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GOLUZIN INEQUALITITIES AND MINIMUM ENERGY FOR MAPPINGS ONTO NONOVERLAPPING REGIONS

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In a previous paper [3], we considered sets of functions f_j which map the unit disk conformally onto nonoverlapping regions D_j, and we established the sharp inequality

$$\sum_{j=1}^{n} \sum_{k=1, j\neq k}^{n} x_j x_k \log\bigl|f_j(0) - f_k(0)\bigr| + \sum_{j=1}^{n} x_j^2 \log\bigl|f_j'(0)\bigr| \leq s^2 \log R,$$

where the x_j are arbitrary real parameters with sum s, and R is the transfinite diameter of $\cup_{j=1}^{n} D_j$. Further hypotheses then led to a stronger inequality where the right-hand side is decreased by adding a certain negative-definite quadratic form depending upon the geometry of the regions D_j.

The purpose of the present paper is to extend the stronger result to a version of the Goluzin inequalities, and to interpret the bound physically as the minimum energy of a system of conductors with prescribed electrostatic charges. A remarkable invariance property of the energy functional is then noted for certain domains bounded by lemniscates. Finally, the energy functional is studied under interior variation and is found to have a perfect square in its variational formula. Applications to extremal problems will be given in a later paper.

1. Goluzin-type inequalities

Let Ω be a finitely connected domain in the extended complex plane $\hat{\mathbf{C}}$, containing the point at infinity and bounded by disjoint rectifiable Jordan curves $\Gamma_1, \dots, \Gamma_m$. Let Δ_k be the compact set bounded by Γ_k. The *harmonic measure* ω_k is the bounded harmonic function in Ω which has the boundary value 1 on Γ_k and 0 on Γ_j for all $j \neq k$. The period of the harmonic conjugate of ω_k around Γ_j is

$$P_{kj} = \frac{1}{2\pi} \int_{\Gamma_j} \frac{\partial \omega_k}{\partial n} \, |dz|, \tag{1}$$

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where $\partial/\partial n$ denotes the inner normal derivative. It is known (see [9, Chapter 1]) that $((P_{jk}))$ for $j,k = 1,2,\ldots, m-1$ is a symmetric nonsingular matrix; and it generates a negative-definite quadratic form. The same is therefore true for the inverse matrix $((p_{jk}))$.

Green's function of Ω with pole at infinity has the form

$$g(z) = g(z,\infty) = \log|z| - \log R + O(1/|z|)$$

near infinity. It is harmonic in Ω except for the logarithmic pole at infinity, and it vanishes on the boundary. The *Robin constant* of Ω is $-\log R$, where R is the *transfinite diameter*, or *logarithmic capacity* of the complementary region

$$\tilde{\Omega} = \Delta_1 \cup \Delta_2 \cup \cdots \cup \Delta_m.$$

Further information about transfinite diameter may be found in Goluzin [6, Chapter VII]; Hille [7, Chapter 16]; and Tsuji [13].

More generally, Green's function of Ω with pole at $\zeta \in \Omega$ is the function harmonic in $\Omega \setminus \{\zeta\}$ which vanishes on $\partial\Omega$ and has the form

$$g(z,\zeta) = -\log|z-\zeta| + \cdots$$

near ζ. The normal derivative of Green's function is a resolvent kernel for the Dirichlet problem. In particular,

$$\omega_k(\zeta) = \frac{1}{2\pi}\int_{\Gamma_k} \frac{\partial g(z,\zeta)}{\partial n}\,|dz| \tag{2}$$

and

$$\omega_k(\infty) = \frac{1}{2\pi}\int_{\Gamma_k} \frac{\partial g}{\partial n}\,|dz|, \qquad k = 1,2,\ldots,m, \tag{3}$$

where

$$g(z) = g(z,\infty).$$

Now choose integers $n \geq m$ and n_k with $1 \leq n_1 < n_2 < \cdots < n_m = n$. For convenience, let $n_0 = 0$. Divide the integers from 1 to n into the blocks

$$I_k = \{\nu : n_{k-1} < \nu \leq n_k\}, \qquad k = 1,\ldots,m.$$

Next choose integers $N \geq n$ and N_ν with $1 \leq N_1 < N_2 < \cdots < N_n = N$. Again let $N_0 = 0$ and define the blocks

$$J_\nu = \{j : N_{\nu-1} < j \leq N_\nu\}, \qquad \nu = 1,\ldots,n.$$

Let $x_1, x_2, \ldots, x_N$ be arbitrary real numbers with sum s, and let

$$\sigma_k = \sum_{\nu \in I_k} \sum_{j \in J_\nu} x_j \tag{4}$$

be the sum of the x_j's associated with the kth block I_k of indices ν. Hence $\sigma_1 + \sigma_2 + \cdots \sigma_m = s$.

For $1 \le \nu \le n$, let the functions f_ν map the unit disk $\mathbf{D}$ conformally onto nonoverlapping domains $D_\nu = f_\nu(\mathbf{D})$ with $D_\nu \subset \Delta_k$ for all $\nu \in I_k$, $k = 1, 2, \ldots, m$. Such a vector $(f_1, \ldots, f_n)$ of univalent functions will be called *admissible* with respect to the region Ω. Next choose points $\zeta_1, \zeta_2, \ldots, \zeta_N$ in $\mathbf{D}$ which are distinct in each prescribed block: $\zeta_i \ne \zeta_j$ if $i, j \in J_\nu$ and $i \ne j$, where $\nu = 1, 2, \ldots, n$. A vector $(\zeta_1, \ldots, \zeta_N)$ of such points will again be called *admissible.*

The following theorem may be regarded as a generalization of the Goluzin inequalities (see [2, Chapter 4] or [10, Chapter 3]. It gives information on the values $f_\nu(\zeta_j)$ of functions with nonoverlapping ranges in $\tilde{\Omega}$ in terms of certain conformal invariants of Ω.

Theorem 1. *Let Ω be an m-tuply connected domain in $\hat{\mathbf{C}}$, containing the point at infinity. Choose integers $n \ge m$ and $N \ge n$, and integers n_k and N_ν as above. Let $x_1, \ldots, x_N$ be arbitrary real parameters with sum s, and let σ_k be defined by (4). Choose an admissible vector of points $\zeta_j \in \mathbf{D}$. Then for each admissible vector $(f_1, \ldots, f_n)$ of functions f_ν univalent in $\mathbf{D}$, the sharp inequality*

$$\begin{aligned}
&\sum_{\nu=1}^{n}\Big\{\sum_{i\in J_\nu}\sum_{j\in J_\nu, i\ne j} x_i x_j \log\left|\frac{f_\nu(\zeta_i)-f_\nu(\zeta_j)}{\zeta_i-\zeta_j}\right| + \sum_{j\in J_\nu} x_j^2 \log\left|f_\nu'(\zeta_j)\right|\Big\} \\
&\qquad + \sum_{\nu=1}^{n}\sum_{\mu=1,\nu\ne\mu}^{n}\sum_{i\in J_\nu}\sum_{j\in J_\mu} x_i x_j \log\left|f_\nu(\zeta_i)-f_\mu(\zeta_j)\right| \\
&\qquad \le \sum_{\nu=1}^{n}\sum_{i\in J_\nu}\sum_{j\in J_\nu} x_i x_j \log\frac{1}{1-\overline{\zeta_i}\zeta_j} + s^2\log R \\
&\qquad + \sum_{j=1}^{m-1}\sum_{k=1}^{m-1} p_{jk}\big(s\omega_j(\infty)-\sigma_j\big)\big(s\omega_k(\infty)-\sigma_k\big)
\end{aligned} \tag{5}$$

holds, where R is the transfinite diameter of $\tilde{\Omega}$, ω_k are the harmonic measures of the boundary components of Ω, and $((p_{jk}))$ is the inverse of the period matrix of their harmonic conjugates.

In our previous paper [3], we established a special case of this theorem in which all $\zeta_j = 0$. Earlier work of Alenicyn [1] and Kühnau [8] had led to related results. Note that the last term in (5) is a negative-definite quadratic form, so our inequality improves upon the weaker version (still sharp; *cf.* [3]) in which this term is absent.

The proof is similar to that in [3] but is more complicated; it will be given in two stages. In Section 2 we pose an appropriate extremal problem and prove the existence of an extremal configuration. (This rather delicate argument was omitted in [3].) Then in Section 3 we use the method of boundary variation to describe the extremal configuration and thus to establish the sharp inequality (5). In Section 4 we give a physical interpretation of the last term in (5).

2. Existence of an extremal configuration

It is convenient to pose the problem of maximizing the functional

$$\varphi = \sum_{\nu=1}^{n} G_\nu + \sum_{\nu=1}^{n} \sum_{\mu=1,\nu\neq\mu}^{n} H_{\nu\mu}, \tag{6}$$

where

$$G_\nu = \sum_{j\in J_\nu} \sum_{k\in J_\nu, j\neq k} x_j x_k \log\left|f_\nu(\zeta_j) - f_\nu(\zeta_k)\right| + \sum_{j\in J_\nu} x_j^2 \log\left|f_\nu'(\zeta_j)\right| \tag{7}$$

and

$$H_{\nu\mu} = \sum_{j\in J_\nu} \sum_{k\in J_\mu} x_j x_k \log\left|f_\nu(\zeta_j) - f_\mu(\zeta_k)\right|. \tag{8}$$

We regard Ω as the conformal image of some fixed domain of the same type under a mapping of the form

$$w = F(z) = z + \sum_{k=0}^{\infty} b_k z^{-k} \tag{9}$$

near infinity. The transfinite diameter R of $\tilde{\Omega}$ is invariant under such mappings, as are the period matrix of Ω and the harmonic measures $\omega_k(\infty)$. For fixed choices of the parameters n_k, N_k, x_j, and ζ_j, an *extremal configuration* will be understood to be a domain Ω in the prescribed equivalence class together with an admissible vector of univalent functions $(f_1, \ldots, f_n)$, such that the functional φ attains a maximum value.

Because the functional φ is translation-invariant, no generality is lost in restricting consideration to functions F for which $b_0 = 0$ in the expansion (9). These functions form a compact normal family. The corresponding sets of admissible vectors of univalent functions $(f_1, \ldots, f_n)$ are uniformly bounded. Thus for each index ν $(1 \le \nu \le n)$, the resulting family of functions f_ν is normal but not compact. Indeed, even if the domain Ω is held fixed, it may well happen that a sequence $\{f_{\nu 1}, f_{\nu 2}, \ldots\}$ of such functions tends locally uniformly to a constant limit. We want to show that this cannot happen for an extremal sequence.

Under the assumption that $b_0 = 0$, the conformal mappings of the form (9) produce uniformly bounded regions $\tilde{\Omega}$, so that (assuming each component Δ_k contains an open set) the associated families of admissible vectors of univalent functions are also uniformly bounded. It follows that for all conformal images Ω and for all associated admissible vectors, the functional φ has a finite supremum M. Choose a sequence of normalized ($b_0 = 0$) mappings F_l with ranges Ω_l and a sequence of vectors $(f_{1l}, \ldots, f_{nl})$ admissible with respect to Ω_l, such that $\varphi_l = \varphi(f_{1l}, \ldots, f_{nl}) \to M$ as $l \to \infty$. By normality we may assume that $F_l \to F$ and $f_{\nu l} \to f_\nu$ $(\nu = 1, \ldots, n)$ locally uniformly. By the Carathéodory convergence theorem, the regions converge to their kernel Ω, the range of F.

Suppose now that for each set $E \subset I_k$ of indices in the same block I_k,

$$\sum_{\nu \in E} \sum_{j \in J_\nu} x_j \neq 0. \tag{10}$$

Then we claim that all of the limit functions $f_1, \ldots, f_n$ are univalent (nondegenerate), so that the vector $(f_1, \ldots, f_n)$ is admissible with respect to Ω. Since φ is a continuous functional, this will show that it attains a maximum value; *i.e.*, there exists an extremal configuration.

If any of the limit functions f_ν are not univalent, they must be constant. Thus for some set $E \subset I_k$ of indices ν, the derivatives $f'_{\nu l}(0) \to 0$ as $l \to \infty$; while $f'_{\nu l}(0) \to f'_\nu(0) \neq 0$ for all other $\nu \in I_k$. For $\nu \in E$, the functions $f_{\nu l}$ make a total contribution to φ_l of

$$\psi_l = \sum_{\nu \in E} G_\nu^{(l)} + \sum_{\nu \in E} \sum_{\mu \in E, \nu \neq \mu} H_{\nu\mu}^{(l)}.$$

Under the assumption (10), we are going to show that $\psi_l \to -\infty$ as $l \to \infty$. Since every term of φ_l is bounded above, this will show that $\varphi_l \to -\infty$, a contradiction.

In view of the Koebe distortion theorem, it is readily shown that

$$G_\nu^{(l)} \le A t_\nu^2 \log d_\nu + B$$

and

$$H_{\nu\mu}^{(l)} \le A t_\nu t_\mu \log(d_\nu + d_\mu) + B,$$

where $d_\nu = |f'_\nu(0)|$, $t_\nu = \sum_{j \in J_\nu} x_j$, and A and B are constants depending only on the numbers x_j and ζ_j. Thus

$$\psi_l \le A \Big\{ \sum_{\nu \in E} \sum_{\mu \in E, \nu \neq \mu} t_\nu t_\mu \log(d_{\nu l} + d_{\mu l}) + \sum_{\nu \in E} t_\nu^2 \log d_{\nu l} \Big\} + C, \tag{11}$$

where $d_{\nu l} = |f'_{\nu l}(0)|$. The proof that $\psi_l \to -\infty$ is now completed by the following lemma.

Lemma. *Let $a_1, a_2, \ldots, a_n$ be arbitrary real numbers and let $u_1, u_2, \ldots, u_n$ be positive numbers. Then*

$$\begin{aligned} S_n &= \sum_{j=1}^{n} a_j^2 \log u_j + \sum_{j=1}^{n} \sum_{k=1, j\neq k}^{n} a_j a_k \log(u_j + u_k), \\ &\leq (a_1 + a_2 + \cdots + a_n)^2 \log\big(\max\{u_1, \ldots, u_n\}\big) + k_n. \end{aligned}$$

where k_n depends only on $a_1, \ldots, a_n$.

Proof. The proof will proceed by induction. For $n = 1$ there is nothing to be shown. Suppose the inequality is true for some n, and suppose for convenience that $0 < u_1 \leq u_2 \leq \cdots \leq u_{n+1}$. Our inductive hypotheses says that

$$S_n \leq b_n^2 \log u_n + k_n, \qquad \text{where} \quad b_n = a_1 + \cdots + a_n.$$

But

$$\begin{aligned} S_{n+1} &= S_n + a_{n+1}^2 \log u_{n+1} + 2a_{n+1} \sum_{j=1}^{n} a_j \log(u_j + u_{n+1}) \\ &= S_n + a_{n+1}^2 \log u_{n+1} + 2a_{n+1} b_n \log u_{n+1} \\ &\quad + 2a_{n+1} \sum_{j=1}^{n} a_j \log(1 + u_j/u_{n+1}) \\ &\leq (b_n^2 + a_{n+1}^2 + 2a_{n+1} b_n) \log u_{n+1} + k_{n+1} \\ &= b_{n+1}^2 \log u_{n+1} + k_{n+1}, \end{aligned}$$

where k_{n+1} is again independent of the u_j because $1 < 1 + u_j/u_{n+1} \leq 2$. Thus the lemma is proved.

In view of the lemma, it follows from (11) that

$$\psi_l \leq A\Bigg(\sum_{\nu \in E} \sum_{j \in J_\nu} x_j\Bigg)^2 \log\Big(\max_{\nu \in E} |f'_{\nu l}(0)|\Big) + K,$$

where A and K depend only on the parameters x_j and ζ_j. Thus $\psi_l \to -\infty$ under the assumption (10). This contradiction shows that $f'_\nu(0) \neq 0$ for every ν. In other words, each limit function f_ν is univalent. This completes the proof of the existence of an extremal configuration if (10) holds.

It was shown in [3] by a simple example that an extremal configuration need not exist if a sum of the form (10) is allowed to vanish.

3. Variational proof of the theorem

Using the method of boundary variation, we shall now give a proof of Theorem 1. We make the initial assumption (10), thus ensuring the existence of an extremal configuration.

We again use the notation Ω, f_ν, D_ν to indicate an extremal configuration. Let

$$\Gamma = \tilde{\Omega} \cap \tilde{D}_1 \cap \tilde{D}_2 \cap \cdots \cap \tilde{D}_n$$

be the set of points which lie outside Ω and outside the range of every function f_ν. Fix a point $z_0 \in \Gamma$ and construct the boundary variation ([11]; see [2, Chapter 10])

$$z^* = V_\varrho(z) = z + \frac{a\varrho^2}{z - z_0} + O(\varrho^3),$$

where $\varrho > 0$ is so small that V_ϱ is analytic and univalent outside a small part of Γ near z_0. Thus the functions $f_\nu^* = V_\varrho \circ \varphi_\nu$ are univalent and map the disk onto nonoverlapping regions

$$D_\nu^* = V_\varrho(D_\nu) \subset \Delta_k^* = V_\varrho(\Delta_k) \subset \tilde{\Omega}^*$$

for $\nu \in I_k$, where $\Omega^* = V_\varrho(\Omega)$ and $\tilde{\Omega}^* = V_\varrho(\tilde{\Omega})$. Observe that $\tilde{\Omega}^* = (\Omega^*)\tilde{}$.

Let φ^* be the value of the functional (6) for the perturbed configuration (Ω^*, f_ν^*). Note that $\varphi^* \le \varphi$ because (Ω, f_ν) is an extremal configuration.

Our next task is to calculate φ^*. Since

$$f_\nu^*(\zeta) = f_\nu(\zeta) + a\varrho^2\left[f_\nu(\zeta) - z_0\right]^{-1} + O(\varrho^3), \tag{12}$$

it follows that

$$\log\left|f_\nu^{*\prime}(\zeta)\right| = \log\left|f_\nu'(\zeta)\right| - \mathrm{Re}\left\{a\varrho^2\left[f_\nu(\zeta) - z_0\right]^{-2}\right\} + O(\varrho^3). \tag{13}$$

It also follows from (10) that

$$\begin{aligned} \log\left|f_\nu^*(\zeta) - f_\mu^*(\eta)\right| &= \log\left|f_\nu(\zeta) - f_\mu(\eta)\right| \\ &\quad - \mathrm{Re}\left\{a\varrho^2\left[f_\nu(\zeta) - z_0\right]^{-1}\left[f_\mu(\eta) - z_0\right]^{-1}\right\} + O(\varrho^3), \end{aligned} \tag{14}$$

where $\zeta \ne \eta$ if $\nu = \mu$.

The variational formulas (13) and (14) show that the expressions G_ν and $H_{\nu\mu}$ as given by (7) and (8) are deformed to

$$\begin{aligned} G_\nu^* = G_\nu &- \mathrm{Re}\Big\{a\varrho^2 \sum_{j\in J_\nu} x_j^2\left[f_\nu(\zeta_j) - z_0\right]^{-2}\Big\} \\ &- \mathrm{Re}\Big\{a\varrho^2 \sum_{j\in J_\nu}\ \sum_{k\in J_\nu, j\ne k} x_j x_k\left[f_\nu(\zeta_j) - z_0\right]^{-1}\left[f_\nu(\zeta_k) - z_0\right]^{-1}\Big\} + O(\varrho^3) \end{aligned} \tag{15}$$

and

$$(16)\quad H^*_{\nu\mu} = H_{\nu\mu} - \mathrm{Re}\Big\{a\varrho^2 \sum_{j\in J_\nu}\sum_{k\in J_\mu} x_j x_k \big[f_\nu(\zeta_j)-z_0\big]^{-1}\big[f_\mu(\zeta_k)-z_0\big]^{-1}\Big\} + O(\varrho^3).$$

Inserting the expressions (15) and (16) into (6), we see by a straightforward calculation that

$$(17)\qquad \varphi^* = \varphi - \mathrm{Re}\Big\{a\varrho^2\Big[\sum_{\nu=1}^{n}\sum_{j\in J_\nu} x_j\big(f_\nu(\zeta_j)-z_0\big)^{-1}\Big]^2\Big\} + O(\varrho^3).$$

Using the inequality $\varphi^* \le \varphi$ and invoking the basic lemma in the theory of boundary variation (see [11] or [2, p. 297]), we now conclude from (17) that the points $z_0 \in \Gamma$ lie on trajectories of a quadratic differential:

$$(18)\qquad -\Big(\sum_{\nu=1}^{n}\sum_{j\in J_\nu}\frac{x_j}{z-f_\nu(\zeta_j)}\Big)^2 dz^2 > 0.$$

The perfect square allows us to draw an immediate conclusion. Taking the square root in (18) and integrating, we find that

$$(19)\qquad F(z) = \sum_{\nu=1}^{n}\sum_{j\in J_\nu} x_j \log\big|z-f_\nu(\zeta_j)\big| = c_k$$

for all $z \in \Gamma \cap \Delta_k$, where the c_k are real constants. It also follows that for each k, the regions $D_\nu = f_\nu(\mathbf{D})$ which lie in Δ_k (i.e., for which $\nu \in I_k$) actually fill this region, leaving no open set uncovered. Their boundaries are formed by arcs of a lemniscate; in fact, they lie on level curves of the function F defined in (19).

Consider now the functions

$$(20)\qquad \mathcal{H}_\nu(\zeta) = \sum_{\mu=1}^{n}\sum_{j\in J_\mu} x_j \log\big|f_\nu(\zeta)-f_\mu(\zeta_j)\big| - \sum_{j\in J_\nu} x_j \log\Big|\frac{\zeta-\zeta_j}{1-\overline{\zeta_j}\zeta}\Big|,$$

$\nu = 1, 2, \ldots, n$, which are harmonic in $\mathbf{D}$. It follows from (19) that $\mathcal{H}_\nu(\zeta) \equiv c_k$ for $|\zeta| = 1$ if $\nu \in I_k$. Thus $\mathcal{H}_\nu(\zeta) = c_k$ for all $\zeta \in \mathbf{D}$ if $\nu \in I_k$. Choosing $\zeta = \zeta_i$ for some $i \in J_\nu$, we deduce from (20) that

$$\begin{aligned} c_k = \mathcal{H}_\nu(\zeta_i) = &\sum_{\mu=1,\mu\ne\nu}^{n}\sum_{j\in J_\mu} x_j \log\big|f_\nu(\zeta_i)-f_\mu(\zeta_j)\big| \\ &+ \sum_{j\in J_\nu} x_j \log\Big|\frac{f_\nu(\zeta_i)-f_\nu(\zeta_j)}{\zeta_i-\zeta_j}\Big| + x_i \log\big|f'_\nu(\zeta_i)\big| \\ &+ \sum_{j\in J_\nu} x_j \log\big|1-\overline{\zeta_j}\zeta_i\big|, \qquad i \in J_\nu, \quad \nu \in I_k. \end{aligned}$$

Now multiply by x_i and sum over all $i \in J_\nu$, recalling the expressions (7) and (8) for G_ν and $H_{\nu\mu}$. The resulting formula is

$$G_\nu + \sum_{\mu=1,\mu\neq\nu}^{n} H_{\nu\mu} = c_k \sum_{j\in J_\nu} x_j - \sum_{j\in J_\nu} x_j^2 \log(1-|\zeta_j|^2) + \sum_{i\in J_\nu} \sum_{j\in J_\nu, i\neq j} x_i x_j \log\left|\frac{\zeta_i - \zeta_j}{1-\overline{\zeta_j}\zeta_i}\right|,$$

where $\nu \in I_k$. Now sum over all $\nu \in I_k$ and then over all k ($1 \le k \le m$), bearing in mind the definition (6) of the basic functional φ and the definition (4) of σ_k. This gives the final expression

$$\varphi = \sum_{k=1}^{m} c_k \sigma_k - \sum_{\nu=1}^{n} \sum_{j\in J_\nu} x_j^2 \log(1-|\zeta_j|^2) + \sum_{\nu=1}^{n} \sum_{i\in J_\nu} \sum_{j\in J_\nu, i\neq j} x_i x_j \log\left|\frac{\zeta_i - \zeta_j}{1-\overline{\zeta_j}\zeta_i}\right|. \tag{21}$$

It remains to determine the constants c_k. For this purpose we again consider the function F defined by (19), which is harmonic in the domain Ω except at infinity, where $F(z) - s\log|z|$ is harmonic. (Here $s = x_1 + \cdots + x_N$.) By (19), $F(z)$ has the constant boundary values c_k for all $z \in \Gamma_k = \partial\Delta_k$. We now introduce Green's function of Ω,

$$g(z) = g(z,\infty) = \log|z| - \log R + o(1),$$

as defined in Section 1. In terms of the harmonic measures ω_k of Γ_k with respect to Ω, we construct the function

$$G(z) = F(z) - sg(z) - \sum_{k=1}^{m} c_k \omega_k(z).$$

Observe that G is harmonic in Ω (even at infinity), and it vanishes identically on each boundary component Γ_k. Thus

$$F(z) = sg(z) + \sum_{j=1}^{m} c_j \omega_j(z), \qquad z \in \Omega. \tag{22}$$

Letting z tend to infinity, we deduce from (22) and the asymptotic formula

$$F(z) = s\log|z| + O\big(1/|z|\big)$$

that

$$\sum_{j=1}^{m} c_j \omega_j(\infty) = s \log R. \tag{23}$$

Recall now that by construction the points $f_\nu(\zeta_j)$ lie inside Γ_k if $\nu \in I_k$. Thus from (19) and (22) we calculate

$$\sigma_k = \frac{1}{2\pi} \int_{\Gamma_k} \frac{\partial F}{\partial n} |dz| = s\omega_k(\infty) + \sum_{j=1}^{m} c_j P_{jk}, \tag{24}$$

where (3) and (1) have been used.

The remaining calculations are the same as those in [3]. Since the columns of the period matrix sum to zero, we may rewrite (24) in the form

$$s\omega_k(\infty) - \sigma_k = \sum_{j=1}^{m-1} (c_m - c_j) P_{jk}.$$

In terms of the inverse matrix $((p_{jk}))$, this gives

$$c_m - c_j = \sum_{k=1}^{m-1} p_{jk} \left[s\omega_k(\infty) - \sigma_k \right], \qquad j = 1, 2, \ldots, m-1. \tag{25}$$

Similarly, the identity $\omega_1(z) + \cdots + \omega_m(z) \equiv 1$ allows (23) to be rewritten as

$$c_m = s \log R + \sum_{k=1}^{m-1} (c_m - c_k) \omega_k(\infty). \tag{26}$$

Introducing (25) into (26), we find that

$$c_m = s \log R + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} p_{jk} \left[s\omega_j(\infty) - \sigma_j \right] \omega_k(\infty). \tag{27}$$

The formulas (25) and (27) allow us to compute

$$\begin{aligned} \sum_{k=1}^{m} \sigma_k c_k &= s c_m + \sum_{k=1}^{m-1} \sigma_k (c_k - c_m) \\ &= s^2 \log R + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} p_{jk} \left[s\omega_j(\infty) - \sigma_j \right] \left[s\omega_k(\infty) - \sigma_k \right]. \end{aligned}$$

This formula may now be combined with (21) to express φ in terms of the invariants of Ω and the given parameters. Since this is the maximum value of φ over all admissible configurations, we have established the inequality (5) of Theorem 1 and have proved its sharpness under the assumption (10). Finally, the condition (10) can be removed by a continuity argument, and the inequality remains sharp.

4. Minimum energy

We shall now discover a physical interpretation of the functional

$$\psi = \log R + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} p_{jk}\big(\omega_j(\infty) - \sigma_j\big)\big(\omega_k(\infty) - \sigma_k\big), \tag{28}$$

which occurs as a bound in the Goluzin-type inequalities of Theorem 1. Here we shall assume for convenience that the σ_j are real numbers with sum $s = 1$.

We consider again a multiply connected domain $\Omega \subset \hat{\mathbf{C}}$ bounded by disjoint rectifiable Jordan curves $\Gamma_1, \Gamma_2, \ldots, \Gamma_m$. Let Δ_k be the compact region bounded by Γ_k. Then the complement of Ω is $\tilde{\Omega} = \Delta_1 \cup \cdots \cup \Delta_m$. Let $\Gamma = \partial\Omega = \Gamma_1 \cup \cdots \cup \Gamma_m$. Green's function of Ω is again denoted by

$$g(z) = \log|z| - \log R + O\big(1/|z|\big). \tag{29}$$

We regard the boundary curves $\Gamma_1, \ldots, \Gamma_m$ as electrostatic conductors for charges obeying a force law with a logarithmic potential. We place a total charge σ_k on Γ_k, where $\sigma_1 + \cdots + \sigma_m = 1$. These charges will reach an *equilibrium distribution* $\{\varrho_k(z)\}$ which minimizes the total energy

$$E = \tfrac{1}{2}\int_\Gamma \int_\Gamma \log\frac{1}{|z-\zeta|}\, d\varrho(z)\, d\varrho(\zeta) = \tfrac{1}{2}\sum_{k=1}^{m}\sum_{j=1}^{m} \int_{\Gamma_k}\int_{\Gamma_j} \log\frac{1}{|z-\zeta|}\, d\varrho_j(z)\, d\varrho_k(\zeta)$$

under the constraint

$$\int_{\Gamma_k} d\varrho_k(z) = \sigma_k, \qquad k = 1, \ldots, m.$$

Let

$$\gamma(z) = \int_\Gamma \log|z-\zeta|\, d\varrho(\zeta) \tag{30}$$

be the *equilibrium potential*; that is, the *conductor potential* induced by the equilibrium distribution of charge. This function is harmonic in Ω, except for a logarithmic singularity at infinity. It has the properties

(i) $\gamma(z) = \log|z| + O\big(1/|z|\big)$ as $z \to \infty$;
(ii) $\gamma(z) \equiv \alpha_k$ for $z \in \Gamma_k$, $k = 1, \ldots, m$.

Property (i) is easily seen to hold for an arbitrary logarithmic potential. Property (ii), that the equilibrium potential is constant on each boundary component, is a special property of the equilibrium distribution; it is proved as follows.

Consider the simple variation $\varrho^*(z) = \varrho(z) + \varepsilon\tau(z)$ of the equilibrium distribution, where

$$\int_{\Gamma_k} d\tau(z) = 0, \qquad k = 1, \ldots, m, \tag{31}$$

and ε is a small real parameter.

This induces a variation

$$\begin{aligned} E^* &= E + \varepsilon \int_\Gamma \int_\Gamma \log \frac{1}{|z-\zeta|}\, d\varrho(\zeta)\, d\tau(z) + O(\varepsilon^2) \\ &= E - \varepsilon \sum_{k=1}^{m} \int_{\Gamma_k} \gamma(z)\, d\tau(z) + O(\varepsilon^2), \end{aligned}$$

where the definition (30) has been used. Since $E^* \geq E$, it follows that

$$\int_{\Gamma_k} \gamma(z)\, d\tau(z) = 0, \qquad k = 1, \ldots, m.$$

Thus in view of (31), $\gamma(z)$ is constant on Γ_k.

It is now clear that the equilibrium potential has a close relation to Green's function:

$$\gamma(z) = g(z) + \sum_{k=1}^{m} \alpha_k \omega_k(z), \tag{32}$$

where ω_k is the harmonic measure of Γ_k with respect to Ω. In particular, by (29) and property (i) of the equilibrium potential,

$$\sum_{k=1}^{m} \alpha_k \omega_k(\infty) = \log R. \tag{33}$$

Note finally that by Gauss' theorem

$$\frac{1}{2\pi} \int_{\Gamma_k} \frac{\partial \gamma}{\partial n} |dz| = \sigma_k, \qquad k = 1, \ldots, m, \tag{34}$$

where $\partial/\partial n$ indicates the inner normal derivative with respect to Ω.

Using (1), (3), and (34), we find by integration of (32) that

$$\sigma_k = \omega_k(\infty) + \sum_{j=1}^{m} \alpha_j P_{jk}, \qquad k = 1, \ldots, m. \tag{35}$$

On the other hand, since $\gamma(z) \equiv \alpha_k$ on Γ_k, it follows directly from the definition that the minimum energy is given by

$$2E = -\sum_{k=1}^{m} \alpha_k \sigma_k. \tag{36}$$

In order to determine the α_k, we use the fact that $\sum_{k=1}^{m} P_{jk} = 0$ to rewrite (35) as

$$\sigma_k - \omega_k(\infty) = \sum_{j=1}^{m-1} (\alpha_j - \alpha_m) P_{jk}.$$

Introducing the inverse $((p_{jk}))$ of the matrix $((P_{jk}))$, we conclude that

$$\sum_{k=1}^{m-1} p_{ki}\big(\sigma_k - \omega_k(\infty)\big) = \alpha_i - \alpha_m. \tag{37}$$

In view of (28), (33), (37), and (36), a simple calculation now gives

$$\psi = \sum_{k=1}^{m} \alpha_k \omega_k(\infty) + \sum_{j=1}^{m-1} (\alpha_j - \alpha_m)\big(\sigma_j - \omega_j(\infty)\big) = \sum_{j=1}^{m} \alpha_j \sigma_j = -2E.$$

This identifies the functional ψ defined in (28) as twice the negative of the minimum electrostatic energy with charge σ_k on the conductor Γ_k $(k = 1, \ldots, m)$, where $\sigma_1 + \cdots + \sigma_m = 1$.

5. Invariance of the energy functional

Up to this point we have required that the number n of functions f_ν be greater than or equal to the connectivity m of the domain Ω, so that their ranges D_ν can "fill all of the holes". We now assume that $n < m$ and that all of the points $\zeta_j = 0$. Specifically, we require that the range $D_k = f_k(\mathbf{D})$ lie inside Γ_k for $k = 1, 2, \ldots, n$; and we pose the problem of maximizing the sum

$$\varphi = \sum_{j=1}^{n} \sum_{k=1, j\neq k}^{n} x_j x_k \log\big|f_j(0) - f_k(0)\big| + \sum_{j=1}^{n} x_j^2 \log\big|f_j'(0)\big|,$$

where the x_j are nonzero real parameters with sum s. The variational argument given in Section 3 (or in [3]) is still valid, and we find that in the extremal configuration the boundary components Γ_k of Ω are arcs of a lemniscate. Adapting the formula (19) to this special case, we find more precisely that

$$F(z) = \sum_{j=1}^{n} x_j \log\big|z - f_j(0)\big| = c_k, \qquad z \in \Gamma_k. \tag{38}$$

The function F is harmonic in the whole plane except for singularities at the points $f_j(0)$, $j=1,\dots,n$. Thus it follows from the maximum principle that Γ_k is a rectifiable Jordan curve for $1 \le k \le n$, while Γ_k degenerates to a slit for $n < k \le m$.

We may now conclude as before (*cf.* formulas (22) and (23)) that

$$F(z) = sg(z) + \sum_{j=1}^{m} c_j \omega_j(z), \qquad z \in \Omega, \tag{39}$$

and

$$\sum_{j=1}^{m} c_j \omega_j(\infty) = s \log R, \tag{40}$$

where g is Green's function of Ω and ω_j is the harmonic measure of Γ_j, while R is the transfinite diameter of $\tilde{\Omega}$.

Now comes the main observation. Since $n < m$, we may also represent F with respect to the domain $\hat{\Omega} \supset \Omega$ bounded only by $\Gamma_1, \dots, \Gamma_n$. Let $\hat{g}$ be Green's function of $\hat{\Omega}$, let $\hat{\omega}_j$ be the harmonic measure of Γ_j with respect to $\hat{\Omega}$, and let $\hat{R}$ be the transfinite diameter of $(\hat{\Omega})\tilde{}$. Then because F is actually harmonic in $\hat{\Omega}$ except for a logarithmic singularity at infinity, we may conclude in similar fashion that

$$F(z) = s\hat{g}(z) + \sum_{j=1}^{n} c_j \hat{\omega}_j(z), \qquad z \in \hat{\Omega}, \tag{41}$$

and

$$\sum_{j=1}^{n} c_j \hat{\omega}_j(\infty) = s \log \hat{R}. \tag{42}$$

Taking normal derivatives in (39) and integrating around Γ_k, we find by (38), (1), and (3) that

$$s\omega_k(\infty) + \sum_{j=1}^{m} c_j P_{jk} = \begin{cases} x_k, & 1 \le k \le n, \\ 0, & n < k \le m. \end{cases} \tag{43}$$

In similar fashion, the expression (41) gives

$$s\hat{\omega}_k(\infty) + \sum_{j=1}^{n} c_j \hat{P}_{jk} = x_k, \qquad 1 \le k \le n. \tag{44}$$

The equations (40), (42), (43), and (44) now allow the calculation of the energy functionals

$$(45)\qquad \psi = s^2 \log R + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} p_{jk}\big(s\omega_j(\infty) - x_j\big)\big(s\omega_k(\infty) - x_k\big)$$

and

$$(46)\qquad \hat{\psi} = s^2 \log \hat{R} + \sum_{j=1}^{n-1} \sum_{k=1}^{n-1} \hat{p}_{jk}\big(s\hat{\omega}_j(\infty) - x_j\big)\big(s\hat{\omega}_k(\infty) - x_k\big)$$

where $\big((p_{jk})\big)$ and $\big((\hat{p}_{jk})\big)$ are the inverse matrices of $\big((P_{jk})\big)$ and $\big((\hat{P}_{jk})\big)$, respectively. In (45) it is understood that $x_k = 0$ for $n < k \leq m$. Proceeding exactly as in Section 3, one finds

$$\psi = \sum_{k=1}^{m} x_k c_k,$$

where $x_k = 0$ for $n < k \leq m$. But the same calculation gives

$$\hat{\psi} = \sum_{k=1}^{n} x_k c_k.$$

Thus $\psi = \hat{\psi}$.

This is a remarkable result. Given a fixed lemniscatic domain Ω bounded by n level curves of a function

$$(47)\qquad F(z) = \sum_{k=1}^{n} x_k \log|z - \alpha_k|$$

each surrounding a single point α_k, plus $m-n$ level arcs of the same function, we may compare the functionals ψ for Ω and $\hat{\psi}$ for the larger domain $\hat{\Omega}$ obtained from Ω by removing the $m-n$ boundary slits. The Riemann matrix $\big((P_{jk})\big)$, the harmonic measures $\omega_k(\infty)$, and the transfinite diameter R of the boundary will all vary strongly from Ω to $\hat{\Omega}$, yet the combinations ψ and $\hat{\psi}$ are always equal. The result may be stated as follows.

Theorem 2. *Let Ω be an m-tuply connected domain in the extended complex plane, containing the point at infinity. Let its boundary consist of n closed lemniscates $\Gamma_1, \ldots, \Gamma_n$ which are level curves of a function $F(z)$ of the form (47), each surrounding a single point α_k, plus $m-n$ slits $\Gamma_{n+1}, \ldots, \Gamma_m$ which are level arcs of the same function $F(z)$. Let $\hat{\Omega}$ be the larger domain bounded only by $\Gamma_1, \ldots, \Gamma_n$, where $1 \leq n < m$. Let ψ and $\hat{\psi}$ be the energy functionals of Ω and $\hat{\Omega}$, defined by (45) and (46), respectively, where $x_k = 0$ for $n < k \leq m$. Then $\psi = \hat{\psi}$.*

We may now use the conformal invariance of the functionals considered to formulate a more general result.

Theorem 3. *Let $\hat{\Omega}$ be a domain containing the point at infinity and bounded by n disjoint continua $\Gamma_1, \ldots, \Gamma_n$; and let $\hat{g}(z) = \hat{g}(z, \infty)$ be Green's function of $\hat{\Omega}$. Let Ω be a subdomain of $\hat{\Omega}$ bounded by the same continua $\Gamma_1, \ldots, \Gamma_n$ and by $(m-n)$ slit continua $\Gamma_{n+1}, \ldots, \Gamma_m$ with equations $\hat{g}(z) = c_k$, $k = n+1, \ldots, m$. Then the corresponding energy functionals $\hat{\psi}$ and ψ are equal.*

One simple example may be noted. Choose $n = 1$, $x_1 = 1$, and $x_k = 0$ for $2 \le k \le m$. Then the equations (38) define a domain bounded by a circle of radius $\hat{R}$ and $m-1$ concentric circular arcs of larger radius. Thus we may conclude that $\psi = \log \hat{R}$ for every domain of this type, regardless of the sizes and positions of the circular slits.

6. Variation of the energy functional

We shall now study the behavior of the energy functional

$$\psi = \psi(\Omega) = \log R + \sum_{j=1}^{m-1} \sum_{k=1}^{m-1} p_{jk} \big(\omega_j(\infty) - x_j\big) \big(\omega_k(\infty) - x_k\big) \tag{48}$$

under a variation of the domain Ω. (We suppose for convenience that $s = 1$.) It will turn out that ψ has a perfect square in its variation, so that the variational method is an effective tool for solving extremal problems.

Choose a point $z_0 \in \Omega$ and consider the interior variation

$$z^* = z + \frac{\varepsilon}{z - z_0}, \tag{49}$$

where ε is a small complex parameter. This leads to the well-known variational formulas [12]

$$\log R^* = \log R - \operatorname{Re}\{\varepsilon p'(z_0)^2\} + O(\varepsilon^2), \tag{50}$$

$$\omega_k^*(\infty) = \omega_k(\infty) + \operatorname{Re}\{\varepsilon p'(z_0) w_k'(z_0)\} + O(\varepsilon^2), \tag{51}$$

$$P_{jk}^* = P_{jk} + \operatorname{Re}\{\varepsilon w_j'(z_0) w_k'(z_0)\} + O(\varepsilon^2), \tag{52}$$

where $p(z)$ and $w_k(z)$ are the analytic completions of Green's function $g(z)$ and the harmonic measure $\omega_k(z)$, respectively. From (52) it is easy to deduce the corresponding formula

$$p_{jk}^* = p_{jk} - \operatorname{Re}\{\varepsilon v_j'(z_0) v_k'(z_0)\} + O(\varepsilon^2), \tag{53}$$

for the inverse matrix, where

$$v_j(z) = \sum_{k=1}^{m-1} p_{jk} w_k(z). \tag{54}$$

A straightforward calculation, using (48), (50), (51), (53), and (54), now reveals the elegant formula

$$\psi^* = \psi - \mathrm{Re}\left\{\varepsilon\left(p'(z_0) - \sum_{j=1}^{m-1} v_j'(z_0)\big(\omega_j(\infty) - x_j\big)\right)^2\right\} + O(\varepsilon^2). \tag{55}$$

In a forthcoming paper [5], we apply the formula (55) to the solution of various extremal problems. The lengthy calculations involve *Robin's function* of a multiply connected domain, roughly described as a kind of interpolation between Green's function and Neumann's function.

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