorname{cap}(F)$$

where

$$F = U \setminus \bigcup_{\lambda \in \Gamma_0} U_\lambda.$$

To complete the proof we show that $\delta = 0$. Suppose, on the contrary, that $\delta > 0$. Since $B_{\alpha,p}(F) > 0$, it follows from the Kellogg property that there is a point $x \in F \cap b(F) \subset U$. Then choose $\lambda \in \Lambda$ such that $x \in U_\lambda$. Now, since $F \setminus U_\lambda$ is (α,p)-thin and F (α,p)-fat at x, Lemma 2.1 enables us to pick a ball B_k with rational center and radius such that

$$B_{\alpha,p}\big((F \setminus U_\lambda) \cap B_k\big) < B_{\alpha,p}(F \cap B_k)$$

whence

$$\operatorname{cap}(F \setminus U_\lambda) < \operatorname{cap}(F) = \delta$$

which is a contradiction. The theorem follows.

3. Local connectedness

This section is devoted to proving that the (α,p)-fine topology is locally connected provided $\alpha p > 1$. This is not true if $\alpha p \le 1$.

Recall that the (α,p)-base of the set E is

$$b(E) = \big\{x \in \mathbf{R}^n : E \quad \text{is } (\alpha,p)\text{-fat at } x\big\}.$$

We have

3.1. Proposition. *The (α,p)-base $b(E)$ of a set $E \subset \mathbf{R}^n$ is a G_δ-set.*

Proof. The sets

$$G_k = \Big\{x \in \mathbf{R}^n : \int_0^1 \big(r^{\alpha p - n} B_{\alpha,p}(E \cap B(x,r))\big)^{1/(p-1)} \frac{dr}{r} > k\Big\}$$

are easily seen to be open and, clearly,

$$b(E) = \bigcap_{k=1}^{\infty} G_k.$$

The lemma follows.

We call a set $U \subset \mathbf{R}^n$ *(α,p)-finely regular* if $U^c = b(U^c)$. Proposition 3.1 immediately yields

3.2. Corollary. *An (α,p)-finely regular set U is (α,p)-finely open and of type F_σ.*

3.3. Remark. It is easily seen that the (α,p)-fine interior of any (α,p)-finely closed set is (α,p)-finely regular.

Next we establish the Lusin–Menchoff property (or binormality) for the (α,p)-fine topology.

3.4. Theorem. *Suppose that $F \subset \mathbf{R}^n$ is (euclidean) compact and $T \subset \mathbf{R}^n$ (α,p)-finely closed with $T \cap F = \emptyset$. Then there is an open set $G \subset \mathbf{R}^n$ such that*

$$T \subset G \subset G \cup b(G) \subset F^c.$$

Proof. We proceed with the proof inductively. Put $T_0 = T$ and let $k \in \mathbf{N}$. Then choose a finite set $\mathcal{Z}_k \subset F$ such that

$$F \subset \bigcup_{z\in\mathcal{Z}_k} B(z, 2^{-k-2}).$$

Write

$$P_k = \{x \in \mathbf{R}^n : \operatorname{dist}(x,F) \le 3\cdot 2^{-k-3}\}.$$

For each $j = 1,2,\dots,k$ and $z \in \mathcal{Z}_j$ choose an open neighborhood $G_{k,j,z}$ of T_{k-1} such that

$$B_{\alpha,p}\big(G_{k,j,z} \cap B_j(z)\big) \le (2-2^{-k})\, B_{\alpha,p}\big(T \cap B_j(z)\big)$$

where $B_j(z) = B(z, 3\cdot 2^{-j-2})$. Then putting

$$T_k = T_{k-1} \cup \Big(\bigcap_{j,z} G_{k,j,z} \setminus P_k\Big)$$

we obtain

$$B_{\alpha,p}\big(T_k \cap B_j(z)\big) \le (2-2^{-k})\, B_{\alpha,p}\big(T \cap B_j(z)\big) \tag{3.5}$$

for every $j = 1,2,\dots,k+1$ and $z \in \mathcal{Z}_j$ since

$$T_k \cap B_{k+1}(z) \subset T_k \cap P_k \subset T.$$

To complete the proof write

$$G = \bigcup_{k=1}^{\infty} T_k.$$

Clearly, G is open and $T \subset G \subset F^c$. Moreover, it follows from (3.5) that

$$B_{\alpha,p}\big(G \cap B_j(z)\big) \le 2\, B_{\alpha,p}\big(T \cap B_j(z)\big)$$

for all $j \in \mathbf{N}$ and $z \in \mathcal{Z}_j$. Hence Lemma 2.1 implies that

$$b(G) \cap F = \emptyset$$

and the theorem is proved.

3.6. Remarks. (a) Stated differently, Theorem 3.4 says that F has an (α,p)-finely open neighborhood U and T an open neighborhood G such that $U \cap G = \emptyset$.

(b) Theorem 3.4 holds also if F is assumed to be closed instead of compact, see Corollary 3.8 below and the proof of [LMZ, 10.25].

The next two corollaries follow using [LMZ, 3.13 and 3.14].

3.7. Corollary. *The (α,p)-fine topology is completely regular.*

3.8. Corollary. *If U is an (α,p)-finely open F_σ-set, then there is an upper semicontinuous and (α,p)-finely continuous function $f:\mathbf{R}^n \to [0,1]$ such that*

$$U = \{x \in \mathbf{R}^n : f(x) > 0\}.$$

3.9. Remarks. (a) It also follows that $(\mathbf{R}^n, \tau_{\alpha,p})$ is a Baire space; see [LMZ, 3.16].

(b) Corollary 3.7 implies that (α,p)-regular open sets form a base for the (α,p)-fine topology $\tau_{\alpha,p}$, cf. Remark 3.3.

We say that a property holds (α,p)-*quasieverywhere*, abbreviated (α,p)-q.e., if it holds except on a set of (α,p)-capacity zero. A function f, defined (α,p)-q.e., is called (α,p)-*quasicontinuous* if for every $\varepsilon > 0$ there is an open set G such that $B_{\alpha,p}(G) < \varepsilon$ and that the restriction $f\,|_{G^c}$ is continuous. Then a function is (α,p)-quasicontinuous if and only if it is (α,p)-finely continuous (α,p)-q.e. [HW, Theorem 8].

3.10. Theorem. *Suppose that f is an (α,p)-finely continuous function on $\mathbf{R}^n$. Then each $x_0 \in \mathbf{R}^n$ has an (α,p)-fine neighborhood W of x_0 such that $f\,|_W$ is continuous.*

Proof. Similarly to [HKM, 3.17] one can easily show that there is a set $E \subset \mathbf{R}^n$ which is (α,p)-thin at x_0 such that

$$\lim_{\substack{x \to x_0 \\ x \notin E}} f(x) = f(x_0).$$

Fix $j \in \mathbf{N}$. Since f is quasicontinuous, there is an open set $G_j \subset B(x_0, 2^{-j+2}) \setminus B(x_0, 2^{-j-1})$ such that the restriction

$$f|_{\overline{B}(x_0,2^{-j+1})\setminus(B(x_0,2^{-j})\cup G_j)}$$

is continuous and

$$B_{\alpha,p}(G_j) < 2^{-2jn}. \tag{3.11}$$

Writing

$$A = \bigcup_{j=1}^{\infty} G_j \cup E$$

and $W = B(x_0, 1) \setminus A$, (3.11) implies that W is an (α, p)-fine neighborhood of x_0. Moreover, $f|_W$ is continuous as desired.

For the next lemma we need to assume that $\alpha p > 1$.

3.12. Lemma. *Suppose that $\alpha p > 1$ and that U is an (α, p)-fine neighborhood of x_0. Then there is an (α, p)-fine neighborhood V of x_0, $V \subset U$, such that V is connected in the euclidean topology.*

Proof. By [AL, Theorem 2] there is an (α, p)-fine neighborhood V' of x_0, $V' \subset U$, such that for each $x \in V'$ there is an arc γ_x joining x to x_0 in U. Then

$$V = \bigcup_{x \in V'} \gamma_x$$

is the desired (α, p)-fine neighborhood of x_0.

3.13. Remark. Lemma 3.12 is false if $\alpha p \leq 1$. Indeed, let

$$U = \Big(\bigcup_{j=1}^{\infty} \partial B(0, 1/j) \Big)^c .$$

Since $B_{\alpha,p}(\partial B(0, 1/j)) = 0$ for $\alpha p \leq 1$ [M1, Theorem 21], U is (α, p)-finely open. On the other hand, $\{0\}$ is the euclidean 0-component of U. Hence Lemma 3.12 fails to hold if $\alpha p \leq 1$.

3.14. Lemma. *Suppose that $\alpha p > 1$ and that U is an (α, p)-finely regular set. Then the family*

$$Clop\, U = \{V \subset U : V \text{ and } U \setminus V \text{ are } (\alpha, p)\text{-finely open}\}$$

is a σ-algebra on U.

Proof. Let $V_k \in Clop\, U$, $k = 1, 2, \ldots$, and $x_0 \in \cap_{k=1}^{\infty} V_k$. It suffices to show that $\cap V_k$ is an (α, p)-fine neighborhood of x_0. Since for each integer k, the sets V_k and $U \setminus V_k$ are (α, p)-finely regular, Corollaries 3.2 and 3.8 allow us to choose upper semicontinuous and (α, p)-continuous functions f_k and g_k, $0 \leq f_k, g_k \leq 1$, such that

$$V_k = \{x \in \mathbf{R}^n : f_k(x) > 0\}$$

and

$$U \setminus V_k = \{x \in \mathbf{R}^n : g_k(x) > 0\}.$$

Write

$$f = \sum_{k=1}^{\infty} 2^{-k}(f_k + g_k).$$

Then f is (α, p)-finely continuous whence, by Theorem 3.10, there is an (α, p)-fine neighborhood $\tilde{W} \subset U$ of x_0 such that $f|_{\tilde{W}}$ is (euclidean) continuous. In light of Lemma 3.12 we may pick an (α, p)-fine neighborhood W of x_0, $W \subset \tilde{W}$, such that W is connected. Then fix $k \in \mathbf{N}$. Since $f|_W$ is continuous and f_k and g_k are upper semicontinuous, the restrictions $f_k|_W$ and $g_k|_W$ are continuous. Thus the sets $V_k \cap W$ and $(U \setminus V_k) \cap W$ are relatively open in W. Since W is connected and $x_0 \in V_k \cap W$ it follows that $W \subset V_k$. Consequently,

$$W \subset \bigcap_{k=1}^{\infty} V_k$$

whence $\cap_{k=1}^{\infty} V_k$ is an (α, p)-fine neighborhood of x_0 as desired.

Now we are ready to prove our principal theorem.

3.15. Theorem. *If $\alpha p > 1$, then the (α, p)-fine topology is locally connected.*

Proof. Let U_0 be an (α, p)-fine neighborhood of x_0. Choose an (α, p)-finely regular neighborhood U of x_0 such that $U \subset U_0$. Write

$$\begin{aligned} Clop_{x_0} U = \{V \subset U \, : x_0 \in V, V \text{ is } (\alpha, p)\text{–finely open} \\ \text{and} \quad U \setminus V \text{ is } (\alpha, p)\text{–finely open}\}. \end{aligned}$$

Using the quasi-Lindelöf property (Theorem 2.3) we find a sequence $V_k \in Clop_{x_0} U$ such that the set

$$F = \bigcup \{U \setminus V : \, V \in Clop_{x_0} U\} \setminus \bigcup_{k=1}^{\infty} (U \setminus V_k)$$

has the (α, p)-capacity zero. Then Lemma 3.14 implies that

$$W = \bigcap_{k=1}^{\infty} V_k \setminus F$$

is an (α, p)-fine neighborhood of x_0. On the other hand, W is the (α, p)-fine component containing x_0 since

$$W = \bigcap \{V : \, V \in Clop_{x_0} U\}.$$

The proof is complete.

3.16. Remark. The example in Remark 3.13 shows that the (α, p)-fine topology is not locally connected if $\alpha p \leq 1$.

4. Variational capacity and Hausdorff measures

Throughout this section let $\alpha = m$ be a positive integer and $p > 1$ such that $mp \leq n$. We present some results concerning capacity and measure densities; these results, mostly known, will be needed in Section 5.

Let U and Ω be open sets in $\mathbf{R}^n$ with $U \subset\subset \Omega$. Define the *variational* (m,p)*-capacity* of U in Ω to be the number

$$\operatorname{cap}_{m,p}(U,\Omega) = \inf \sum_{|\alpha|=m} \int_\Omega |D^\alpha \varphi|^p \, dx$$

where the infimum is taken over all $\varphi \in C_0^\infty(\Omega)$ with $\varphi \geq 1$ in U. If $E \subset\subset \Omega$ is any set we define

$$\operatorname{cap}_{m,p}(E,\Omega) = \inf_{\substack{E \subset U \\ U \subset\subset \Omega \text{ open}}} \operatorname{cap}_{m,p}(U,\Omega).$$

Then there are constants $c_1 = c_1\big(n,m,p,\operatorname{dist}(E,\partial\Omega)\big)$ and $c_2 = c_2\big(n,m,p,\operatorname{diam}(\Omega)\big)$ such that

$$c_1 \, B_{m,p}(E) \leq \operatorname{cap}_{m,p}(E,\Omega) \leq c_2 \, B_{m,p}(E),$$

see [R2, Section 6].

Let $h : [0,\infty) \to [0,\infty)$ be a continuous nondecreasing function with $h(0) = 0$ and $\lim_{r\to\infty} h(r) = \infty$. We define the h*-Hausdorff measure* (or *content*) of a set E by

$$H_h(E) = \inf\Big\{ \sum_i h(r_i) : \; E \subset \bigcup_{i=1}^\infty B(x_i,r_i) \Big\}.$$

The following theorem is due to Yu.G. Reshetnyak [R1, Lemma 6] and [R2, Theorem 4.1]; see also [Mar]. We briefly indicate how to deduce it from Reshetnyak's results.

4.1. Theorem. *Suppose that*

$$\int_0^{2r} \left(\frac{h(t)}{t^{n-mp}} \right)^{1/p} \frac{dt}{t} = I(r) < \infty$$

for all $0 < r \leq r_0 \leq 1$. *Then for each open set* $E \subset \mathbf{R}^n$ *and* $x \in \mathbf{R}^n$

$$H_h(E \cap B(x,r)) \leq c \, I(r)^p \operatorname{cap}_{m,p}\big(E \cap B(x,r), B(x,2r)\big)$$

for $r \leq r_0$. *Here* $c = c(n,m,p)$.

The proof is based on the following lemma [R1, Lemma 6].

4.2. Lemma. *If $u \in L^p(B(2r))$ is nonnegative, $\operatorname{spt} u \subset B(2r)$ and*

$$v(x) = \int_{B(2r)} \frac{u(y)}{|x-y|^{n-m}}\, dy,$$

then there are constants $K_1 = K_1(n,m,p)$, $K_2 = K_2(n,p)$ and $K_3 = K_3(n)$ such that

$$H_h\Big(\Big\{x : v(x) > \frac{K_1\, I(r)}{\delta} + K_2\, r^{m-n/p}\|u\|_p\Big\}\Big) \le K_3\, \delta^p \|u\|_p^p$$

for all $\delta > 0$.

Proof of Theorem 4.1. Fix $r \le r_0$. Choose $\varphi \in C_0^\infty(B(2r))$ such that $\varphi = 1$ in $E \cap B(r)$. Then, by [R2, Lemma 6.2],

$$\varphi(x) \le c_1(n,m) \int_{B(2r)} \sum_{|\alpha|=m} \frac{|D^\alpha \varphi(y)|}{|x-y|^{n-m}}\, dy.$$

Write

$$u(y) = c_1 \sum_{|\alpha|=m} \big|D^\alpha \varphi(y)\big|.$$

Since there is a constant $c = c(n,m) > 0$ such that

$$c^{-1} \inf \int_{B(2r)} |u|^p \, dm \le \operatorname{cap}_{m,p}\big(E \cap B(r), B(2r)\big) \le c \inf \int_{B(2r)} |u|^p \, dm$$

where the infimum is taken over all such u, we may assume that

$$\|u\|_p < \tfrac{1}{2} K_2^{-1} r^{(n-mp)/p}$$

where K_2 is the constant of Lemma 4.2.

Now choosing $\delta = 2K_1 I(r)$ Lemma 4.2 yields

$$H_h\big(E \cap B(r)\big) \le H_h\big(\{x : \varphi(x) \ge 1\}\big) \le c(n,m,p) I(r)^p \|u\|_p^p.$$

This completes the proof.

4.3. Remark. As well known, the converse inequality for Theorem 4.1 holds with the function $h(r) = r^{n-mp}$ if $mp < n$, $h(r) = \big(\log(2/r)\big)^{1-p}$ if $mp = n$, see e.g. [MK], [R2, Theorem 4.2] and [Mar, 4.1]. A survey of comparison theorems can be found in [Hed].

4.4. Remark. If $mp < n$ it follows from the Sobolev embedding theorem that for $r \le 1$

$$\operatorname{cap}_{m,p}\big(E, B(2)\big) \le \operatorname{cap}_{m,p}\big(E, B(2r)\big) \le c \operatorname{cap}_{m,p}\big(E, B(2)\big) \tag{4.5}$$

whenever $E \subset B(r)$. Here $c = c(n,m,p)$, cf. [Maz, Proposition 9.1.1/3]. Hence

$$c_1 B_{m,p}(E) \le \operatorname{cap}_{m,p}\big(E, B(2r)\big) \le c_2 B_{m,p}(E) \tag{4.6}$$

whenever $E \subset B(r)$. Here $c_i = c_i(n,m,p)$, $i = 1,2$.

If $mp = n$, the assertions (4.5) and (4.6) do not hold.

5. Comparison between different types of connectedness

Our main result in this section reads: for $\alpha p > n-1$ the (α,p)-finely open set is (α,p)-finely connected, arcwise connected, and (euclidean) connected at the same time.

We start with two auxiliary results. The first is a consequence of [AH, Theorem B].

5.1. Lemma. *Suppose that $\alpha p > n-1$. Then there is $q_0 > 1$ such that*

$$\tau_{\alpha,p} \subset \tau_{n-1,q}$$

for all $q \in (1, q_0)$.

Proof. We show that the assertion follows from [AH, Theorem B].

We may assume that $\alpha p < n$. Then we need only to show that there is $q > 1$ such that

$$\frac{(n-1)(p-1)+n}{p} \le \frac{\alpha(q-1)+n}{q}. \tag{5.2}$$

Since $p > 1$, the left side of (5.2) is less than n. Hence there is $q_0 > 1$ such that (5.2) holds for $q \le q_0$ because

$$\frac{\alpha(q-1)+n}{q} \to n$$

as $q \to 1$. The lemma is proved.

The next lemma is essential, see [HK3, 3.4], [LM, 3.16] and [MS] for special cases.

5.3. Lemma. *Suppose that $\alpha p > n-1$. If U is an (α,p)-fine neighborhood of x_0, then there is a sequence of radii $r_i \to 0$ such that*

$$\partial B(x_0, r_i) \subset U.$$

Proof. Write $B(r) = B(x_0, r)$ for $r > 0$. In the light of Lemma 5.1 we may assume that $\alpha = n-1$ and that $\alpha p < n$. Let $E = U^c$. We may further assume that E is open. Let $q = \frac{1}{2}(\alpha p + n - 1) > n-1$ and write $h(r) = r^{n-q}$, $r > 0$. Then

$$I(r) = \int_0^{2r} \left(\frac{h(t)}{t^{n-\alpha p}}\right)^{1/p} \frac{dt}{t} = c(n,p) r^{(n-q)/p}\, r^{(\alpha p - n)/p}$$

and hence it follows from Theorem 4.1 and Remark 4.4 that, for $r \le 1$,

$$\begin{aligned} \frac{H_h(E \cap B(r))}{r^{n-q}} &\le c\frac{\operatorname{cap}_{\alpha,p}\big(E \cap B(r), B(2r)\big)}{r^{n-\alpha p}} \\ &\le c(n,p)\frac{B_{\alpha,p}\big(E \cap B(r)\big)}{r^{n-\alpha p}}. \end{aligned} \tag{5.4}$$

Fix $r \leq \frac{1}{2}$ and suppose that for each $\rho \in [\frac{1}{2}r, r]$ the sphere $\partial B(\rho)$ meets E. Then it follows easily, cf. Remark 5.7 (b) below, that

$$H_h(E \cap B(r)) \geq H_h([\tfrac{1}{2}re_1, re_1]) \geq c(n,p)r^{n-q} \tag{5.5}$$

where $e_1 = (1, 0, \ldots, 0)$. Then (5.4) and (5.5) yield

$$\frac{B_{\alpha,p}(E \cap B(r))}{r^{n-\alpha p}} \geq c(n,p) > 0. \tag{5.6}$$

On the other hand, since E is (α, p)-thin at x_0 we can find an integer j_0 such that for $r = 2^{-j}$, $j = j_0, j_0 + 1, \ldots$,

$$\frac{B_{\alpha,p}(E \cap B(r))}{r^{n-\alpha p}} < c$$

where c is the constant of (5.6). This proves the lemma.

5.7. Remarks. (a) Lemma 5.3 fails to hold for $\alpha p \leq n - 1$. In fact, the line segment $E = (0, e_1]$ is of (α, p)-capacity zero [M1, Theorem 21] whence E^c is an (α, p)-fine neighborhood of 0.

(b) To establish (5.5) above we made use of the following simple symmetrization property of Hausdorff measures: Let $E \subset \mathbf{R}^n$ and write $E^* = \{|x|\, e_1 : x \in E\}$. Then $H_h(E^*) \leq H_h(E)$.

The main result of this section is

5.8. Theorem. *Suppose that $\alpha p > n - 1$ and that U is an (α, p)-finely open set. Then the following are equivalent.*
(1) *U is (α, p)-finely connected.*
(2) *U is arcwise connected.*
(3) *U is (euclidean) connected.*

Proof. The implication of (1)$\Longrightarrow$(2) was proved in [AL, Corollary 2] for all $\alpha p > 1$. Since (2) trivially implies (3) we need only to show that (3) implies (1). This, in turn, immediately follows from [LMZ, 5.4] and the next lemma.

5.9. Lemma. *Let $\alpha p > n - 1$. If V and W are disjoint (α, p)-finely open euclidean connected sets, then*

$$V \cap \overline{W} = \emptyset.$$

Proof. Suppose, on the contrary, that $x_0 \in V \cap \overline{W}$. Let $x_1 \in W$. In the light of Lemma 5.3 we can pick a radius $0 < r < |x_1 - x_0|$ such that

$$\partial B(x_0, r) \subset V \subset W^c.$$

Then let $x_2 \in W \cap B(x_0, r)$. Since $\partial B(x_0, r)$ separates x_1 and x_2 in W, W cannot be connected, and the lemma follows.

5.10. Remarks. (a) Let U be an (α,p)-finely open and (α,p)-finely connected set with $\alpha p > 1$. It follows from [AL, Theorem 2] that each two points in U can be joined by a coordinate path in U.

(b) Suppose that $\alpha p \leq n-1$. Then the statement (2) (and hence (3)) of Theorem 5.8 does not imply (1). To see this, suppose first that $\alpha p > 1$. Let E be the line segment $(0, e_1]$. Since E is of (α,p)-capacity zero we may choose an open connected neighborhood D of E such that D is (α,p)-thin at 0. Thus D^c is an (α,p)-fine neighborhood of 0. Let V be the (α,p)-fine component of the (α,p)-fine interior of D^c containing 0. Then V is (α,p)-finely open and arcwise connected by Theorem 3.15 and [AL, Corollary 2]. Thus

$$U = D \cup V$$

is (α,p)-finely open and arcwise connected. However, U is not (α,p)-finely connected.

Next, using the inclusion relations among fine topologies [AH, Theorem B] we obtain a counterexample also for $\alpha p \leq 1$ if $n \geq 3$. The plane case follows from a slightly more careful but similar reasoning.

(c) For $\alpha p \leq 1$ the implication of $(1) \Longrightarrow (2)$ in Theorem 5.8 is false as shown in [AL, p. 62], cf. Remark 3.13.

(d) Theorem 5.8 is known in the plane for the classical fine topology, cf. [F3, Theorem 3] and [GL].

6. Asymptotic paths for $\mathcal{A}$-subharmonic functions

In this final section we give an application of Theorem 5.8. We show that there is a coordinate path along which an entire $\mathcal{A}$-subharmonic function which is not bounded from above tends to infinity. However, due to Theorem 5.8, we are confined to the case $p > n-1$.

Recall that the $(1,p)$-fine topology, $1 < p \leq n$, is intimately connected to the (nonlinear) potential theory of $\mathcal{A}$-subharmonic functions. More precisely, let $\mathcal{A} : \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ be a mapping which satisfies the usual assumptions of measurability, boundedness, ellipticity, coercivity, and homogeneity (that is, the assumptions (2.1) – (2.5) in [HKM] or in [H]). Continuous weak solutions to the equation (1.3) are called *$\mathcal{A}$-harmonic*, and an upper semicontinuous function u in an open set Ω is termed *$\mathcal{A}$-subharmonic* if for each domain D, compactly contained in Ω, and each $\mathcal{A}$-harmonic $h \in C(\overline{D})$, $h \geq u$ in ∂D implies $h \geq u$ in D.

For basic properties of $\mathcal{A}$-subharmonic functions and their potential theory we refer to [HK 1–3], [K].

It was proved in [H] that if u is an entire $\mathcal{A}$-subharmonic function in $\mathbf{R}^n$, and not bounded above, then there is a path Γ, $\Gamma(t) \to \infty$ as $t \to \infty$, such that $u(x) \to \infty$ as x tends to ∞ along Γ. In the classical theory of subharmonic functions it is known that the path Γ can be chosen to be polygonal; this was first proved by L. Carleson. We refer to [F3] for a lucid survey on the subject.

We do not know whether a polygonal path can always be found for general $\mathcal{A}$-subharmonic functions. However, we apply Theorem 5.8 and show that this is the case at least for $p > n-1$.

6.1. Theorem. *Suppose that $p > n-1$ and that u is $\mathcal{A}$-subharmonic in $\mathbf{R}^n$, unbounded from above. Then there is a coordinate path Γ going to infinity such that*

$$\lim_{\substack{x \to \infty \\ x \in \Gamma}} u(x) = \infty.$$

By a coordinate path we mean a path which is a countable union of (possibly degenerated) line segments parallel to the coordinate axes.

Before we indicate how Theorem 5.8 can be used to deduce Theorem 6.1 some remarks about $(1,p)$-fine topology and $\mathcal{A}$-subharmonic functions are due.

In [HKM] the $\mathcal{A}$-fine topology $\tau_{\mathcal{A}}$ was defined to be the coarsest topology in $\mathbf{R}^n$ making all $\mathcal{A}$-subharmonic functions in $\mathbf{R}^n$ continuous. It was then shown in [HKM] that

$$\tau_{\mathcal{A}} = \tau_{1,p}.$$

In effect, in [HKM] a seemingly different Wiener criterion was used, namely

$$\int_0^1 \left(r^{p-n} \operatorname{cap}_{1,p}(E \cap B(x_0,r), B(x_0,2r))\right)^{1/(p-1)} \frac{dr}{r} < \infty. \tag{6.2}$$

However, this integral converges simultaneously with the integral in (1.1). For $p < n$ this is an immediate consequence of (4.6) and for completeness we provide a proof in the case $p = n$ (the fact that the two Wiener criteria coincide also when $p = n$ is evidently part of the folklore). Thus, let $p = n$ and define the capacity

$$C_{1,n}(E) = \inf\Big\{ \int_{\mathbf{R}^n} \left(|\nabla v|^n + |v|^n\right) dm : v \in C_0^\infty(\mathbf{R}^n),\ v \geq 1 \text{ in } E \Big\}.$$

Then the $C_{1,n}$-capacity is equivalent to the $B_{1,n}$-capacity, cf. [AM] or [Hed]. We prove

6.3. Proposition. *Let $E \subset \mathbf{R}^n$ and $x \in \mathbf{R}^n$. Then*

$$\sum_{k=1}^{\infty} \left(\operatorname{cap}_{1,n}(E \cap B_k, B_{k-1})\right)^{1/(n-1)} < \infty$$

if and only if

$$\sum_{k=1}^{\infty} \left(C_{1,n}(E \cap B_k)\right)^{1/(n-1)} < \infty.$$

Here $B_k = B(x, 2^{-k})$.

Proof. The "only if" part being a trivial consequence of Poincaré's inequality, we only prove the converse. For $k \in \mathbf{N}$ let u_k be the $C_{1,n}$-capacitary potential of $E \cap B_k$. Let $d_k = \sup_{\partial B_{k-1}} u_k$. We show that $\lim_{k\to\infty} d_k = 0$. This will complete the proof since then for k big enough $d_k < \frac{1}{2}$, and it follows that

$$C_{1,n}(E \cap B_k) \geq 2^{-n} \operatorname{cap}_{1,n}(E \cap B_k, B_{k-1})$$

which implies the assertion.

To this end, suppose that $d = \limsup d_k > 0$. Then using the Harnack inequality for u_k outside $\overline{B}_k$ and the minimum principle (see [S]) we obtain an infinite set $I \subset \mathbf{N}$ such that

$$\inf_{\overline{B}_{k-1}} u_k \geq c(n)d$$

for every $k \in I$.

Write

$$S = \bigcup_{k \in I} \partial B_{k-1}.$$

Fix $k \in \mathbf{N}$ and pick the least $j \in I$ such that $j \geq k$. Then

$$\begin{aligned} C_{1,n}(E \cap B_k) &\geq C_{1,n}(E \cap B_j) \geq (c\,d)^n C_{1,n}(B_{j-1}) \\ &\geq (c\,d)^n C_{1,n}(S \cap B_k). \end{aligned}$$

Then the set S is $(1,n)$-thin at x, contradicting Lemma 3.12. Thus $d = 0$ and the proposition is proved.

Proof of Theorem 6.1. Let u be an $\mathcal{A}$-subharmonic function in $\mathbf{R}^n$, unbounded from above. Then there is a number $L_0 > 0$ such that for each $L > L_0$ u is unbounded in the set $K(x,L)$, where $K(x,L)$ is the union of all continua which contain x and on which u is $\geq L$, see [H, 4.5]. Pick points x_j in $\mathbf{R}^n$ inductively as follows. Let x_1 be any point such that $u(x_1) > 2L_0$, and suppose that $x_1, \ldots, x_j$ have been chosen. Let x_{j+1} be a point in $K\big(x_j, u(x_j)\big)$ such that $u(x_{j+1}) > 2^{j+1}L_0$. Then x_j and x_{j+1} can be joined by a continuum in the finely open set $\{u > 2^j L_0\}$. Since $p > n-1$, Theorem 5.8 implies that x_j and x_{j+1} belong to the same fine component U_j of $\{u > 2^j L_0\}$; in particular, there is a coordinate path Γ_j joining x_j and x_{j+1} in U_j [AL, Theorem 2]. Then $\Gamma = \cup \Gamma_j$ is the required path, and the theorem is proved.

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