Annales Academiæ Scientiarum Fennicæ
Series A. I. Mathematica
Volumen 15, 1990, 293–306

ON CONFORMAL WELDING HOMEOMORPHISMS ASSOCIATED TO JORDAN CURVES

Y. Katznelson, Subhashis Nag, and Dennis P. Sullivan

Abstract

For any Jordan curve C on the Riemann sphere, the conformal welding homeomorphism, Weld(C), of the circle or real line onto itself is obtained by comparing the boundary values of the Riemann mappings of the unit disk or upper half-plane onto the two complementary Jordan regions separated by C.

In Part I we show that a family of infinitely spiralling curves lie intermediate between smooth curves and curves with corners, in the sense that the welding for the spirals has a non-differentiable but Lipschitz singularity. We also show how the behaviour of Riemann mapping germs under analytic continuation implies that spirals were the appropriate curves to produce such weldings. A formula relating explicitly the rate of spiralling to the extent of non-differentiablity is proved by two methods, and related to known theorems for chord-arc curves.

Part II studies families of Jordan curves possessing the same welding. We utilise curves with positive area and the Ahlfors–Bers generalised Riemann mapping theorem to build an infinite dimensional "Teichmüller space" of curves all having the same welding homeomorphism. A point of view on welding as a problem of extension of conformal structure is also derived in this part.

Introduction

In this paper we address two questions regarding the conformal welding homeomorphism, Weld(C), associated to any Jordan curve C on the Riemann sphere $\hat{\mathbf{C}}$. Weld(C) is the homeomorphism of the real line on itself obtained by comparing the boundary values of the two Riemann mappings of the upper half-plane U onto the two complementary Jordan regions D_1 and D_2 separated by C. (We assume that C is an oriented Jordan curve, so that D_1 is chosen to be the region to the left of C.)

Precisely, let $f_i\colon U \to D_i$, $i = 1,2$, denote any two Riemann mappings onto the relevant domains. By Caratheodory's theorem one knows that f_1 and f_2 extend continuously to the boundaries. Let $\partial f_i\colon \partial U = \mathbf{R} \cup \{\infty\} \to C$, $i = 1,2$,

Classification: 30C20, 30C35.—Keywords: Conformal mapping, boundary values, Jordan curves.

doi:10.5186/aasfm.1990.1517

denote these two boundary homeomorphisms. Then $\mathrm{Weld}(C)$ is the orientation reversing self-homeomorphism of the circle $(\mathbf{R} \cup \{\infty\})$ given by

$$\mathrm{Weld}(C) = \partial f_2^{-1} \circ \partial f_1. \tag{1}$$

Note that reversal of orientation on C replaces the welding by its inverse.

Since f_1 and f_2 are ambiguous up to pre-composition by arbitrary elements of $\mathrm{Aut}(U) = \mathrm{PSL}_2(\mathbf{R})$, we note that $\mathrm{Weld}(C)$ as defined is ambiguous up to arbitrary pre and post compositions by elements of $\mathrm{PSL}_2(\mathbf{R})$. Moreover, if the curve C is replaced by its image under any Möbius automorphism of $\hat{\mathbf{C}}$, the corresponding welding homeomorphism class remains precisely the same. Thus, one should think of the welding as a map:

$$\begin{aligned} \mathrm{Weld}\colon &\{\text{Möbius equivalence classes of Jordan curves on } \hat{\mathbf{C}}\} \\ &\to \mathrm{PSL}_2(\mathbf{R}) \setminus \mathrm{Homeo}(S^1) \,/\, \mathrm{PSL}_2(\mathbf{R}). \end{aligned} \tag{2}$$

By normalising the choice of f_1 and f_2 we may assume that $\mathrm{Weld}(C)$ fixes each of the points 0, 1, ∞. No further normalisations are then possible using pre and post compositions by $\mathrm{PSL}_2(\mathbf{R})$ transformations. It is also sometimes convenient to make $\mathrm{Weld}(C)$ orientation preserving by taking f_2 to be a Riemann mapping from the lower half-plane L onto D_2. We will assume, without loss of generality, that C passes through ∞, and that $f_1(\infty) = \infty$, $f_2(\infty) = \infty$.

The conformal welding arises rather ubiquitously in various contexts in geometric function theory. As is well-known, two standard ways of viewing the universal Teichmüller space $T(1)$ are:

$$T(1) = \{\text{Jordan curves that are quasicircles}\} \,/\, \mathrm{Mob}(\hat{\mathbf{C}}) \tag{3}$$

and

$$T(1) = \{\text{Quasisymmetric homeomorphisms of } \mathbf{R} \text{ fixing } 0, 1, \infty\}. \tag{4}$$

The passage between these two descriptions of the complex Banach manifold $T(1)$ is precisely via conformal welding. In particular note that $\mathrm{Weld}(C)$ for C a quasicircle is quasisymmetric, and this correspondence is a nice bijective one.

One may also note that the homeomorphism $\mathrm{Weld}(C)$, (and its barycentric extension), was utilised critically for arbitrary Jordan curves C in the paper [E-N].

With the foregoing background we are ready to explain our work in this article. Inside the universal Teichmüller space $T(1)$ one may conveniently distinguish certain special subfamilies of quasicircles. First of all there are the smooth (C^∞) Jordan curves—whose corresponding welding homeomorphisms are C^∞ diffeomorphisms. Because, by classical results of S. Warshawski, the Riemann mapping to a Jordan domain with C^∞ boundary extends C^∞ to the boundary.

In contrast, suppose the curve C has a "corner" of positive angle $\alpha\pi$, $(0 < \alpha < 1)$, at some point. Weld(C) will then have a "power law" behaviour ($t \mapsto t^{(2-\alpha)/\alpha}$, near $t = 0$) at the corresponding point ($t = 0$ say). Thus Weld(C) will have vanishing or infinite derivate there. A natural inquiry therefore is to seek Jordan curves which are in a sense *intermediate* between the C^∞ ones and the ones with corners by requiring the welding homeomorphisms to have non-differentiable but Lipschitz behaviour at some point(s).

(5) *Table.*

Curve	*Welding*
C^∞	C^∞ diffeomorphism
?	Lipschitz
Corner	Power law

In Part I of this paper we will fill in the ? slot in the above table by exhibiting a large family of *infinitely spiralling Jordan curves* for which the welding homeomorphism, at the point corresponding to the eye of the spirals, has the requisite behaviour. The derivatives from the left and right of Weld(C) will exist, be finite and non-zero, but are unequal. Moreover, we find an explicit formula (equation (8)) relating the rate of spiralling parameter to the ratio of the left and right derivatives. We deduce that the discrepancy between the left and right derivatives will decrease down to zero as the rate of spiralling increases to infinity.

This last assertion ties in intimately with certain general results of David et al. regarding the welding homeomorphisms of "chord-arc" curves with small chord-arc constants. The "chord-arc" curves are a well-known subfamily of quasicircles, and our spirals fall within this subfamily.

In Section I.2 we develop a criterion for Weld(C) to possess a non-differentiable but Lipschitz point in terms of *analytic continuation properties of the Riemann mappings involved.* This leads to a rather surprising explanation, from a very different point of view, of why the spirals were the right Jordan curves for producing the type of Weld(C) desired. The explicit formula (8) mentioned above is reproved by the new method.

In marked contrast to the previous part, Part II of this paper looks at Jordan curves quite outside the realm of quasicircles. We consider the question of non-injectivity of the welding correspondence (2).

We reinterpret welding as a problem of extension of conformal structure. This gives us a criterion for a homeomorphism to be a welding, and also for there to be only one curve from which it arises as welding homeomorphism.

One then observes that there exist Jordan curves C with positive two-dimensional Lebesgue measure. The next idea is to *use the "μ-trick"* (i.e. the Ahlfors–Bers generalized Riemann mapping theorem), *to construct a large family of Jordan curves possessing the same welding homeomorphism as C.*

It is natural to consider then a "Teichmüller space of C"—comprising the various quasiconformal deformations of C with dilatation supported on C. We conclude by conjecturing that the Jordan curves sharing the given welding homeomorphism are precisely the members of this Teichmüller space.

Acknowledgements. T. Wolff and S.C. Bagchi are thanked for useful conversations. The second author would like to heartily thank the Institut des Hautes Etudes Scientifiques for their kind hospitality during Spring 1989, when much of this work was done.

Part I. Welding homeomorphisms for spiralling curves

1. A main result

We start with a simple but crucial observation on the "locality" of the welding correspondence. Suppose we wish to concentrate at a point z_0 on the curve C. The nature of Weld(C) at the corresponding point $t_0 = \partial f_1^{-1}(z_0)$ is then at issue. The point of the following Lemma is that although the Riemann mappings are global constructs, the smoothness/non-smoothness of Weld(C) at t_0 are only dependent on the local nature of C around z_0.

Lemma I.1. *Locally the welding homeomorphism is determined, up to pre and post composition by real analytic diffeomorphisms of real intervals, by an arbitrarily small neighbourhood of the curve.*

Proof. Consider two different conformal mappings, f and g, of a standard half-disk D onto a topological half-disk B bounded by a portion of the curve around the relevant point $z_0 \in C$.

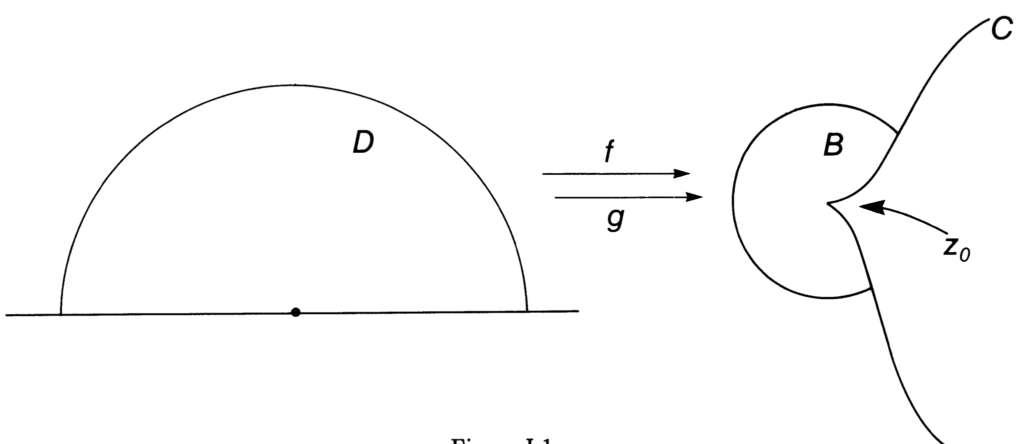

Figure I.1.

The self-map $g^{-1} \circ f$ on D will extend conformally, by the Schwarz reflection principle, throughout a full disk containing the real interval ∂D. It follows that the welding $\tilde{w}$ determined by looking at any small portion of C arounf z_0 is obtained from the actual welding w, (determined by the global Riemann mappings), via the relation:

$$\tilde{w} = \varrho_2 \circ w \circ \varrho_1 \tag{6}$$

where ϱ_1 and ϱ_2 are two real-analytic diffeomorphisms (i.e., real-analytic coordinate changes) of the relevant segments on the real axis. □

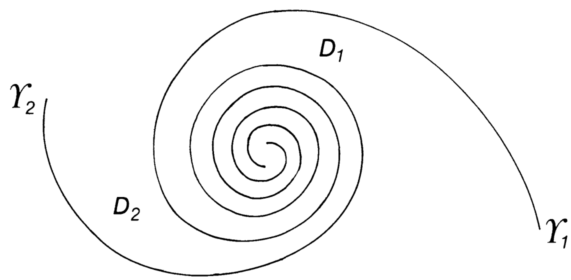

Figure I.2.

Consider now the Jordan curves C_a, $a > 0$, given by $C_a = \gamma_1 \cup \{0\} \cup \gamma_2$, where in polar coordinates:

$$\gamma_1 : r = e^{-a\theta}, \qquad \gamma_2 : r = -e^{-a\theta}, \qquad \theta \to +\infty \tag{7}$$

so each C_a is a disjoint union of two logarithmic spirals that join up at the vortex point $z = 0$ (Figure I.2). This point is our focus of interest—we will call it the "eye" of these spiralling curves. The quantity a, $0 < a < \infty$, is the "rate of spiralling" parameter.

Theorem I.2. *For each* a, $0 < a < \infty$, *the homeomorphism* Weld(C_a), *in a neighbourhood of the point (say* t_0*) corresponding to the eye of* C_a, *is Lipschitz but not differentiable at* t_0. *The left derivative (*λ_1, *say) and the right derivative (*λ_2, *say) for* Weld(C_a) *both exist, are finite and non-vanishing at* t_0, *but do not match.*

In fact, the rate of spiralling parameter can be related to the ratio λ_1/λ_2 *explicitly:*

$$a = \left| \frac{2\pi}{\log(\lambda_1/\lambda_2)} \right|. \tag{8}$$

Thus, the discrepancy between the derivatives from the opposite sides decreases monotonically to zero as the rate of spiralling blows up to ∞.

Remark. Weld(C_a) is, of course, diffeomorphic everywhere except at t_0 since C_a is a C^∞ curve away from the eye. The last statement asserts that as the rate of spiralling becomes larger and larger the curve, or more precisely its welding, looks more and more smooth even at the eye. This assertion will be ramified in two further ways in following sections.

Proof. The proof is by a direct computation keeping in mind the locality lemma. Consider the entire function

$$G(z) = e^{-(a+i)z}. \tag{9}$$

G maps the positive x-axis and certain parallel horizontal half-lines onto the spirals γ_1 and γ_2.

In fact, let R denote the half-strip region below and above the horizontal half-lines $y = 0$ and $y = a\pi/(1+a^2)$, and bounded to the left by the line segment joining the origin to

$$\lambda = -\frac{\pi}{1+a^2} + i\frac{a\pi}{1+a^2}.$$

Then the maps $G_1 = G$ and $G_2 = -G$ restricted to R are conformal mappings onto relevant portions of the complementary Jordan domains D_1 and D_2. The boundary point "$\infty + iy$" of R corresponds to the eye of C_a.

Now, the welding for C_a, looked at on the boundary of R, becomes

$$z \mapsto \begin{cases} z+\lambda, & \text{on } x\text{-axis}, \\ z-\lambda, & \text{on } y = a\pi/(1+a^2). \end{cases}$$

Making a conformal change of variables, (using a holomorphic logarithm), we can replace the domain R by the upper half-plane and get the critical boundary point "$\infty + iy$" to correspond to the real number $t_0 = 1$. Normalising so that Weld(C_a) maps t_0 to itself, we get finally the following expression for the welding:

$$t \overset{\text{Weld}(C_0)}{\longmapsto} \begin{cases} \left((\alpha\sqrt{t}-\beta)/(\beta\sqrt{t}-\alpha)\right)^2 & \text{for } t \geq 1, \\ \left((\alpha\sqrt{t}+\beta)/(\beta\sqrt{t}+\alpha)\right)^2 & \text{for } t \leq 1 \end{cases} \tag{10}$$

in a neighbourhood of $t_0 = 1$ on the real line. Here $\beta > 1 > \alpha > 0$ are determined from the spiralling parameter by the relations $\beta = 1 + e^{-\pi/a}$, $\alpha = 1 - e^{-\pi/a}$.

[This expression (10) for Weld(C_a) is up to real-analytic changes of coordinates, as explained in Lemma I.1.]

Now, calculating the derivatives of (10) from the left and right of $t = 1$, one obtains $(\alpha-\beta)/(\alpha+\beta) = -e^{-\pi/a}$ from the left, and its reciprocal, $-e^{\pi/a}$, from the right. But equation (6) in the proof of Lemma I.1 shows that the *ratio* λ_1/λ_2 does *not* depend on making real-analytic coordinate changes. Consequently one gets $\lambda_1/\lambda_2 = e^{-2\pi/a}$,—and equation (8) follows. □

Part I.2. Lipschitz welding and analytic continuation of Riemann mappings

There is an interesting and distinctly different way to explain why the spirals were the right coordinates to produce Lipschitz welding homeomorphisms, utilising an idea about analytic continuation of Riemann map germs. Even formula (8) emerges naturally. We explain this method below.

Assume that Weld(C): $\mathbf{R} \to \mathbf{R}$ has a graph with a corner at zero—say it is $x \mapsto \lambda_1 x$ for small negative values of x and $x \mapsto \lambda_2 x$ for small positive values of x. (Namely, λ_1 and λ_2 are the derivatives of Weld(C) form the left and the right at 0. Remember that we are interested only in the local considerations.)

Let $f_1\colon U \to D_1$ and $f_2\colon L \to D_2$ be Riemann mappings onto the complementary regions separated by C, with $f_1(0) = f_2(0) =$ the point of interest (p) on C. We assume that the arc of the curve C on each side of p is real-analytic up to, but not including, the point p. Thus the curve C in a neighbourhood of p looks like $\gamma_1 \cup \{p\} \cup \gamma_2$ where γ_1 and γ_2 are two open real-analytic arcs meeting at p.

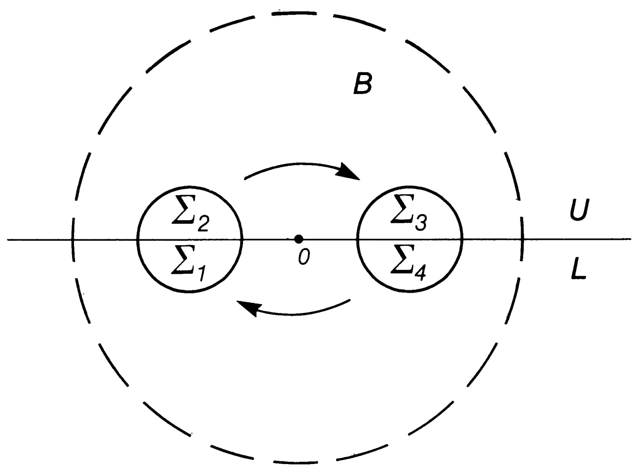

Figure I.3.

Consider the restriction of the conformal map f_2 on a small half-disk Σ_1 (in the lower half-plane) located infinitesimally to the left of the origin (see Figure I.3).

The idea is to use the stipulated nature of the welding homeomorphism to obtain a relation between f_2 restricted to Σ_1 and its analytic continuation in a circuit around the origin. In what follows 'A' will be generic notation to denote analytic continuation along some path.

Since f_2 maps the real segment on the boundary of Σ_1 to a piece of the real-analytic arc γ_1, one knows that f_2 must extend conformally to the full disk $\Sigma_2 \cup \partial\Sigma_1 \cup \Sigma_1$ (see, for example, Nehari [N, p. 186]). Comparing this extended

f_2, call it Af_2 (on Σ_2) with the Riemann map f_1 on Σ_2 we see that

$$f_1^{-1} \circ Af_2 = K_1, \qquad \text{on } \Sigma_2 \tag{11}$$

where K_1 is some conformal map on Σ_2. We assume that K_1 is analytic in a ball B around 0. Then the power series for K_1 must be

$$K_1(z) = \frac{1}{\lambda_1} z + O(z^2), \qquad \text{around} \quad z = 0, \tag{12}$$

because (11), together with the stipulated nature of Weld(C) on the left of zero, imply that the derivative of K_1 at 0 is λ_1^{-1}.

Thus the analytic continuation of $f_2|_{\Sigma_1}$ up to the region Σ_3 will produce the function $f_1 \circ K_1$. Now, again by an argument as above, $f_1 \circ K_1$ extends analytically to the disk $\Sigma_3 \cup \Sigma_4$, and we may compare this extension, call it $A(f_1 \circ K_1) = Af_1 \circ K_1$ with f_2 itself on Σ_4. We get

$$f_2^{-1} \circ Af_1 \circ K_1 = K_2, \qquad \text{on } \Sigma_4 \tag{13}$$

where K_2 has a power series expansion

$$K_2(z) = \frac{\lambda_2}{\lambda_1} z + O(z^2), \qquad \text{around} \quad z = 0. \tag{14}$$

This follows because Weld(C) $= \partial f_2^{-1} \circ \partial f_1$ has slope λ_2 to the right of the origin.

Thus, $f_2|_{\Sigma_2}$, analytically continued in a clockwise circle around the origin, ends up as $f_2 \circ K_2$. We will neglect terms $O(z^2)$ in the Riemann mappings, since we are interested only in local structure around 0. *We therefore see that if the welding has slopes λ_1 and λ_2 from the left and right of zero respectively, then the germ of the Riemann map f_2 restricted to a region infinitesimally to the left of zero will satisfy the funtional equation:*

$$Af_2(z) = f_2\left(\frac{\lambda_2}{\lambda_1} z\right) \tag{15}$$

where A denotes analytic continuation along a clockwise circuit encircling zero.

Notice that the analytic functions in L satisfying (15) clearly form an *algebra*. The countable family of functions:

$$\varphi_k(z) = \exp(\beta_k \log z), \tag{16}$$

with

$$\beta_k = \frac{2k\pi i}{\log(\lambda_1/\lambda_2) - 2i\pi}, \qquad k = 0, \pm 1, \pm 2, \ldots \tag{17}$$

where $\log z$ is any holomorphic log in the lower half-plane, *all satisfy* (15), (easy to verify).

Moreover, these φ_k linearly span a subalgebra of the algebra satisfying (15). (This is because $\varphi_j \varphi_k = \varphi_{j+k}$, since the β_k constitute an additive subgroup of $\mathbf{C}$.)

To see therefore what kind of curve C will produce the type of welding prescribed, we only need to *calculate the images of small portions of the positive real axis* ($\mathbf{R}^+$) *and of the negative real axis* ($\mathbf{R}^-$) *under any non-trivial (i.e.* $k \neq 0$*) one of these functions* φ_k*. Indeed one obtains two logarithmic spirals with the same rate of spiralling from the two sides joining up at their common eye (the point 'p' on* C*).* We indicate the calculation below.

Setting $\lambda_1/\lambda_2 = \lambda$, one gets from (17)

$$\beta_k = \Big(\frac{-4k\pi^2}{\log^2 \lambda + 4\pi^2}\Big) + i\Big(\frac{2k\pi \log \lambda}{\log^2 \lambda + 4\pi^2}\Big),$$

so that

$$\frac{\operatorname{Re} \beta_k}{\operatorname{Im} \beta_k} = -\frac{2\pi}{\log(\lambda_1/\lambda_2)} \tag{18}$$

is independent of k. Now the portion γ_1 of C is $\varphi_k(t)$ for $t < 0$, $t \to 0-$ and the portion γ_2 of C is $\varphi_k(t)$ for $t > 0$, $t \to 0+$. These turn out to be spirals, and the sign of k has to be controlled so that the eye of these spirals as $t \to 0$ is not at ∞. (This depends on the sign of $\operatorname{Im} \beta_k$.) In polar coordinates one obtains the curves γ_1 and γ_2 to be of the form (assuming $\lambda_1 > \lambda_2$):

$$\gamma_1 : r = A_1 e^{-a\theta}, \qquad \gamma_2 : r = A_2 e^{-a\theta}, \qquad \theta \to +\infty. \tag{19}$$

Here

$$a = \Big|\frac{2\pi}{\log(\lambda_1/\lambda_2)}\Big| \tag{20}$$

comes out independent of the choice of k. A_1 and A_2 are unequal real constants.

Note that (19) *and* (20) *tally exactly with* (7) *and* (8) *in the previous section.*

Remark. If $\lambda_1 < \lambda_2$ one would find the spirals going in clockwise to their common eye, i.e. as $\theta \to -\infty$. The equations for the spirals are essentially the same as (19) (with a replaced by $-a$). By reflecting the curve C, and changing the orientation on C if necessary, one can get the spirals to match the form (19) or (7) exactly again. (Reflecting C replaces the welding $h(t)$ by $-h(-t)$; reversing orientation replaces h by h^{-1}.)

Somewhat more generally, it is easy to show that even if one works with a linear combination φ of various φ_k's, then one still obtains infinitely spiralling curves as the image under under φ of $\mathbf{R}^+$ and $\mathbf{R}^-$. This is because the term in φ with one particular β_k will eventually dominate as $t \to 0$, and consequently the image curve will look asymptotically exactly like the logarithmic spirals (19) obtained from that dominating φ_k. (To be precise, we call a curve infinitely spiralling around an "eye" p on it, if the argument function on the curve (measured with p as origin) becomes unbounded in every neighbourhood of p.)

Our results of the preceding section are thus exactly corroborated, and somewhat generalised, by the present analysis. Note that the rate of spiralling always satisfies (20), *(i.e.* (8) *of Theorem* I.2*), and depends only on the ratio of slopes of* Weld(C) *from the two sides of the 'singular' point.*

Remark. The functional equation (15) for the approximate Riemann mapping makes contact with J. Ecalle's theory of "resurgent functions".

Part I.3. Chord-arc spirals and their weldings

The spiral family we have been dealing with are "chord-arc" curves, and we want to indicate a relationship between our results above and known facts about the welding homeomorphism for chord-arc curves.

Recall that a Jordan curve (normalised to go through ∞) is called a *chord-arc curve* if it is a rectifiable curve $z(s)$ (s is an arc-length parameter), such that there exists $K \geq 0$ so that

$$|z(s_1) - z(s_2)| \geq \frac{1}{1+K}|s_1 - s_2|, \tag{21}$$

for all s_1 and s_2 real; see Semmes [S].

Condition (21) implies Ahlfors' well-known condition for being a quasi-circle. In fact, chord-arc curves are the images of $\mathbf{R}$ under bi-Lipschitz homeomorphisms of the plane, where quasicircles are the images of $\mathbf{R}$ under the more general quasiconformal homeomorphisms of the plane.

Our double-spiralling curves C_a of Theorem I.2 are chord-arc curves. This is best seen by remembering (Semmes, loc. cit.) that *chord-arc curves are in one-one correpondence with a certain open region Ω in the real Banach space* BMO($\mathbf{R}$) (real valued functions on $\mathbf{R}$ of bounded mean oscillation). The connection is that given $b \in \Omega$ one associates the chord-arc curve such that

$$\arg(z_b'(s)) = b(s). \tag{22}$$

For the choice

$$b(s) = -\frac{1}{a}\log|s|, \qquad -\infty < s < \infty, \tag{22}$$

one obtains precisely the double log-spiral C_a (up to a global rotation and magnification). *Thus, the subfamily of quasicircles comprising the chord-arc curves is a real Banach manifold Ω containing the one-parameter family of curves C_a.*

The welding homeomorphism h for any chord-arc curve has derivative a.e. and $\log(h')$ is known to lie again in $\mathrm{BMO}(\mathbf{R})$. (This is implied by the fact that h' itself is known to lie in the class A_∞ of weights of Muckenhoupt; see [S] and [JK].) One thus has, from the welding correspondence (2), a well-defined map

$$W\colon \Omega \to \mathrm{BMO}(\mathbf{R}) \tag{24}$$

by associating to any $b \in \Omega$ the BMO function $\log(\mathrm{Weld}'(z_b))$.

It is known that W maps a small neighbourhoold of the origin into a small neighbourhood of the origin. Indeed, the curves z_b arising from small BMO functions b, (which are precisely the chord-arc curves with small chord-arc constant K of (21)), have been shown (David [D]) to be characterized by having small BMO norm for $\log(h')$. See [S], [D] and work of Coifman and Meyer.

This ties in exactly with what we established in the preceding two sections about the spirals with very large rate of spiralling a. Indeed the BMO norm of b in equation (23) and the corresponding K for the spiral, is small precisely when a is very large. But then $\log(h')$ has a jump discontinuity at 0 of jump size $|\lambda_1 - \lambda_2|$, where λ_1 and λ_2 are as usual the derivatives from the left and right of 0 of $\mathrm{Weld}(C_a)$. The BMO norm of a function with a jump discontinuity is easily estimated (locally around the jump) as approximately half the jump size. Consequently, David's result that $\log(h')$ should have small BMO norm for large rate of spiralling fits precisely with the last assertion of Theorem I.2—which is what equations (8) and (20) implied.

Remark. If one knew that the map W in (24) was continuous (in the BMO norms) then one could generalise the result of Theorem I.2 to an infinite-dimensional family of chord-arc spirals. In fact then all chord-arc curves close to the family C_a would possess weldings that are (in BMO norm) close to the weldings for the C_a. Such BMO functions, which are close to functions with a jump discontinuity, have also a characteristic kind of discontinuity. Unfortunately, the map W is not known to be continuous (except at the origin, as we noted).

Remark. In concluding Part I we would like to recall that spiralling curves figured prominently in the well-known paper of Gehring [G].

Part II. Jordan curves sharing the same welding

We study here the non-injectivity of the welding correspondence (2). To handle the inverse map to (2) we consider the following point of view.

Given any (orientation preserving) homeomorphism $h\colon S^1 \to S^1$ we create a 2-sphere S^2_h by joining two copies of the closed unit disk along the boundary circle—the attaching map used on the boundary being h. More precisely, let us identify the circle S^1 as $\mathbf{R} \cup \{\infty\} = \hat{\mathbf{R}}$ on the sphere $\hat{\mathbf{C}}$. Then S^2_h is the identification space

$$S^2_h = (U \cup \hat{\mathbf{R}}) \cup_h (L \cup \hat{\mathbf{R}}) \tag{25}$$

where U and L are the upper and lower half-planes respectively. Notice that S^2_h is a topological 2-sphere with a well-defined conformal structure on the complement of the circle $\hat{\mathbf{R}}$. The following proposition can be seen straight from the definitions.

Proposition II.1. *The homeomorphism* $h\colon \hat{\mathbf{R}} \to \hat{\mathbf{R}}$ *occurs as a welding homeomorphism for some Jordan curve* C *if and only if* S^2_h *carries a conformal structure that extends the standard conformal structure already existing on the open regions* U *and* L. *For any such conformal structure on* S^2_h *(supposing at least one exists) the Riemann surface* S^2_h *becomes conformally equivalent to* $\hat{\mathbf{C}}$. *The image of* $\hat{\mathbf{R}}(\subset S^2_h)$ *under any such biholomorphic equivalence is a Jordan curve* C *whose welding is* h.

Corollary II.2. *The Jordan curve* C *on* $\hat{\mathbf{C}}$ *is the unique one (up to Möbius transformations) that will produce the welding* $h = \mathrm{Weld}(C)$ *if and only if any homeomorphism of* $\hat{\mathbf{C}}$ *which is conformal on the complement of* C *is necessarily conformal on all of* $\hat{\mathbf{C}}$, *i.e., is a Möbius transformation.*

Proof. Suppose C_1 and C_2 are two Jordan curves producing the same welding h. Let φ_1 and φ_2 be two conformal equivalences of S^2_h (with possibly different conformal structures) onto $\hat{\mathbf{C}}$—as in the above proposition; φ_i throws $\hat{\mathbf{R}}$ onto C_i, $i = 1, 2$. Then $\sigma = \varphi_2 \circ \varphi_1^{-1}$ is a self-homeomorphism of the Riemann sphere $\hat{\mathbf{C}}$ which maps C_1 onto C_2 and which is conformal on the complement of C_1. The corollary follows. □

Remark. If C is as in the above corollary we call C a removable set for conformal mapping. In the analogous sense one can also talk of Jordan curves that are removable sets for quasiconformal mappings. This concept is relevant to what follows, and is important in some further work.

Having noted the criterion for injectivity of (2) we now show how to apply the Ahlfors–Bers generalised Riemann mapping theorem in a simple but fruitful fashion to construct many inequivalent curves sharing the same welding. Since on $T(1)$ the welding correspondence is bijective, we are working outside the realm of quasicircles.

A classical fact. There exists Jordan curves possessing positive two-dimensional Lebesgue measure. Indeed, there exist Jordan curves on $\mathbf{R}^2$ that contain the set $A \times A$ where A is a Cantor set in $[0, 1]$ of linear measure arbitrarily close to 1.

References. See Gelbaum and Olmsted [GO], pp. 135–138. For the original paper see Osgood [O].

The idea is now to take a Jordan curve γ of positive area as above and consider Beltrami coefficients supported on γ. Namely, look at the complex Banach space $L^\infty(\gamma)$ and consider it isometrically embedded in $L^\infty(\mathbf{C})$ by extending functions to the whole plane by setting them identically zero off γ. Let $L^\infty(\gamma)_1 \hookrightarrow L^\infty(\mathbf{C})_1$ denote the open unit balls.

By Ahlfors–Bers [AB], for every μ in $L^\infty(\mathbf{C})_1$ there is a unique quasiconformal homeomorphism of $\mathbf{C}$ solving the Beltrami equation:

$$\bar{\partial} w = \mu \cdot \partial w \tag{26}$$

and fixing the points $0,\ 1,\ \infty$. Call the solution $w = w^\mu$.

Theorem II.3. *For any $\mu \in L^\infty(\gamma)_1$ the Jordan curve γ^μ which is the image of γ under w^μ possesses the same welding homeomorphism as γ itself. If γ has positive areal measure $L^\infty(\gamma)_1$ is infinite dimensional, and consequently one has many Jordan curves with the same welding* Weld(γ).

Remark. If γ has zero two-dimensional measure then w^μ is Möbius and the theorem is trivially true.

Proof. w^μ is conformal on the complement of γ. So if f_1 and f_2 were two Riemann mappings U onto the two regions D_1 and D_2 separated by γ, then $\tilde{f}_1 = w^\mu \circ f_1$ and $\tilde{f}_2 = w^\mu \circ f_2$ will be Riemann mappings of U onto the respective regions separated by γ^μ. It follows that $\partial \tilde{f}_2^{-1} \circ \partial \tilde{f}_1 = \partial f_2^{-1} \circ \partial f_1$. □

In analogy with the usual definitions of Teichmüller theory, it appears natural to define a "Teichmüller space" for the curve γ as $L^\infty(\gamma)_1$ modulo the equivalence relation $\sim$, where $\mu \sim \nu$ if and only if $w^\mu = w^\nu$ on γ. But the equivalence relation is trivial in this situation:

Proposition II.4. *We have $L^\infty(\gamma)_1 /\!\sim\, \equiv L^\infty(\gamma)_1$.*

Proof. If $\mu \sim \nu$ consider $F = (w^\nu)^{-1} \circ w^\mu$. F is a quasiconformal homeomorphism of $\mathbf{C}$ which has dilatation zero away from γ since μ and ν are both supported on γ. But F is given to be the identity on γ itself—hence its dilatation on γ is also zero. Hence F is identity map, so $w^\mu \equiv w^\nu$ on $\mathbf{C}$, and so $\mu \equiv \nu$. □

We end by questioning whether for any Jordan curve γ, all the curves that share the same welding homeomorphism with γ are just the members of this "Teichmüller space" $L^\infty(\gamma)_1$. In particular this would imply that the welding determines uniquely the curve when the curve has zero area. In the case of the well-known quasicircles $\longleftrightarrow$ quasisymmetric correspondence, recall that the process of recovering the quasicircle from its welding is also via a "μ-trick". See Vainio [V] for related work.

References

[AB] AHLFORS, L., and L. BERS: Riemann's mapping theorem for variable metrics. - Ann. of Math. (2) 72, 1960, 385–404.

[D] DAVID, G.: Courbes corde-arc et espaces de Hardy generalises. - Ann. Inst. Fourier (Grenoble) 32:3, 1982, 227–239.

[EN] EARLE, C.J., and S. NAG: Conformally natural reflections in Jordan curves with applications to Teichmüller spaces. - In Holomorphic functions and moduli II, edited by D. Drasin et al., Berkeley Mathematical Sciences Research Institute Conference. Springer-Verlag, New York, 1988, 179–194.

[G] GEHRING, F.W.: Spirals and the universal Teichmüller space. - Acta Math. 141, 1978, 99–113.

[GO] GELBAUM, B.R., and J.M.J. OLMSTED: Counterexamples in analysis. - Holden-Day, Inc., San Francisco, 1964.

[JK] JERISON, D., and C. KENIG: Hardy spaces A_∞, and singular integrals on chord-arc domains. - Math. Scand. 50, 1982, 221–247.

[N] NEHARI, Z.: Conformal mapping. - McGraw-Hill, New York, 1952.

[O] OSGOOD, W.: A Jordan curve of positive area. - Trans. Amer. Math. Soc. 4, 1903, 107–112.

[S] SEMMES, S.W.: The Cauchy integral, chord-arc curves and quasiconformal mappings. - In Bieberbach conjecture symposium volume, edited by A. Baernstein et al., American Mathematical Society, Providence, R.I., 1986, 167–183.

[V] VAINIO, J.V.: Conditions for the possibility of conformal sewing. - Ann. Acad. Sci. Fenn. Ser. A I Math. Dissertationes 53, 1985.

Y. Katznelson
Stanford University
Department of Mathematics
Stanford, CA
U.S.A.
and
I.H.E.S.
Bures-sur-Yvette
France

Subhashis Nag
Indian Statistical Institute
Mathematics Unit
Bangalore-560 059
India
and
I.H.E.S.
Bures-sur-Yvette
France

Dennis P. Sullivan
City University of New York
Graduate Center
New York, NY
U.S.A.
and
I.H.E.S.
Bures-sur-Yvette
France

Received 2 November 1989