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ON THE NUMBER OF ISOMETRY CLASSES OF BILINEAR SPACES IN UNCOUNTABLE DIMENSIONS

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1. Introduction

Of late efforts have been made to estimate the number of isometry classes (similarity classes) of ε-hermitean forms over division rings equipped with anti-isomorphisms $^*\colon K \to K$. (Refer to [3] for standard terminology.) A strong result is the following

Theorem 1. [2, Theorem 2] *For each regular uncountable cardinal α and for K any division ring (of arbitrary characteristic and cardinality) endowed with an involutory anti-isomorphism *, and for ε a central element subject to $\varepsilon \cdot \varepsilon^* = 1$, there are 2^α pairwise non-similar, non-degenerate ε-hermitean forms in dimension α.*

In [6] a special case of Theorem 1 is established by *model theoretic* methods: Let α be any uncountable cardinal and K a commutative field. If $\operatorname{card}(K) < \alpha$ then there are 2^α pairwise non-similar non-degenerate trace-valued ε-hermitean K-spaces of dimension α.

Here we shall prove the following *complementary* result:

Theorem 2. *Let $\gamma \le \beta \le \alpha$ be (not necessarily regular) uncountable cardinals. If K is a commutative field of any characteristic and cardinality α then there is a symmetric form (skew-symmetric form) $\Phi\colon E \times E \to K$ with $\dim E = \beta$ such that E contains 2^γ non-degenerate subspaces of dimension γ that are pairwise non-isometric.*

The idea used in the proof of Theorem 2 is akin to that used in [5] for the construction of spaces with small orthogonal group. Apparently, the relevance for richness results of the constructions given in [5] have passed unnoticed. In our proof of Theorem 2 just a little more work needs to be added in order to obtain the following rather surprising result:

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Theorem 3. *If, in Theorem 2, the form Φ is symmetric and K is a purely transcendental extension K/K_0 of transcendence degree α over some subfield K_0 and if -1 is a square in K and if* $\mathrm{char}(K) \neq 2$ *then the 2^γ subspaces $F \subseteq E$ can be chosen such that in each one of them Witt's cancellation theorem holds (i.e. if $F = A \oplus A^\perp = B \oplus B^\perp$ and $A \simeq B$ (isometric) then $A^\perp \simeq B^\perp$) whereas Witt's extension theorem invariably fails to hold in these spaces (i.e., there always are isometries between subspaces that admit no extension to the whole space).*

Some of our results can be generalized to forms other than symmetric or skew-symmetric. Our reason for staying commutative here has been to give a sharper repoussé pattern to our tricks.

In the last section we add a few short remarks concerning the interrelations between the cancellation theorem, the extension theorem and, of all things, orthomodularity.

2. A lemma on finite dimensional forms

Let K be a commutative field of arbitrary characteristic, K/K_0 a transcendental extension and $(x_{ij})_{0\leq i\leq j\leq n}$ a family of $\frac{1}{2}(n+1)(n+2)$ elements in K that are algebraically independent over K_0. On $K^{n+1}\times K^{n+1}$ we consider the symmetric bilinear form

$$\Psi(\mathbf{x},\mathbf{y}) := \sum_{i,j=0}^{n} \xi_i x_{ij}\eta_j, \qquad \mathbf{x}=(\xi_0,\ldots,\xi_n), \quad \mathbf{y}=(\eta_0,\ldots,\eta_n)$$

with $x_{ji} := x_{ij}$ when $i<j$ (so $x_{ij}=x_{rs}$ if and only if $\{i,j\}=\{r,s\}$).

Lemma 4. *The form $\Psi(\mathbf{x},\mathbf{y})$ is a square in K if and only if $\mathbf{x}$ is the zero vector.*

Proof ([8]). We consider first the case where $\mathrm{char}(K)\neq 2$. Assume $\Psi(\mathbf{x},\mathbf{x}) = N_0^2$, where $\mathbf{x}=(\xi_0,\xi_1,\ldots,\xi_n)$ and $\xi_i\neq 0$ for all i. We shall use the following abbreviations: For $i,j,k\in\{0,1,\ldots,n\}$ and $\mathbf{e}_i$ the canonical basis vectors in K^{n+1} we set

$$\begin{aligned}
(x_0)_{ij} &:= x_{ij} && (=\Psi(\mathbf{e}_i,\mathbf{e}_j))\\
(x_{k+1})_{ij} &:= (x_k)_{ki}\cdot(x_k)_{kj} - (x_k)_{kk}\cdot(x_k)_{ij} &&\\
x_k &:= (x_k)_{kk}, && x_{-1} := 0.
\end{aligned}$$

Instead of investigating the equation

$$\sum_{i,j=0}^{n} \xi_i\xi_j x_{ij} = N_0^2 \tag{1}$$

we shall study the solvability of equations that have nontrivial solutions provided that (1) admits a nontrivial solution. More precisely, we claim that the equation

$$\text{(2)} \qquad \sum_{i,j=k}^{n} \xi_i \xi_j (x_k)_{ij} = N_k^2 - x_{k-1}N_{k-1}^2 + x_{k-1}x_{k-2}N_{k-2}^2 - + - \cdots \\ + (-1)^k x_{k-1}x_{k-2}x_{k-3}\cdots x_1 x_0 N_0^2,$$

which reduces to (1) for $k=0$, entails the equation

$$\text{(3)} \qquad \sum_{i,j=k+1}^{n} \xi_i \xi_j (x_{k+1})_{ij} = N_{k+1}^2 - x_k N_k^2 + x_k x_{k-1} N_{k-1}^2 - + - \cdots \\ + (-1)^{k+1} x_k x_{k-1} x_{k-2} \cdots x_1 x_0 N_0^2$$

if $4N_{k+1}^2$ is defined to be the discriminant of the quadratic equation (2) with unknown ξ_k. As $\xi_k \in K$ we have $N_{k+1} \in K$. By induction we find that we may choose $k := n$ in (2):

$$\xi_n^2 x_n = N_n^2 - x_{n-1}N_{n-1}^2 + x_{n-1}x_{n-2}N_{n-2}^2 - + \cdots + (-1)^n x_{n-1}\cdots x_0 N_0^2.$$

Thus, the following equation, with unknowns $X, X_1, X_2, \ldots$,

$$\text{(4)} \quad X_n^2 = x_n X^2 + x_{n-1}X_{n-1}^2 - x_{n-1}x_{n-2}X_{n-2}^2 + - \cdots - (-1)^n x_{n-1}\cdots x_0 X_0^2$$

admits a nontrivial (integral) solution ($X = \xi_n$ and all ξ_i are nonzero). Next, we define recursively a sequence of equations that admit nontrivial solutions provided (4) possesses a nontrivial solution. In one case the contradiction will become plain. More precisely, we claim: If for $k < n$ the equation

$$\text{(5)} \quad X_{n-k}^2 = (x_{n-k})_{nn} X^2 + x_{n-(k+1)} X_{n-(k+1)}^2 - x_{n-(k+1)} x_{n-(k+2)} X_{n-(k+2)}^2 \\ + - \cdots - (-1)^{n-k} x_{n-(k+1)} \cdots x_0 X_0^2$$

has a nontrivial solution (this is actually the case if $k=0$) then the same holds true for $k+1$ in lieu of k. In order to prove this we first mention that for $t \le s$ the degree of the polynomial $(x_t)_{ij}$ in the unknown x_{ss} is precisely one if and only if $i = j = s$ and zero otherwise; this is readily established by induction on t. Furthermore, the degree of x_{ss} in the polynomial $(x_{s+1})_{nn}$ is invariably one, because $s < n$ in this case. From this we recognize that the coefficients of the squares in (5) are linear, throughout, in the unknown $x := x_{n-(k+1),n-(k+1)}$; so

$$\begin{aligned} (x_{n-k})_{nn} &= A_{n-k}x + B_{n-k} \\ x_{n-(k+1)} &= A_{n-(k+1)}x + B_{n-(k+1)} \\ x_{n-(k+1)}x_{n-(k+2)} &= A_{n-(k+2)}x + B_{n-(k+2)} \\ &= \ldots \\ x_{n-(k+1)}x_{n-(k+2)} \cdot \cdots \cdot x_1 \cdot x_0 &= A_0 x + B_0 \end{aligned}$$

with all A_j nonzero. We therefore conclude that

$$(6)\qquad \begin{aligned} A_j &= A_{n-(k+1)}x_{n-(k+2)}x_{n-(k+3)}\cdots x_{j+1}x_j \quad \text{when } j < n-(k+1),\\ A_{n-k} &= A_{n-(k+1)}\big(-(x_{n-(k+1)})_{nn}\big). \end{aligned}$$

The second relation follows by considering that

$$(x_{n-k})_{nn} = (x_{n-(k+1)})^2_{n-(k+1),n} - x_{n-(k+1)}(x_{n-(k+1)})_{nn}.$$

Now, the equation (5) can be written as follows:

$$(7)\qquad \begin{aligned} X^2_{n-k} &= x\big(A_{n-k}X^2 + A_{n-(k+1)}X^2_{n-(k+1)} - + \cdots - (-1)^{n-k}A_0X_0^2\big)\\ &\quad + B_{n-k}X^2 + B_{n-(k+1)}X^2_{n-(k+1)} - + \cdots - (-1)^{n-k}B_0X_0^2. \end{aligned}$$

By our assumption that (5) have a nontrivial solution we know that (7) has a nontrivial solution consisting of polynomials X, X_0, X_1, ... Let m be the maximal degree of the unknown x occuring in the polynomials X, X_0, X_1, ... and a_j (a, respectively) the coefficient of x^m in X_j (in X, respectively). Thus, by definition, not all among a_j, a are zero. On the other hand,

$$(8)\qquad A_{n-k}a^2 + A_{n-(k+1)}a^2_{n-(k+1)} - + \cdots - (-1)^{n-k}A_0a_0^2 = 0$$

because the left hand side in (8) is the coefficient of x^{2m} in the bracket term of (7) and this coefficient must vanish for, otherwise, the right hand side in (7) would show the odd degree $2m+1$ in the unknown x, a possibility that is manifestly ruled out by the left hand side of (7). Rewriting (8) by utilization of our relations (6) the (nonzero) term $A_{n-(k+1)}$ cancels and we obtain

$$0 = -(x_{n-(k+1)})_{nn}a^2 + a^2_{n-(k+1)} - + \cdots - (-1)^{n-k}x_{n-(k+2)}\cdots x_1x_0a_0^2.$$

By shifting $a^2_{n-(k+1)}$ to the left hand side we recognize that the a, a_j provide a nontrivial solution of (5) with $k+1$ in lieu of k. We have thus shown that the equations (5) have nontrivial solutions for all $k \le n$. In particular, letting $k = n$ there is a nontrivial solution X_0, X of the equation $X_0^2 = (x_0)_{nn}X^2$, i.e., $X_0^2 = x_{nn}X^2$. But this is a contradiction. Therefore (1) has no nontrivial solution when $\mathrm{char}(K) \neq 2$.

Finally, if $\mathrm{char}(K) = 2$ we have to discuss the equation $\sum_{i=0}^n \xi_i^2 x_{ii} = N_0^2$ instead of (1). Multiplication by a common denominator allows us to assume the ξ_i and N_0 to be polynomials with coefficients in K. Each x_{ii} shows up once on the left hand side with an odd exponent. As the right hand side is a square, we must have $\xi_i = 0$ for all i. Q.E.D.

3. The setup (Definition of the form Φ in Theorem 2)

In this section K is a commutative field of any characteristic and with $\alpha :=$ card(K) uncountable. Let K_0 be the prime field of K and $\mathcal{Y}$ a transcendence basis of K/K_0. Thus, card($\mathcal{Y}$) $= \alpha$ and K is an algebraic extension of $K_0(\mathcal{Y})$.

For β a fixed uncountable cardinal $\leq \alpha$ we select some arbitrary injective family $\chi = (x_{\iota\kappa})_{0\leq\iota<\kappa<\beta}$ with $x_{\iota\kappa} \in \mathcal{Y}$.

Let $(\mathbf{e}_\iota)_{\iota<\beta}$ be the basis of some fixed β-dimensional K-vector space E. We look at symmetric forms (skew-symmetric forms) Φ: $E \times E \to K$ that have

$$\Phi(\mathbf{e}_\iota, \mathbf{e}_\kappa) = x_{\iota\kappa}, \qquad \text{for each } \iota, \kappa : 0 \leq \iota < \kappa < \beta. \tag{9}$$

Notice that, in contrast to the form Ψ of Lemma 4, (9) does not require the diagonal coefficients $\Phi(\mathbf{e}_\iota, \mathbf{e}_\iota)$ of Φ to belong to $\mathcal{Y}$.

Notation. If $K' \supseteq K$ is an arbitrary commutative overfield then $E' := K' \otimes_K E$, regarded as a K'-vector space, and Φ' is the form $E' \times E' \to K'$ with $\Phi'(\sum_i \lambda_i \otimes \mathbf{x}_i, \sum_j \mu_j \otimes \mathbf{y}_j) = \sum_{ij} \lambda_i \Phi(\mathbf{x}_i, \mathbf{y}_j)\mu_j$ for all λ_i, $\mu_j \in K'$ and all $\mathbf{x}_i$, $\mathbf{y}_j \in E$ ("K'-ification").

The following facts are proved in [5]:

Proposition 5. *Let Φ: $E \times E \to K$ be a symmetric or skew-symmetric form with (9). If $K' \supseteq K$ is an arbitrary commutative overfield then the K'-ification (E', Φ') enjoys the following properties:*

(i) If F is any linear subspace in E' and $\dim F \geq \aleph_0$ then $\dim F^\perp \leq \aleph_0$ in E'.

(ii) The full group $\mathcal{O}(E')$ of isometries Ψ: $E' \to E'$ consists of all finite products $\pm\Omega_1 \circ \Omega_2 \circ \cdots$ where the Ω_i are symmetries about non-degenerate hyperplanes of E'. In other words, for each isometry $\varphi \in \mathcal{O}(E')$ either $\mathrm{Ker}(\varphi - \mathbf{1})$ *or* $\mathrm{Ker}(\varphi + 1)$ *is of finite codimension in E'.*

In order to prove (i) quote [5, Theorem 1.1, p. 514]; in order to prove (ii) quote [5, Theorem 2.3 (with n:=1), Theorem 2.2, p. 516 and Theorem 1.1, p. 514].

4. Proof of Theorem 2

Let $E = K(\mathbf{e}_\iota)_{\iota<\beta}$ and the form Φ be as in Proposition 5. Pick some subset $J \subseteq \beta$ with uncountable $\gamma :=$ card(J) and set $E_J := \mathrm{span}\{\mathbf{e}_\iota | \iota \in J\} \subseteq E$. The next lemma shows that any isometric injection

$$\varphi \colon E_J \to E \tag{10}$$

can move E_J as a subspace of E^n only by a little bit:

Lemma 6. *If (1) is an isometric injection then there exists a finite subset $M \subseteq \beta$ such that for each $\iota \in J$ we have $\varphi \mathbf{e}_\iota \in Ke_\iota + F$ where $F := \mathrm{span}\{\mathbf{e}_\iota | \iota \in M\} \subseteq E$. In particular $\dim(E_J + \varphi E_J)/E_J \leq \dim F < \infty$.*

Proof. For all $\iota \in J$ we set $\varphi\mathbf{e}_\iota = \sum \alpha_{\iota\mu}\mathbf{e}_\mu \in E$; the summation extends over the finite sets $M(\iota) := \{\mu \in \beta \mid \alpha_{\iota\mu} \neq 0\}$. Let $M = \cup_{\iota\in J}[M(\iota) \setminus \{\iota\}]$. We show that M is finite. Assume by way of contradiction that M is infinite. There is a denumerably infinite subset $S \subset J$ and a map κ that assigns to every $\iota \in S$ a $\kappa(\iota) \in \beta$ with $\kappa(\iota) \in M(\iota) \setminus \{\iota\}$ and $\kappa(\iota) \neq \kappa(\nu)$ for all $\iota \neq \nu$ in S. There is a countable subset $A \subseteq \mathcal{Y}$ (the transcendence basis of K/K_0 as introduced at the beginning of the previous section) such that $\alpha_{\iota\kappa} \in \overline{K_0(A)}$ (the relative algebraic closure of $K_0(A)$ in K) for all $\kappa \in M(\iota)$, $\iota \in S$. Let $N = \cup_{\iota\in S}M(\iota)$, $\operatorname{card} N = \aleph_0$. By the pigeonholing principle there is a $\nu \in J$ and for it a $\mu_0 \in M(\nu)$ such that $\mu_0 \notin N \cup S$ and

$$\{x_{\iota\mu_0} \in \chi \mid \iota \in N\} \cap A = \emptyset. \tag{11}$$

For $\varphi\mathbf{e}_\nu = \sum_{\mu\in M(\nu)} \alpha_{\nu\mu}\mathbf{e}_\mu$ and $\iota \in S$ we have (φ being an isometry) that

$$\Phi(\varphi\mathbf{e}_\iota, \varphi\mathbf{e}_\nu) = x_{\iota\nu} = \alpha_{\iota\kappa(\iota)}\alpha_{\nu\mu_0}x_{\kappa(\iota)\mu_0} + \sum_{\kappa\mu} \alpha_{\iota\kappa}\alpha_{\nu\mu}x_{\kappa\mu}. \tag{12}$$

The sum in (12) extends over the set $[M(\iota) \times M(\nu)] \setminus \{(\kappa(\iota), \mu_0)\}$. There is a finite subset $B \subseteq \mathcal{Y}$ such that $\alpha_{\nu\mu} \in \overline{K_0(B)}$ for all $\mu \in M(\nu)$.

Since S is infinite, there is a $\sigma \in S$ with $x_{\kappa(\sigma)\mu_0} \notin B$. As $\kappa(\sigma) \neq \sigma$ by the choice of the map κ and since $\mu_0 \neq \sigma$ (because $\mu_0 \notin N \cup S$) we have $x_{\sigma\nu} \neq x_{\kappa(\sigma)\mu_0}$. Let

$$C = A \cup B \cup \{x_{\sigma\nu}, x_{\kappa\mu} \mid (\kappa, \mu) \in [M(\sigma) \times M(\nu)] \setminus \{(\kappa(\sigma), \mu_0)\}\}.$$

By (11) we have $x_{\kappa(\sigma)\mu_0} \notin A$, hence $x_{\kappa(\sigma)\mu_0} \notin C$. All quantities in equation (12) equated for $\iota = \sigma$ are contained in $\overline{K_0(C)}$ with the exception of $x_{\kappa(\sigma)\mu_0}$. The coefficient of $x_{\kappa(\sigma)\mu_0}$ in (12) is not zero. Hence we should have $x_{\kappa(\sigma)\mu_0} \in \overline{K_0(C)}$; so $x_{\kappa(\sigma)\mu_0}$ is algebraically dependent over C which is a contradiction. We have thus shown that M is finite. Q.E.D.

Proof of Theorem 2. In the power set $\mathcal{P}(\beta)$ we select a system $\mathcal{S} \subseteq \mathcal{P}(\beta)$ such that $\operatorname{card}(\mathcal{S}) = 2^\gamma$ and, for each J, $J' \in \mathcal{S}$, $\operatorname{card}(J) = \gamma = \operatorname{card}(J')$ and $\operatorname{card}((J \setminus J') \cup (J' \setminus J)) \geq \aleph_0$. Of the 2^γ subspaces $E_J (J \in \mathcal{S})$ no two can be isometric by Lemma 6.

5. Proof of Theorem 3

In this section the commutative field is a transcendental extension K/K_0 of transcendence degree $\alpha > \aleph_0$. Let $(\mathbf{e}_\iota)_{\iota<\gamma}$ be the basis of some fixed γ-dimensional K-vector space E and $\aleph_0 < \gamma \leq \alpha$. Let $\Phi: E \times E \to K$ be a *symmetric* form such that

$$\{\Phi(\mathbf{e}_\iota, \mathbf{e}_\kappa) \in K \mid 0 \leq \iota \leq \kappa < \gamma\} \text{ is algebraically independent over } K_0. \tag{13}$$

Thus, in contrast to Sections 3 and 4 the diagonal elements $\Phi(\mathbf{e}_\iota, \mathbf{e}_\iota)$ of the form are taken into consideration here. From Lemma 4 we obtain:

(14) If Φ enjoys (13) then Φ is anisotropic, i.e., if $\Phi(\mathbf{x},\mathbf{x}) = 0$ then $\mathbf{x} = \mathbf{0}$.

Lemma 7. *If Φ enjoys (13) and if* $\operatorname{char}(K) \neq 2$ *and if -1 is a square in K then Witt's cancellation theorem holds:*

(15) *If $E = A \oplus A^\perp = B \oplus B^\perp$ and $A \simeq B$ (isometry) then $A^\perp \simeq B^\perp$.*

Proof. If $E = A + A^\perp$ then by Proposition 5(i) either $\dim A$ or $\dim A^\perp$ is finite. Of course, if $\dim A$ is finite then (15) follows by the finite dimensional cancellation theorem. We are left with the case that $\dim A^\perp$ is finite. We first convince ourselves that $\dim A^\perp = \dim B^\perp$. Assume by way of contradiction that $\dim B^\perp > \dim A^\perp$. So there is a nonzero $a \in B^\perp \cap A$. Fix an isometry $\varphi_0\colon A \xrightarrow{\sim} B$. Since $\varphi_0 a \in B$ we get a nonzero isotropic vector $\lambda a + \varphi_0 a$ if $\lambda^2 = -1$. But this contradicts (14) whence we must have $\dim A^\perp = \dim B^\perp$. Let K' be the algebraic closure of K. We pass to the K'-ifications E', A', B', $(A')^\perp = (A^\perp)'$, $(B')^\perp = (B^\perp)'$. As $E' = A' \oplus A'^\perp = B' \oplus B'^\perp$ and $\dim_{K'}(A'^\perp) = \dim_K(A^\perp) = \dim_K(B^\perp) = \dim_{K'}(B'^\perp)$ it is clear that $A'^\perp$ and $B'^\perp$ are isometric and that therefore φ_0 induces an isometry $\varphi_0'\colon A' \xrightarrow{\sim} B'$ that admits an extension φ' to all of E'. Of course, the subset E of the K'-space E' need not be invariant under the metric automorphism φ' of E'.

By Proposition 5(ii) we may assume that $L := \operatorname{Ker}(\varphi' - \mathbf{1})$ is of finite codimension in E'. (We may replace φ' by $-\varphi'$ if necessary.)

Since φ' and $\mathbf{1}$ are both $\sigma(\Phi')$-continuous (that is "weakly ortho-continuous", cf. [3, p. 33f]) L is $\sigma(\Phi)$-closed in E'. Because E is non-degenerate (by(14)) so is E'; furthermore $\dim E'/L < \infty$; therefore $\dim(L \cap L^\perp) < \infty$. Ergo, there is a non-degenerate $L_0 \subseteq L$ with $L_0^{\perp\perp} = L_0$ and with finite $\dim L/L_0$ (use a Witt decomposition for $L \cap L^\perp$ in E'). It follows that

(16) $$E' = L_0 \oplus L_0^\perp, \qquad \dim L_0^\perp < \infty, \qquad L_0 \subseteq \operatorname{Ker}(\varphi' - \mathbf{1}).$$

We can find a finite dimensional non-degenerate subspace $P \subseteq E$ such that $P' := K' \otimes P \supseteq L_0^\perp$. Consequently, $E' = P'^\perp \oplus P'$ and $E = (P^\perp \cap E) \oplus P$, $P^\perp \subseteq \operatorname{Ker}(\varphi' - \mathbf{1})$. Set $Q := P^\perp \cap A$. Again $Q^{\perp\perp} = Q$ in E (being the intersection of two ortho-closed subspaces of E). Furthermore $\dim Q \cap Q^\perp < \infty$ because E is non-degenerate and $\dim E/Q \leq \dim P + \dim E/A < \infty$. Thus, once more, we can find a subspace $Q_0 \subseteq Q$ with

(17) $$E = Q_0 \oplus Q_0^\perp \text{ and } Q_0 \subseteq \operatorname{Ker}(\varphi_0 - \mathbf{1}),\ \dim Q_0^\perp < \infty.$$

Now we readily see that φ_0 also admits of an extension to the space E: Since $Q_0 \subseteq A$, $Q_0 = \varphi_0 Q_0 \subseteq B$ we obtain

$$Q_0 \oplus (Q_0^\perp \cap B) = B = \varphi_0 A = \varphi_0\big(Q_0 \oplus (Q_0^\perp \cap A)\big) = Q_0 \oplus \varphi_0(Q_0^\perp \cap A).$$

Therefore $\varphi_0(Q_0^\perp \cap A) = Q_0^\perp \cap B$ and the restriction $\varphi_0|Q_0^\perp \cap A$ can be extended to all of $Q_0^\perp$ by the classical finite dimensional case of Witt's theorem.

The following is a simple but nonetheless rather interesting observation (cf. Theorem 9 below):

Lemma 8. *We assume that* $\mathrm{char}(K) \neq 2$. *If an anisotropic space* (E, Φ) *satisfies Witt's extension theorem (i.e., each isometry* $\varphi_0\colon F \tilde{\to} G$ *between subspaces* F, $G \subseteq E$ *extends to a metric automorphism* φ *of* E*) then* (E, Φ) *is orthomodular, i.e., all* $\perp$*-closed subspaces of* E *are splitting:*

$$\text{for each } X \subseteq E\colon X^{\perp\perp} \oplus X^\perp = E. \tag{18}$$

Proof. For X a fixed subspace of E let φ_0 be the metric automorphism of the subspace $F := X^{\perp\perp} \oplus X^\perp$ with $\varphi_0|X^{\perp\perp} = \mathbf{1}$ and $\varphi_0|X^\perp = -\mathbf{1}$. Let φ be the extension of φ_0 to E. If $\mathbf{z} \in E$, $\mathbf{x}' \in X^\perp$, $\mathbf{x}'' \in X^{\perp\perp}$ are typical elements we have

$$\Phi(\mathbf{z}, \mathbf{x}') = \Phi(\varphi\mathbf{z}, \varphi\mathbf{x}') = \Phi(\varphi\mathbf{z}, -\mathbf{x}'),$$

so $\mathbf{z} + \varphi\mathbf{z} \in X^{\perp\perp}$ and

$$\Phi(\mathbf{z}, \mathbf{x}'') = \Phi(\varphi\mathbf{z}, \varphi\mathbf{x}'') = \Phi(\varphi\mathbf{z}, \mathbf{x}''),$$

so $\mathbf{z} - \varphi\mathbf{z} \in X^\perp$. Addition yields $2\mathbf{z} \in X^{\perp\perp} + X^\perp$ hence $E \subseteq X^{\perp\perp} + X^\perp$ as $\mathbf{z}$ was arbitrary.

Proof of Theorem 3. In Section 4 choose a symmetric form Φ that has $\{\Phi(e_\iota, e_\kappa) \mid 0 \leq \iota \leq \kappa < \beta\}$ algebraically independent over K_0 (instead of merely satisfying (9)). The spaces E_J selected in the proof of Theorem 2 are then all of the kind described in Lemma 7. Now as to the validity of the extension theorem our argument is as follows. Recall that by Proposition 5(i) all splitting subspaces $X \subseteq E_J$ have one among $\dim X$, $\dim X^\perp$ finite. Assume by way of contradiction that the extension theorem holds. By Lemma 8 *all* $\perp$-closed subspaces X ($X = X^{\perp\perp}$) of E_J are splitting, hence $\dim X$ or $\dim X^\perp$ is finite for each $\perp$-closed subspace X of E_J. Therefore, the lattice $\mathcal{L}_{\perp\perp}(E_J)$ of all $\perp$-closed subspaces in E_J is modular; indeed the sum of two $\perp$-closed subspaces is again $\perp$-closed since $U + V$ is $\perp$-closed if U is $\perp$-closed and V is finite dimensional. Ergo $\dim E_J$ is finite by a classical theorem of H.A. Keller [7]. Contradiction! We have shown that none of the spaces E_J satisfy Witt's extension theorem.

The proof of Theorem 3 is complete.

6. Remarks on orthomodularity, cancellation and the extension theorem

The interrelation between orthomodular spaces and Witt's extension theorem is much stronger than is apparent from Lemma 8. To delve into this matter here would leave us too far astray. However, it is appropriate to mention the following result of ours:

Theorem 9. *Assume that* (E, Φ) *is an infinite dimensional, non-degenerate* ε*-hermitean sesquilinear space over an involutorial division ring* K *of arbitrary characteristic. (We make no assumptions on separability, i.e., on the existence of* $\aleph_0$*-dimensional subspaces* $F \subseteq E$ *with* $F^{\perp\perp} = E$*). If Witt's extension theorem holds in* (E, Φ) *then* Φ *is anisotropic. Furthermore, if the subgroup* $\varepsilon := \{\xi \in \text{center}(K) | 0 \neq \xi \text{ and } \xi\xi^* = 1\}$ *of* K *is not the trivial group* $\{1\}$ *(thus, in particular, when* $\text{char}(K) \neq 2$*), then the validity of the extension theorem for* (E, Φ) *implies that* (E, Φ) *is orthomodular, i.e. satisfies* (18).

For several years we had been inclined to believe that there are no infinite dimensional spaces (E, Φ) in which Witt's extension theorem holds unconditionally. But then I succeeded in giving a construction of such spaces, see [4, Remark following Theorem 4]. Of course, such spaces are rather scarce in view of Theorem 9 as orthomodular spaces are not exceedingly abundant. Nevertheless it may be difficult to *prove* that a specific space violates the extension theorem.

Here we have shown, among other things, that in sharp contrast to the finite dimensional case, the cancellation theorem does not imply the extension theorem.

In [8] the cancellation property was investigated; we have come across a gap in the existence proof for infinite dimensional spaces with cancellation, a gap that seems difficult to close; for this reason I have given rather detailed proofs here.

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