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COMPLETIONS OF H-CONES

Sirkka-Liisa Eriksson-Bique

Introduction

H-cones ([2]) and hyperharmonic cones ([5]) are ordered convex cones possessing order properties similar to those of positive superharmonic and hyperharmonic functions, respectively, on harmonic spaces. An H-cone can always be extended to a hyperharmonic cone by adjoining to it an element ∞. This extension does not generally have potential-theoretic properties. In this paper we construct a completion of an H-cone which resembles a set of positive hyperharmonic functions on an $\mathcal{S}$-harmonic space. We recall that a harmonic space X is $\mathcal{S}$-harmonic if for any $x \in X$ there exists a positive superharmonic function on X which is strictly positive at x.

In $\mathcal{S}$-harmonic spaces every positive hyperharmonic function is a pointwise supremum of an upward directed family of positive superharmonic functions [3, Corollary 2.3.1]. In our completion of an H-cone S, every element is a supremum of an upward directed family of elements in S.

We present three characterizations of a completion. A completion of an H-cone S is a set of some functions in S (Theorem 2.7). This idea of a completion is stated in [4, p. 18]. Moreover, a completion is a set of upward directed families for which an equivalence relation is defined (Theorem 2.8). This extension was considered in [6, Proposition 2.2.]. Lastly a completion of an H-cone S is a set of some subsets of S (Theorem 2.9).

If infima of pairs of functions and suprema of upward directed families are pointwise in an H-cone S of functions, then its completion is a set of functions that are pointwise suprema of upward directed families of functions of S. It is an open question whether this fact holds without the assumption that infima of pairs of functions are pointwise. A completion of the dual of an H-cone is given in [4, Proposition 2.6].

1. Preliminaries

Our basic structure is a partially ordered abelian semigroup $(W, +, \leq)$ with a neutral element 0 and having the properties

(1.1) $$u \geq 0$$

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and

$$u \le v \quad \Longrightarrow \quad u + w \le v + w \tag{1.2}$$

for all u, v, $w \in W$.

Along with the initial order ($\le$), we use another partial order $\preceq$, called *specific order*, defined as follows:

$$u \preceq v \quad \text{if} \quad v = u + u' \qquad \text{for some } u' \in W.$$

A structure $(W, +, \le)$ satisfying (1.1) and (1.2) is called an *ordered convex cone* if it admits an operation of multiplication by strictly positive real numbers such that for all α, $\beta \in \mathbf{R}_+ \setminus \{0\}$ and x, $y \in W$

$$\alpha(x + y) = \alpha x + \alpha y, \qquad (\alpha + \beta)x = \alpha x + \beta x$$

$$(\alpha\beta)x = \alpha(\beta x), \qquad 1x = x,$$

$$x \le y \qquad \Longrightarrow \qquad \alpha x \le \alpha y.$$

A mapping φ from an ordered convex cone C onto an ordered convex cone D is called an *isomorphism* if it satisfies

$$s \le t \quad \Longleftrightarrow \quad \varphi(s) \le \varphi(t),$$

$$\varphi(s + t) = \varphi(s) + \varphi(t),$$

$$\varphi(\alpha s) = \alpha\varphi(s),$$

for all s, $t \in C$ and $\alpha \in \mathbf{R}_+ \setminus \{0\}$. Ordered convex cones C and D are called *isomorphic* if there exists an isomorphism φ from C onto D.

Definition 1.1. An ordered convex cone $(W, +, \le)$ is called a *hyperharmonic cone* if the following axioms hold:

(H1) for any non-empty upward directed family $F \subset W$ there exists a least upper bound $\bigvee F$ satisfying

$$\bigvee (x + F) = x + \bigvee F$$

for all $x \in W$,

(H2) for any non-empty family $F \subset W$ there exists a greatest lower bound $\bigwedge F$ satisfying

$$\bigwedge (x + F) = x + \bigwedge F,$$

(H3) for any u, v_1 and $v_2 \in W$ such that $u \le v_1 + v_2$ there exist u_1 and $u_2 \in W$ satisfying the properties $u = u_1 + u_2$, $u_1 \le v_1$ and $u_2 \le v_2$.

The theory of hyperharmonic cones is developed in [5], [6], [7] and [8]. We need the following result:

Theorem 1.2. *Let* $(W,+,\leq)$ *be an ordered convex cone. The structure* $(W,+,\leq)$ *is a hyperharmonic cone if and only if axiom* (H1) *and the following properties hold:*

(a) *for any* u *and* v *in* W, *the set* $\{w \in W : u \leq v + w\}$ *has a least element denoted by* $S_v u$ *and* $S_v u \preceq u$,

(b) *every non-empty subset* E *of* W *has a greatest lower bound.*

([5,Theorem 2.3]).

A partially ordered abelian semigroup with a neutral element 0 and satisfying (1.1), (1.2) and (a) is called a hyperharmonic structure by Arsove and Leutwiler in [1].

Note that (H3) leads to the inequality

$$u \wedge (v + w) \leq u \wedge v + u \wedge w \tag{1.3}$$

for all u, v and w in a hyperharmonic cone W.

An element $u \in W$ is called *cancellable* if $x + u \leq y + u$ implies $x \leq y$ for all x, $y \in W$. Cancellable elements in hyperharmonic cones are the same as cancellable elements with respect to the specific order [5, Theorem 3.9]. A useful characterization of cancellable elements is the condition

$$u \text{ is cancellable} \quad \Longleftrightarrow \quad \underline{u} = \bigwedge_{n \in \mathbf{N}} \frac{u}{n} = 0. \tag{1.4}$$

The element $\underline{u}$ $(u \in W)$ satisfies the following properties:

$$\underline{u} + u = u, \tag{1.5}$$

$$v \leq \underline{u} \quad \Longleftrightarrow \quad v + u = u, \tag{1.6}$$

$$v \leq u \quad \Longrightarrow \quad \underline{v} \leq \underline{u} \quad \Longrightarrow \quad \underline{v} \preceq u \quad \Longrightarrow \quad \underline{v} + u = u. \tag{1.7}$$

The proof of the above mentioned properties is stated in [5, Theorem 3.9].

The next result is helpful for handling uncancellable elements

Proposition 1.3. *If* $(W,+,\leq)$ *is a hyperharmonic cone and* u *an element of* W *then* $(\underline{u} + W,+,\leq)$ *is also a hyperharmonic cone. Moreover,* u *is cancellable in* $\underline{u} + W$ *([5, Proposition 4.1]).*

Definition 1.4. The set of cancellable elements of a hyperharmonic cone is called an H-cone.

Referring to [5, Remark 2.6 (a)] and [5, Theorem 3.13] our definition of an H-cone is equivalent to one given by Boboc, Bucur and Cornea [2, p. 27]. In the theory of H-cones the notation $R(u-x)$ is used for the greatest lower bound of the set $\{ s : s \geq u - x \}$ (see [2, p. 40]). We prefer the notation $S_x u$, since the subtraction is not generally defined in a hyperharmonic cone. If S is an H-cone then $R(u-x) = S_x u$ for all u, $x \in S$.

Let W be an ordered convex cone. A subset S is called *solid* in W if for any elements u in W and s in S the condition $u \leq s$ implies $u \in S$. A subset S is called *order dense* in W if for any u in W there exists an upward directed subset F of S such that $u = \bigvee F$.

Theorem 1.5. *If an ordered convex cone W satisfies* (H1) *and* (H2) *and has a solid and order dense subset possessing property* (a) *of Theorem* 1.2 *then W is a hyperharmonic cone.*

Proof. Let W be an ordered convex cone satisfying (H1) and (H2). Denote by S a solid and order dense subset of W enjoying property (a) of Theorem 1.2. Let u and x be arbitrary elements of W. In order to prove that W is a hyperharmonic cone it is enough by Theorem 1.2 to show that the set $E = \{ w \in W : u \leq w + x \}$ has a greatest lower bound and $\bigwedge E \preceq u$. Write $u = \bigvee F$ for an upward directed subset F of S. We verify that

$$\bigwedge E = \bigvee_{t \in F} S_{t \wedge x} t. \tag{1.8}$$

Note that $S_{t\wedge x} t$ exists for all $t \in F$ and $x \in W$ since S is solid and (a) holds in S. The set $\{ S_{t\wedge x} t : t \in W \}$ is directed upwards. Indeed, let s, t and r be elements of F such that $r \geq s$ and $r \geq t$. From the inequalities $s \leq S_{r\wedge x} r + s$ and $r \leq S_{r \wedge x} r + r \wedge x$ we infer that

$$s = s \wedge r \leq (S_{r\wedge x} r + s) \wedge (S_{r\wedge x} r + r \wedge x) = S_{r\wedge x} r + s \wedge x.$$

Hence we have $S_{r\wedge x} r \geq S_{s\wedge x} s$. Similarly we see that $S_{r\wedge x} r \geq S_{t\wedge x} t$. Thus the family $\{ S_{s\wedge x} s : s \in F \}$ is directed upwards and by (H1) has the least upper bound denoted by w_0. The element w_0 belongs to E since

$$x + w_0 \geq x \wedge t + S_{t\wedge x} t \geq t$$

for all $t \in F$ and therefore $x + w_0 \geq u$.

Let w be an arbitrary element of W satisfying $x + w \geq u$. Then

$$w + x \wedge t = (w + x) \wedge (w + t) \geq u \wedge (w + t) \geq t$$

for all $t \in F$. There results $w \geq S_{t\wedge x}t$ for all $t \in F$ and further $w \geq w_0$. Hence w_0 is the least element of E, verifying (1.8).

Lastly we show that $w_0 \preceq u$. From $S_{t\wedge x}t \preceq t$ it follows that $t = S_{t\wedge x}t + w_t$ for some $m_t \in S$. Put

$$v_s = \bigwedge_{\substack{t\geq s \\ t\in F}} m_t$$

for $s \in F$. Let s, t and r be elements of F such that $r \geq s$ and $r \geq t$. Then we have

$$v_s + S_{t\wedge x}t \leq m_r + S_{r\wedge x}r = r \leq u.$$

By taking the least upper bounds we obtain

$$w_0 + \bigvee_{s\in F} v_s \leq u.$$

On the other hand,

$$w_0 + m_t \geq S_{t\wedge x}t + m_t = t \geq s$$

for all $t \in F$ with $t \geq s$. This result implies $w_0 + v_s \geq s$ for all $s \in F$, yielding $w_0 + \bigvee_{s\in F} v_s \geq u$. Hence the equality $w_0 + \bigvee_{s\in F} v_s = u$ holds and therefore $w_0 \preceq u$, completing the proof.

Corollary 1.6. *If an ordered convex cone W satisfies* (H1) *and* (H2) *and has a solid order dense subset S which is an H-cone then W is a hyperharmonic cone.*

This Corollary follows from [2, Proposition 2.1.2] and Theorem 1.5.

2. Completion of an H-cone

Let S be an H-cone. A hyperharmonic cone W is called a *completion* of an H-cone S if S is isomorphic with a solid and order dense subset of W and W satisfies the axiom

(H4) $$\bigvee_{f\in F} w \wedge f = w \wedge (\bigvee F)$$

for all upward directed families $F \subset W$ and $w \in W$.

Note that (H4) does not generally hold in hyperharmonic cones. A counter example is given in [5, Remark 4.18]. However, we can prove the following version of (H4):

Lemma 2.1. *Let W be a hyperharmonic cone. Then the identity*

$$\underline{\bigvee F} + \bigvee_{f\in F} w \wedge f = (\bigvee F) \wedge (w + \underline{\bigvee F})$$

holds for any upward directed subset F of W and $w \in W$.

Proof. Let F be an upward directed subset of W and w be an element of W. Without loss of generality we may assume that $w \leq \bigvee F$. Indeed, we have

$$\bigvee_{f\in F} f \wedge w = \bigvee_{f\in F} (f \wedge (w \wedge \bigvee F))$$

and further by (1.5)

$$\begin{aligned}(\bigvee F) \wedge (w + \underline{\bigvee F}) &= (\bigvee F) \wedge (w + \underline{\bigvee F}) \wedge (\bigvee F + \underline{\bigvee F}) \\ &= (\bigvee F) \wedge ((w \wedge \bigvee F) + \underline{\bigvee F}).\end{aligned}$$

The inequality

$$w + \underline{\bigvee F} \geq w \geq \bigvee_{f\in F} w \wedge f$$

is clear. On the other hand $w \leq \bigvee F$ implies that

$$\bigvee F + \bigvee_{f\in F} w \wedge f = \bigvee_{f\in F} ((f + \bigvee F) \wedge (w + \bigvee F)) \geq w + f$$

for all $f \in F$. Hence we have $\bigvee F + \bigvee_{f\in F} w \wedge f \geq w + \bigvee F$. Applying now (1.5) we obtain

$$\underline{\bigvee F} + \bigvee_{f\in F} w \wedge f \geq w + \underline{\bigvee F}.$$

This completes the proof.

Corollary 2.2. *Let S be an H-cone. Then*

$$\bigvee_{f\in F} f \wedge s = (\bigvee F) \wedge s \tag{2.1}$$

for any upward directed bounded subset F of S and $s \in S$.

Proof. If $F \subset S$ is bounded then $\bigvee F$ is cancellable. This assertion follows from the preceding lemma.

Applying an observation stated in [4, p. 183], we will show that a completion of an H-cone is a set of mappings given below:

Definition 2.3. Let S be an H-cone. Denote by $\overline{S}$ the set of mappings $\varphi : S \to S$ satisfying

$$\varphi(u \wedge v) = \varphi(u) \wedge v \tag{2.2}$$

for all u, $v \in S$.

Proposition 2.4. *Let S be an H-cone and φ a mapping from S into itself. Then the following statements are mutually equivalent:*

(i) *φ satisfies (2.2);*
(ii) *$\varphi(u \wedge v) = \varphi(u) \wedge \varphi(v)$ for all u, $v \in S$ and if $s \leq \varphi(u)$ for some s, $u \in S$ then $\varphi(s) = s$;*
(iii) *$\varphi(u) = \bigvee_{s \in S} u \wedge \varphi(s)$ for all $u \in S$.*

Proof. Assume that φ satisfies (2.2) and u, $v \in S$. Then $\varphi(u \wedge v) = u \wedge \varphi(v)$ and $\varphi(u \wedge v) = v \wedge \varphi(u)$, which yields

$$\varphi(u \wedge v) = u \wedge \varphi(v) \wedge v \wedge \varphi(u) = \varphi(u) \wedge \varphi(v).$$

This completes the proof of the first part of (ii). Suppose now that $s \leq \varphi(u)$ for some s and u in S. Since

$$\varphi(u) = \varphi(u) \wedge \varphi(u) = \varphi(\varphi(u) \wedge u) = \varphi(\varphi(u \wedge u)) = \varphi^2(u)$$

we obtain

$$\varphi(s) = \varphi\big(s \wedge \varphi(u)\big) = s \wedge \varphi^2(u) = s \wedge \varphi(u) = s.$$

Hence (ii) holds.

Assume next that (ii) is true. Since $u \wedge \varphi(s) \leq \varphi(s)$ and $\varphi(s) \leq \varphi(s)$ we have $u \wedge \varphi(s) = \varphi\big(u \wedge \varphi(s)\big)$ and $\varphi(s) = \varphi^2(s)$ by the second part of (ii). It follows that

$$\bigvee_{s \in S} u \wedge \varphi(s) = \bigvee_{s \in S} \varphi\big(u \wedge \varphi(s)\big) = \bigvee_{s \in S} \varphi(u) \wedge \varphi^2(s) = \bigvee_{s \in S} \varphi(u) \wedge \varphi(s) = \varphi(u).$$

Lastly assume that (iii) holds. Using Corollary 2.2 we notice that

$$\varphi(u \wedge v) = \bigvee_{s \in S} u \wedge v \wedge \varphi(s) = v \wedge \bigvee_{s \in S} u \wedge \varphi(s) = \varphi(u) \wedge v,$$

completing the proof.

A function φ satisfying (2.2) possesses the following properties:

Proposition 2.5. *Let S be an H-cone. If a mapping $\varphi : S \to S$ satisfies (2.2), then the following properties hold for all u and v in S:*

$$\varphi(u) \leq u, \tag{2.3}$$

$$u \leq v \Longrightarrow \varphi(u) \leq \varphi(v), \tag{2.4}$$

$$\varphi^2(u) = \varphi(u), \tag{2.5}$$

$$\varphi(u+v) \le \varphi(u) + \varphi(v), \tag{2.6}$$

$$\varphi(u+v) = \varphi\big(\varphi(u) + \varphi(v)\big), \tag{2.7}$$

$$\varphi(\bigvee F) = \bigvee_{f\in F} \varphi(f) \quad \text{for all upward directed bounded subsets } F \text{ of } S. \tag{2.8}$$

Proof. The properties (2.3)–(2.5) are obvious. Applying (2.2) we see that $\varphi(u) = \varphi\big((u+v)\wedge u\big) = \varphi(u+v)\wedge u$ and $\varphi(v) = \varphi(u+v)\wedge v$. Hence by (1.3) we have

$$u \wedge \varphi(u+v) + v \wedge \varphi(u+v) \ge (u+v)\wedge \varphi(u+v) = \varphi(u+v).$$

This result gives (2.6).

The inequalities $\varphi(u) \le u$ and $\varphi(v) \le v$ lead by (2.4) to $\varphi(u+v) \ge \varphi\big(\varphi(u) + \varphi(v)\big)$. Since the converse inequality follows from (2.4)–(2.6), the property (2.7) is true.

Lastly Corollary 2.2 and Proposition 2.4 ensure that

$$\varphi(\bigvee F) = \bigvee_{s\in S} (\bigvee F)\wedge \varphi(s) = \bigvee_{s\in S}\bigvee_{f\in F} f\wedge\varphi(s) = \bigvee_{f\in F}\varphi(s)$$

finishing the proof.

Increasing mappings from S into S induce mappings satisfying (2.2).

Lemma 2.6. *Let S be an H-cone and denote by $\mathcal{F}$ the set of increasing mappings $\varphi : S \to S$. Define a mapping $\hat{}: \mathcal{F} \to \mathcal{F}$ by*

$$\hat\varphi(u) = \bigvee_{s\in S} \varphi(s)\wedge u \quad (u \in S).$$

Then the mapping $\hat{}$ possesses the following properties:

$$\hat\varphi \in \overline{S}, \tag{2.9}$$

$$\hat{\hat\varphi} = \hat\varphi, \tag{2.10}$$

$$\mu \le \varphi \implies \hat\mu \le \hat\varphi, \tag{2.11}$$

$$\widehat{\alpha\varphi}(u) = \alpha\hat\varphi(u/\alpha) = \widehat{\alpha\hat\varphi}(u), \tag{2.12}$$

$$(\widehat{\varphi+\mu})(u) = \big(\hat\varphi(u) + \hat\mu(u)\big)\wedge u, \tag{2.13}$$

$$\widehat{\varphi+\mu} = \widehat{\hat\varphi+\hat\mu}, \tag{2.14}$$

for all $u \in S$, φ, $\mu \in \mathcal{F}$ and $\alpha \in \mathbf{R}_+ \setminus \{0\}$.

Proof. Property (2.9) follows from Corollary 2.2, and (2.10) from Proposition 2.4 (iii). Properties (2.11) and (2.12) are clear. To prove (2.13), let φ and μ be elements of $\mathcal{F}$. Since φ is increasing we infer

$$\begin{aligned}(\hat{\varphi}(u) + \hat{\mu}(u)) \wedge u &= \bigvee_{\substack{s\in S\\ t\in S}} (\varphi(s) \wedge u + \mu(t) \wedge u) \wedge u \\ &= \bigvee_{\substack{s\in S\\ t\in S}} (\varphi(s) + \mu(t)) \wedge u \\ &= \bigvee_{s\in S} (\varphi(s) + \mu(s)) \wedge u = (\widehat{\varphi + \mu})(u).\end{aligned}$$

Property (2.14) follows directly from (2.13) and (2.10).

Let us define in $\overline{S}$ multiplication by strictly positive real numbers and addition as follows:

$$\alpha \cdot \varphi = \widehat{\alpha\varphi},$$

$$\varphi \oplus \mu = \widehat{\varphi + \mu},$$

for $\alpha \in \mathbf{R}_+ \setminus \{0\}$ and φ, $\mu \in \overline{S}$.

Theorem 2.7. *Let S be an H-cone and $\leq$ the pointwise order in $\overline{S}$. Then $(\overline{S}, \oplus, \leq)$ is a completion of S.*

Proof. Using Lemma 2.6 it is easy to check that $(\overline{S}, \oplus, \leq)$ is an ordered convex cone. We apply Theorem 1.5 to prove that $\overline{S}$ is a completion of S. Let F be an upward directed family in $\overline{S}$. The mapping $\mu : S \to S$ defined by $\mu(s) = \bigvee_{\varphi\in F} \varphi(s)$ belongs to $\overline{S}$ by Corollary 2.2 and $\bigvee F = \mu$. Hence the least upper bound is translation invariant, and so (H1) holds in $\overline{S}$.

Let F be a subset of $\overline{S}$. Then the mapping $\mu : S \to S$ defined by $\mu(u) = \bigwedge_{\varphi\in F} \varphi(s)$ belongs to $\overline{S}$ and $\bigwedge F = \mu$. Thus (H2) holds in $\overline{S}$.

Let us define the mapping $i : S \to \overline{S}$ by $i(s)(u) = s \wedge u$ for u and s in S. Obviously the mapping i is well-defined. We show that i is a one-to-one mapping from S onto $i(S)$. If $i(s) \leq i(t)$ for s, $t \in S$ then $s \wedge u \leq t \wedge u$ for all $u \in S$. Hence $s \leq t \wedge s \leq t$. There results

$$i(s) \leq i(t) \iff s \leq t.$$

Thus i is a one-to-one mapping from S onto $i(S)$. Since $(s \wedge u + t \wedge u) \wedge u = (s+t) \wedge (u+s) \wedge (t+u) \wedge 2u \wedge u = (s+t) \wedge u$, the mapping i is also additive. Using (2.12) we easily see that $\alpha \cdot i(s) = i(\alpha s)$. Consequently $\big(i(S), \oplus, \leq\big)$ is an H-cone which is isomorphic with $(S, +, \leq)$.

The cone $i(S)$ is solid in $\overline{S}$. Indeed, assume that $\psi \in \overline{S}$ and $\mu \in i(S)$ such that $\psi \le \mu$. Then $\mu(t) = s \wedge t$ for some $s \in S$ and $\psi(t) \le s \wedge t \le s$ for all $t \in S$. Hence $\bigvee_{t\in S} \psi(t)$ exists and

$$\psi(u) = \big(\bigvee_{t\in S} \psi(t)\big) \wedge u$$

for all $u \in S$ which means $\psi = i\big(\bigvee_{t\in S} \psi(t)\big)$. To prove that $i(S)$ is order dense, suppose that $\psi \in \overline{S}$. Then $\psi(u) = \bigvee_{t\in S} \psi(t) \wedge u$ for all $u \in S$ by Proposition 2.4(iii) and further $\psi = \bigvee_{t\in S} i\big(\psi(t)\big)$.

Collecting the material proved above we establish by Theorem 1.5 the assertion that $\overline{S}$ is a hyperharmonic cone. We still have to show that (H4) holds in $\overline{S}$. Let $F \subset \overline{S}$ be directed upwards. Using the results stated earlier we notice

$$\big(\psi \wedge (\bigvee F)\big)(u) = \psi(u) \wedge (\bigvee F)(u) = \psi(u) \wedge \bigvee_{\mu\in F} \mu(u).$$

Since $\psi(u) \le u$ by Proposition 2.5 we obtain by Corollary 2.2

$$\big(\psi \wedge (\bigvee F)\big)(u) = \bigvee_{\mu\in F} \mu(u) \wedge \psi(u).$$

Thus

$$\psi \wedge (\bigvee F)(u) = \bigvee_{\mu\in F} (\psi \wedge \mu)(u) = (\bigvee_{\mu\in F} \psi \wedge \mu)(u),$$

completing the proof.

A different type of an extension of an H-cone is constructed in [6, Proposition 2.2]. Next we shall show that it is also a completion.

Theorem 2.8. *Let S be an H-cone. Denote by Ω a family of upward directed subsets of S. An equivalence relation $\sim$ in Ω is defined by*

$$F \sim G \quad \Longleftrightarrow \quad \bigvee_{f\in F} s \wedge f = \bigvee_{g\in G} s \wedge g \qquad \text{for all } s \in S.$$

The equivalence classes of the relation $\sim$ is denoted by $[F]$ for $F \in \Omega$ and the set of all equivalence classes by $\mathcal{W}$. Addition, multiplication by strictly positive real numbers and partial ordering are given in $\mathcal{W}$ as follows

$$[F] + [G] = [F + G], \qquad \alpha[F] = [\alpha F],$$

$$[F] \le [G] \quad \Longleftrightarrow \quad \bigvee_{f\in F} s \wedge f \le \bigvee_{g\in G} s \wedge g \qquad \text{for all } s \in S.$$

Then $(\mathcal{W}, +, \le)$ is a completion of S.

Proof. We show that $\mathcal{W}$ and $\overline{S}$ are isomorphic. Define a mapping $\Gamma : \overline{S} \to \mathcal{W}$ by

$$\Gamma(\varphi) = \big[\varphi(S)\big], \qquad \varphi \in \overline{S}.$$

The mapping Γ is well-defined, since $\varphi(S)$ is directed upwards for all $\varphi \in \overline{S}$. Indeed, if s and t belong to S then $\varphi(s+t) = \varphi\big(\varphi(s) + \varphi(t)\big)$ by (2.7). Hence $\varphi(s+t) \in \varphi(S)$. Moreover, by (2.4) and (2.5), $\varphi(s+t) \geq \varphi(s)$ and $\varphi(s+t) \geq \varphi(t)$. Thus $\varphi(S)$ is directed upwards.

Assume that $\mu \leq \psi$ for μ, $\psi \in \overline{S}$. Then Proposition 2.4(iii) leads to

$$\bigvee_{s\in S} \mu(s) \wedge u = \mu(u) \leq \psi(u) = \bigvee_{s\in S} \psi(s) \wedge u$$

for all $u \in S$. Therefore we have $\big[\mu(S)\big] \leq \big[\psi(S)\big]$. The implication

$$\big[\mu(S)\big] \leq \big[\psi(S)\big] \quad \Longrightarrow \quad \mu \leq \psi$$

can be proved similarly. Hence we have established the relation

$$\mu \leq \psi \quad \Longleftrightarrow \quad \Gamma(\mu) \leq \Gamma(\psi)$$

for all μ, $\psi \in \overline{S}$. Let now $F \in \Omega$ and define $\varphi : S \to S$ by $\varphi(u) = \bigvee_{f\in F} f \wedge u$. Corollary 2.2 results in $\varphi \in \overline{S}$. Hence the mapping Γ is a one-to-one mapping from $\mathcal{W}$ onto $\overline{S}$.

The mapping Γ is also additive, since

$$\begin{aligned}
\Gamma(\mu+\psi) &= \Big[\Big\{\,\big(\mu(u)+\psi(u)\big) \wedge u : u \in S\,\Big\}\Big] \\
&= \Bigg[\Big\{\,\big(\bigvee_{s\in S} \mu(s) \wedge u + \bigvee_{t\in S} \psi(t) \wedge u\big) \wedge u : u \in S\,\Big\}\Bigg] \\
&= \Bigg[\Big\{\,\bigvee_{\substack{s\in S\\ t\in S}} \big(\mu(s) + \psi(t)\big) \wedge u : u \in S\,\Big\}\Bigg] = [\mu(S)] + [\psi(S)].
\end{aligned}$$

Using Lemma 2.6 we notice that

$$\Gamma(\alpha\cdot\varphi) = \big[\widehat{\alpha\varphi}(S)\big] = \big[\{\,\alpha\varphi(t/\alpha) : t \in S\,\}\big] = \alpha\Gamma(\varphi).$$

Consequently, $\mathcal{W}$ is a hyperharmonic cone satisfying (H4) and isomorphic with $\overline{S}$. It is obvious that $\mathcal{W}$ is a completion of S.

Popa has found a presentation for the preceding set $\mathcal{W}$ in terms of solid subsets A of S satisfying the following property:

(2.15) If $B \subseteq A$ and $\bigvee B$ exists in S, then $\bigvee B \in A$.

Now we will state and prove this result differently.

Theorem 2.9. *Let S be an H-cone. Denote by $\mathcal{W}_1$ the set of solid subsets of S satisfying* (2.15)*. Addition, multiplication by strictly positive real numbers and partial order in $\mathcal{W}_1$ is given by*

$$A + B = \{ a + b : a \in A, b \in B \},$$

$$\alpha A = \{ \alpha a : a \in A \},$$

$$A \leq B \iff A \subseteq B.$$

Then $(\mathcal{W}_1, +, \leq)$ is a completion of S.

Proof. Notice that $+$ is well-defined in $\mathcal{W}_1$ by (H3). We show first that $\overline{S}$ and $\mathcal{W}_1$ are isomorphic. Define a mapping $\Gamma : \overline{S} \to \mathcal{W}_1$ by

$$\Gamma(\varphi) = \varphi(S), \qquad \varphi \in \overline{S}.$$

To show that Γ is well-defined, let $F \subseteq \varphi(S)$ such that $\bigvee F$ exists in S. By Proposition 2.5

$$\bigvee F \geq \varphi(\bigvee F) \geq \bigvee_{f \in F} \varphi(f).$$

Proposition 2.4(ii) results in $\varphi(f) = f$ for all $f \in F$, which yields $\varphi(\bigvee F) \geq \bigvee F$. Thus we have $\varphi(\bigvee F) = \bigvee F$, and so $\bigvee F$ belongs to $\varphi(S)$. Hence $\varphi(S)$ satisfies (2.15). Since the set $\varphi(S)$ is also solid by Proposition 2.4(ii), the mapping Γ is well-defined.

Let μ and ψ be mappings in $\overline{S}$ such that $\mu \leq \psi$. Then $\mu(u) \leq \psi(u)$ for all $u \in S$ and further by Proposition 2.4(ii), $\psi(\mu(u)) = \mu(u)$. Hence $\mu(S) \subseteq \psi(S)$. Suppose that $\mu(S) \subseteq \psi(S)$ for some μ, $\psi \in \overline{S}$. Using (2.3) we notice that

$$\mu(u) = \psi(\mu(u)) \leq \psi(u)$$

for all $u \in S$. There results $\mu \leq \psi$. Now we have established the result

$$\mu \leq \psi \iff \Gamma(\mu) \leq \Gamma(\psi).$$

Assume that A is a solid subset of S satisfying (2.15). Then evidently the set A is directed upwards. Define a mapping $\varphi : S \to S$ by

$$\varphi(s) = \bigvee_{f \in A} f \wedge s, \qquad s \in S.$$

Proposition 2.4(iii) assures that $\varphi \in \overline{S}$. Since A satisfies (2.15), $\varphi(s) \in A$ for all $s \in S$ and therefore $\varphi(S) \subseteq A$. On the other hand, $\varphi(f) = f$ for all $f \in A$, which leads to $A \subseteq \varphi(S)$. Hence $\varphi(S) = A$. Thus we have shown that Γ is a one-to-one mapping from $\overline{S}$ onto $\mathcal{W}_1$.

It is easy to check using Lemma 2.6 that $\Gamma(\alpha \cdot \mu) = \alpha\Gamma(\mu)$. Let μ and ψ belong to $\overline{S}$. Then $\Gamma(\mu \oplus \psi) = \{ (\mu(u) + \psi(u)) \wedge u : u \in S \}$. Since $\mu(S) + \psi(S)$ is solid, we have $\Gamma(\mu \oplus \psi) \subseteq \mu(S) + \psi(S)$. But applying Proposition 2.5 we infer

$$\Big(\mu\big(\mu(u) + \psi(u)\big) + \psi\big(\mu(u) + \psi(u)\big)\Big) \wedge \big(\mu(u) + \psi(u)\big) \\ \geq \big(\mu^2(u) + \psi^2(u)\big) \wedge \big(\mu(u) + \psi(u)\big) = \mu(u) + \psi(u).$$

Hence $\mu(S) + \psi(S) \subseteq \Gamma(\mu \oplus \psi)$. We have shown that Γ is additive. Altogether we have verified that Γ is an isomorphism from $\overline{S}$ onto $\mathcal{W}_1$. Consequently, $\mathcal{W}_1$ is a hyperharmonic cone satisfying (H4) and evidently a completion of S.

Theorem 2.10. *Let an H-cone S be a cone of extended real-valued functions on a set X such that*

(a) $f \wedge g = \inf(f, g)$ *for all* f, $g \in S$,

(b) $\bigvee F(x) = \sup_{f \in F} f(x)$ *for any dominated upward directed family* F.

Then the completion of S is the set

$$C = \{ \sup_{f \in F} f : F \subseteq S \text{ is directed upwards} \}.$$

Proof. We show that C and $\overline{S}$ are isomorphic. Define a mapping $\Gamma : \overline{S} \to C$ by

$$\Gamma(\varphi) = \sup_{f \in F} \varphi(f).$$

Clearly if $\varphi \leq \mu$ then $\Gamma(\varphi) \leq \Gamma(\mu)$. Conversely, assume that $\Gamma(\varphi) \leq \Gamma(\mu)$ for φ and μ in $\overline{S}$. Then we have

$$\sup_{f \in F} \varphi(f) = \sup_{f \in S} \varphi(f)$$

and further

$$\sup_{f \in S} \inf\big(\varphi(f), g\big) = \inf\big(\sup_{f \in F} \varphi(f), g\big) \leq \inf\big(\sup_{f \in F} \mu(f), g\big) = \sup_{f \in F} \inf\big(\mu(f), g\big)$$

for all $g \in S$. This implies by (2.2) and (a) that $\varphi(g) \leq \mu(g)$ for all $g \in S$. Hence we have proved that

$$\varphi \leq \mu \quad \Longleftrightarrow \quad \Gamma(\varphi) \leq \Gamma(\mu).$$

Let F be an upward directed subset of S. Define a mapping $\varphi : S \to S$ by $\varphi(g) = \sup_{f \in F} f \wedge g$. Then we have

$$\Gamma(\varphi) = \sup_{f \in F} \varphi(f) = \sup_{f \in F} f.$$

Hence the mapping Γ is a one-to-one mapping from $\overline{S}$ onto C.

Using Lemma 2.6 we easily notice that $\Gamma(\alpha \cdot \varphi) = \alpha\Gamma(\varphi)$. To prove additivity of Γ, let $\varphi,\ \mu \in \overline{S}$. Applying the definitions we obtain

$$\begin{aligned}\Gamma(\mu \oplus \varphi) &= \sup_{f\in F}(\mu \oplus \varphi)(f) = \sup_{f\in S}\big(\mu(f) + \varphi(f)\big) \wedge f \\ &\leq \sup_{f\in F} \mu(f) + \varphi(f) = \Gamma(\mu) + \Gamma(\varphi).\end{aligned}$$

To show the converse, we first note that

$$\begin{aligned}&\sup_{f\in F}\big(\mu(f) + \varphi(f)\big) \\ &\qquad \wedge f \geq \sup_{f\in F}\Big(\mu\big(\mu(f)+\varphi(f)\big) + \varphi\big(\mu(f)+\varphi(f)\big) \wedge \big(\mu(f)+\varphi(f)\big)\Big) \\ &\qquad \geq \sup_{f\in F}\big(\mu^2(f) + \varphi^2(f)\big) \wedge \big(\mu(f)+\varphi(f)\big).\end{aligned}$$

Since by Lemma 2.6 $\mu^2(f) = \mu(f)$ and $\varphi^2(f) = \varphi(f)$ we have

$$\Gamma(\mu \oplus \varphi) \geq \Gamma(\mu) + \Gamma(\varphi).$$

Hence Γ is an isomorphism from $\overline{S}$ onto C. It is therefore obvious that C is a completion of S.

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University of Joensuu
Department of Mathematics
Box 111
SF-80101 Joensuu
Finland

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