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HYPERBOLIC METRICS ON FINITE-DIMENSIONAL TEICHMÜLLER SPACES

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1. Introduction. Statement of result

Teichmüller spaces are important in different fields and also interesting as model examples in the general theory of complex manifolds, especially hyperbolic ones. Thus it is of great interest to establish how the general problems deriving from the theory of complex manifolds are solved for these spaces. One such problem is the question of the coincidence of invariant hyperbolic Carathéodory and Kobayashi metrics; it was posed in [1], [4]. This question proves to be very important also for the variational problems of geometric function theory (see [9]).

For the finite-dimensional Teichmüller spaces $T(g,n)$, which correspond to Riemann surfaces of finite conformal type (g,n) with $2g+n>2$, the situation is as follows. There is an important theorem by Royden [12], asserting that on these spaces the Kobayashi metric coincides with their intrinsic Teichmüller metric, which is defined by using quasiconformal mappings; later Gardiner [13] has extended this result also to the infinite-dimensional Teichmüller spaces.

On the other hand, it was established by the author in [7], [8] that, on the Teichmüller spaces $T(g,n)$ of a dimension greater than two, the Carathéodory metric on the whole is smaller than the Teichmüller–Kobayashi metric, so the problem has a negative solution. The arguments used there assume the existence on the surfaces of three linear independent holomorphic quadratic differentials (with fully determined properties), and thus they are suited for $\dim T(g,n) \geq 3$ only. The question remained open for the Teichmüller spaces of dimension 2; there are the spaces $T(0,5)$ of the spheres with five punctures and $T(1,2)$ of the tori with two punctures, which are biholomorphically equivalent.

The goal of the present paper is to show that the metrics do not coincide for these spaces either. The reasons proposed below are valid simultaneously for all spaces $T(g,n)$ of $\dim > 1$ *with punctures;* for $\dim T(g,n) \geq 3$ that gives a new (and more effective) proof that the above metrics do not coincide.

On account of applications, it is important to have sufficient conditions on holomorphic disks in $T(g,n)$ that provide the coincidence of invariant metrics on these disks. Kra [6] has shown this to be true for the Abelian Teichmüller disks defined by quadratic differentials with zeros of even order. An analogous result is established in [10] for the universal Teichmüller space; it can be extended also to the Teichmüller spaces of finitely many punctured disks.

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Here we prove

Theorem. *Let* $\dim T(g, n+1) = 3g - 2 + n > 1$ $(g \geq 0,\ n \geq 0)$ *and* $\pi\colon T(g, n+1) \to T(g, n)$ *be the canonical fiber space in which the projection is induced by forgetting a puncture on a base surface of type* $(g, n+1)$*. Then, in fibers* $\pi^{-1}(x)$*, the hyperbolic Carathéodory and Teichmüller–Kobayashi metrics (on the space* $T(g, n+1)$*) do not coincide.*

2. Some auxiliary constructions and results

First of all, we recall that if M is a complex manifold (finite-dimensional or even Banach), its Carathéodory metric is

$$c_M(x, y) = \sup\Big\{\varrho\big(h(x), h(y)\big) : h \in \operatorname{Hol}(M, \Delta)\Big\},$$

where ϱ is the hyperbolic metric in the unit disk $\Delta = \big\{z : |z| < 1\big\}$ of curvature -4, i.e., with the differential element $d\varrho = \big(1 - |z|^2\big)^{-1}$, and the Kobayashi metric $d_M(x, y)$ is the greatest of all pseudometrics $d(\cdot, \cdot)$ on M satisfying the inequality

$$d\big(h(z'), h(z'')\big) \leq \varrho(z', z''), \qquad h \in \operatorname{Hol}(\Delta, M).$$

Let X_0 be a given Riemann surface of conformal type $(g, n+1)$, i.e, of genus g with $n+1$ punctures, where $2g-2+n > 0$. Fixing a conformal structure on X_0 we consider this surface as the initial point in $T(g, n+1)$. Let p_0 be a puncture on X_0 and

$$\tilde{X}_0 = X_0 \cup \{p_0\}.$$

We uniformize the surfaces X_0 and $\tilde{X}_0$ by finitely generated Fuchsian groups Γ and $\tilde{\Gamma}$ of the first kind and without torsion, acting discontinuously on upper and lower half-planes

$$U = \{z : \operatorname{Im} z > 0\}, \qquad U^* = \{z : \operatorname{Im} z < 0\}.$$

Then, as well known,

$$T(g, n+1) \cong T(\Gamma) = L_\infty(U, \Gamma)_1 / \big\{\mu \in L_\infty(U, \Gamma)_1 : w^\mu|_R = \mathrm{id}\big\},$$

where

$$L_\infty(U, \Gamma)_1 = \big\{\mu \in L_\infty(\mathbf{C}) : \mu|U^* = 0, \ \ (\mu \circ \gamma)\bar{\gamma}'/\gamma' = \mu, \ \ \gamma \in \Gamma; \ \ \|\mu\|_\infty < 1\big\}$$

and w^μ is the quasiconformal automorphism of $\mathbf{C}$ with the Beltrami coefficient $\mu \in L_\infty(U, \Gamma)_1$ and fixed points $0,\ 1,\ \infty$.

The conformality of w^μ in U^* allows us to consider the mapping

$$\Phi\colon \mu \to \{w^\mu, z\} = \frac{(w^\mu)'''}{(w^\mu)'} - \frac{3}{2}\left[\frac{(w^\mu)''}{(w^\mu)'}\right]^2, \qquad z \in U^*,$$

which correctly defines a biholomorphic isomorphism of the space $T(g, n+1)$ onto a bounded domain in the space $B_2(U^*, \Gamma)$ of holomorphic solutions of the equation $(\psi \circ \gamma)\gamma'^2 = \psi$, $\gamma \in \Gamma$, in U^*, with the norm $\|\psi\| = \sup_{U^*}(\operatorname{Im} z)^2 |\psi(z)|$. Here, Φ itself acts holomorphically from $L_\infty(U, \Gamma)_1$ to $B_2(U^*, \Gamma)$ and, in particular,

$$d\Phi(0)\mu = -\frac{6}{\pi} \iint\limits_U \frac{\mu(\zeta)\, d\xi\, d\eta}{(\zeta - z)^4} \qquad (\zeta = \xi + i\eta, z \in U^*). \tag{1}$$

We identify $T(g, n+1)$ with its embedding in $B_2(U^*, \Gamma)$. An analogous construction is valid, of course, also for $T(g, n) \cong T(\tilde{\Gamma})$; the corresponding projection for this space we will denote by $\tilde{\Phi}$.

The Teichmüller metric on $T(g, n+1)$ is

$$\tau_\Gamma\big(\Phi(\mu), \Phi(\nu)\big) = \tfrac{1}{2} \inf\{\log K(w^{\mu'} \circ (w^{\nu'})^{-1} : \Phi(\mu') = \Phi(\mu), \Phi(\nu') = \Phi(\nu)\},$$

where $K(w^\sigma) = \big(1 + \|\sigma\|_\infty\big)/\big(1 - \|\sigma\|_\infty\big)$.

One could easily establish that

$$c_T(x, y) \le d_T(x, y) \le \tau_T(x, y). \tag{2}$$

The identical embedding $j\colon X_0 \hookrightarrow \tilde{X}_0$ "forgetting a puncture" induces an isometrical isomorphism $j_*(\mu)\colon L_\infty(U, \Gamma) \to L_\infty(U, \tilde{\Gamma})$ by

$$j_*(\mu) \circ J = \mu J'/\bar{J}',$$

where $J\colon U \to U$ is a lifting of j from x_0 onto U; this isomorphism is compatible with the projections Φ and $\tilde{\Phi}$. Thus j determines a holomorphic fiber space

$$\pi\colon T(g, n+1) \to T(g, n), \tag{3}$$

which is mentioned in the theorem (this fibering is a holomorphic disk family admitting only a real local C^∞-trivialization).

With this fibering is connected another fiber space

$$\pi_0\colon F(g, n) \to T(g, n), \tag{4}$$

which was introduced by Bers. Here

$$F(g, n) \cong F(\tilde{\Gamma}) = \big\{\big(\tilde{\Phi}(\mu), z\big) \in T(\tilde{\Gamma}) \times \mathbf{C} : \mu \in L_\infty(U, \tilde{\Gamma})_1, z \in w^\mu(U)\big\},$$

with the projection π_0: $(\tilde{\Phi}(\mu), z) \mapsto \tilde{\Phi}(\mu)$.

According to Bers' isomorphism theorem [2], the fiber spaces (3) and (4) are isomorphic; thus there exists a biholomorphic isomorphism

$$T(g, n+1) \cong F(g, n)$$

compatible with the projections. It is defined by the map

$$\mu \mapsto \left(\tilde{\Phi}(j_*(\mu)), w^{j_*(\mu)}(z_0)\right),$$

where z_0 is a fixed preimage of the point p_0 in U by a universal holomorphic covering $U \to \tilde{X}_0$.

The following result of Kra [6] and Nag [11], which we essentially will use, is established with the aid of the isomorphism theorem.

Proposition (Kra–Nag). *The fibers $\pi^{-1}(x)$ (for $(g, n) \neq (0, 3)$) are not totally geodesic in $T(g, n+1)$ relative to the metric τ_T, i.e., they are not Teichmüller disks, and τ_T is connected with the hyperbolic metric* hyp *on the fiber $\pi^{-1}(x)$ (which is isometrically pull-backed from Δ by a holomorphic embedding $\Delta \to \pi^{-1}(x)$) by the strong inequality*

$$\tau_T(z_1, z_2) < \operatorname{hyp}(z_1, z_2) \qquad (z_1, z_2 \in \pi^{-1}(x)). \tag{5}$$

Recall also that the Teichmüller disks Δ_φ in $T(g, n+1)$ have the form

$$\Delta_\varphi = \{\Phi(t\bar{\varphi}/|\varphi|) : t \in \Delta\},$$

where φ is a holomorphic quadratic differential on X_0. The space of such differentials (with L_1-norm) will be denoted by $Q(x_0)$ or $Q(\Gamma)$; that is the cotangent space of $T(g, n+1)$ at the point X_0.

3. Proof of the theorem

Let us now turn to the proof of the theorem. We model $T(g, n+1)$ and $T(g, n)$ again as $T(\Gamma)$ and $T(\tilde{\Gamma})$, and take their embeddings in $B_2(U^*, \Gamma)$ (note that $B_2(U^*, \tilde{\Gamma})$ is isometrically embedded into $B_2(U^*, \Gamma)$ by $\varphi \mapsto (\varphi \circ J)J'^2$).

It is sufficient to establish the validity of the assertion of the theorem for the initial fiber $\pi^{-1}(0)$ only. The case of a general fiber $\pi^{-1}(\varphi)$, $\varphi \in T(\tilde{\Gamma})$ (and an analogous case of an arbitrary point from $\pi^{-1}(0) \setminus \{0\}$) can be reduced to such ones by passing from a quasifuchsian group $\Gamma_\mu = w^\mu \Gamma (w^\mu)^{-1}$ to its Fuchsian equivalent using a conformal mapping of the domain $w^\mu(U)$ onto U; this leads us to the so-called *admissible bijection* of $T(\Gamma)$, which preserves the metrics.

We will prove that for the point $\varphi = 0 \in T(\Gamma)$ (or, equivalently, for the basepoint x_0, to which the point $(0, z_0) \in F(\tilde{\Gamma})$ corresponds) there must exist a neighbourhood in the fiber $\pi^{-1}(0)$ over $0 \in T(\tilde{\Gamma})$, in which the equality

$$c_T(\psi, 0) < \tau_T(\psi, 0) \tag{6}$$

holds.

Assuming the contrary, we find a sequence of Schwarzians $\{\psi_m\} \subset \pi^{-1}(0)$ which converges to 0 in $B_2(U^*, \Gamma)$, and that sequence corresponds on X_0 to a sequence of points $\{p_m\}$, converging to p_0 (for example, in a hyperbolic metric on $\tilde{X}_0$) such that ψ_m represents in $T(\Gamma)$ the marked Riemann surface

$$X_m = \tilde{X}_0 \setminus \{p_m\}$$

(with puncture p_m on $\tilde{X}_0$ instead of p_0) and

$$c_T(\psi_m, 0) = \tau_T(\psi_m, 0), \qquad m = 1, 2, \dots \tag{7}$$

Let

$$\{h_m\} \subset \operatorname{Hol}(T(\Gamma), \Delta), \qquad h_m(0) = 0,$$

be the corresponding sequence of holomorphic functions on which the distances $c_T(\psi_m, 0)$ are attained, i.e.,

$$\varrho\big(h_m(\psi_m), 0\big) = c_T(\psi_m, 0),$$

and suppose that $k_m \mu_m$ are extremal Beltrami differentials with

$$\mu_m = \frac{\bar{\varphi}_m}{|\varphi_m|}, \qquad (0 < \kappa_m < 1,\ \varphi_m \in Q(\Gamma) \setminus \{0\})$$

on which the Teichmüller distance $\tau_T(\psi_m, 0)$ is realized (i.e.,

$$\tau_T(\psi_m, 0) = \tfrac{1}{2} \log\big[(1 + k_m)/(1 - k_m)\big],$$

$m = 1, 2, \dots$). We normalize the quadratic differentials φ_m so that

$$\|\varphi_m\|_{L_1} = 1 \quad \text{for all } \ m = 1, 2, \dots$$

Consider now the corresponding Teichmüller disks

$$\Delta_{\varphi_m} = \left\{\Phi\Big(t \frac{\bar{\varphi}_m}{|\varphi_m|}\Big) : t \in \Delta\right\} \subset T(\Gamma).$$

One can also assume the functions h_m to be taken such that

$$h_m \circ \Phi\Big(t\frac{\bar{\varphi}_m}{|\varphi_m|}\Big) = t \qquad (t \in \Delta)$$

and hence

$$d(h_m \circ \Phi)(0)t\mu_m = t \qquad (t \in \mathbf{C}). \tag{8}$$

Turning, if necessary, to the subsequences, we may assume that φ_m converges in $Q(\Gamma)$ (and locally uniformly in Δ) to $\varphi_0 \in Q(\Gamma) \setminus \{0\}$ with $\|\varphi_0\| = 1$, and h_m converge locally uniformly in $T(\Gamma)$ to a function $h_0 \in \mathrm{Hol}(T(\Gamma), \Delta) \setminus \{0\}$. Besides, the disks $\Delta - \varphi_m$ converge to Δ_{φ_0} in a Teichmüller metric locally, and by virtue of (8)

$$d(h_0 \circ \Phi)(0)t\mu_0 = t \qquad (\mu_0 = \bar{\varphi}_0/|\varphi_0|, t \in \mathbf{C}), \tag{9}$$

which means $dh_0(0)$ is an isometry on the tangent line $T_0\Delta_{\varphi_0}$ to Δ_{φ_0} in 0.

Let us also show that this tangent cannot be transversal to the tangent line $T_0\pi^{-1}(0)$, i.e., that $T_0\Delta_{\varphi_0}$ and $T_0\pi^{-1}(0)$ must coincide. Indeed, if the equation for the fiber $\pi^{-1}(0)$ in $B_2(U^*, \Gamma)$ is

$$\psi = f_0(t) = \sum_{m=1}^{\infty} t^m f_0^{(m)}(0)/m! \qquad (f_0(t) \equiv f(z,t) \in \mathrm{Hol}(\Delta, T(\Gamma))$$

then, for small $|t|$, we have

$$\psi = tf_0'(0) + O(t^2) \qquad (f_0'(0) \neq 0) \tag{10}$$

and so $T_0\pi^{-1}(0) = \{tf_0'(0) : t \in \mathbf{C}\}$; we assume below that f_0 is the above-mentioned biholomorphic embedding. It follows from (1) that

$$f_0(z,t) = -\frac{6}{\pi}\iint\limits_U \frac{\mu(\xi)\,d\xi\,d\eta}{(\xi - z)^4} + O(\|\mu\|^2) \tag{11}$$

for any $\mu \in L_\infty(U, \Gamma)_1$ with $\Phi(\mu) = f_0(t)$. In particular, due to the well-known Ahlfors–Weill theorem on quasiconformal extension, one can take

$$\mu(\zeta) = -2\eta^2 f_0(\bar{\xi}, t)$$

(and then $\{w^\mu, z\} = f_0(z,t)$).

Let us return to our ψ_m. Let these ψ_m correspond to $t = t_m$ from Δ, i.e., $\psi_m = f_0(t_m)$. Then we have from (11)

$$\frac{12}{\pi}\iint\limits_U \frac{\eta^2 f_0(\bar\zeta, t_m)}{(\zeta - z)^4}\, d\xi\, d\eta + O(t_m^2) = -\frac{6k_m}{\pi}\iint\limits_U \frac{\mu_m(\zeta)\, d\xi\, d\eta}{(\zeta - z)^4} + O(k_m^2)$$

(where $k_m < |t_m|$) or, rememberring (10),

$$\frac{12}{\pi}\iint\limits_U \frac{\eta^2 \bar f_0'(0)}{(\zeta - z)^4}\, d\xi\, d\eta = -\frac{6}{\pi}\frac{k_m}{t_m}\iint\limits_U \frac{\mu_m\, d\xi\, d\eta}{(\zeta - z)^4} + O(t_m) + O\Big(\frac{k_m^2}{t_m}\Big).$$

By applying the Lebesgue theorem on majorized convergence to the integral on the right-hand side and taking into account that the left-hand integral is equal to $f_0'(0)$, we obtain by the well-known reproduction formula for the elements from $B_2(U^*, \Gamma)$

$$f_0'(0) = -\frac{6a}{\pi}\int_U\!\!\int \frac{(\bar\varphi_0/|\varphi_0|)\, d\xi\, d\eta}{(\zeta - z)^4},$$

where $a = \lim_{m\to\infty}(k_m/t_m) \neq 0$ (because of $\varphi_0 \neq 0$ and $f_0'(0) \neq 0$); but that means the coincidence of directions for tangent lines of $\pi^{-1}(0)$ and of Δ_{φ_0} in 0.

Now the proof of the theorem is completed in the following way. For the function

$$h_0 \circ f_0 \colon \Delta \to \pi^{-1}(0) \to \Delta$$

we have, by virtue of (9)

$$\big|d(h_0 \circ f_0)(0)t\big| = \big|dh_0(0)df_0(0)t\big| = |t|.$$

Hence, due to Schwarz' lemma, the equality $h_0 \circ f_0(t) = e^{i\alpha}t$, $\alpha \in \mathbf{R}$, must hold for all $t \in \Delta$, or, equivalently, in the whole fiber $\pi^{-1}(0)$ we have

$$c_T(\psi, 0) = \mathrm{hyp}(\psi, 0).$$

Comparing that with (2), we see that in $\pi^{-1}(0)$

$$\tau_T(\psi, 0) \geq \mathrm{hyp}(\psi, 0)$$

must hold, contrary to (5), (in fact, by virtue of the coincidence of τ_T with d_T, here can only the equality be, which means that the fiber $\pi^{-1}(0)$ is Teichmüllerian).

The contradiction thus obtained proves that for all ψ, closed to 0 in $\pi^{-1}(0)$, the strong inequality (6) should be valid. Thus the theorem is proved.

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