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A ČECH TYPE CONSTRUCTION FOR EQUIVARIANT COHOMOLOGY

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Introduction

A classical result of Dowker [D] states that non-equivariant Alexander–Spanier cohomology is isomorphic to Čech cohomology on arbitrary spaces. In [H] we constructed an equivariant version of Alexander–Spanier cohomology, defined for all G-pairs, G a finite group. The purpose of this paper is to give a suitable equivariant generalization of Čech cohomology, and then to prove the appropriate generalization of the above mentioned result of Dowker.

The contents of the paper are as follows: In Section 1 we present the construction of the equivariant Čech cohomology groups and state the main result about the isomorphism between equivariant Alexander–Spanier and Čech cohomology. The proof of this result generalizes the method outlined in [S], exercises 6 D 1–3, for proving the theorem of Dowker. Our proof occupies Sections 2 and 3 below: in Section 2 we show that it is enough to find chain homotopy equivalences between certain chain complexes of contravariant coefficient systems, and in Section 3 we carry out the construction of such homotopy equivalences by the method of acyclic models. In this paper we have to use a slightly different definition of equivariant Alexander–Spanier cohomology than the one we used in [H]. We ponder this difference in an appendix.

1. The construction and main result

Let G be a finite group; this hypothesis holds in all that follows. As in [B], let $\mathcal{O}_G$ denote the category of the canonical G-orbits G/H, $H \leq G$, and $\mathcal{C}_G$ the category of contravariant coefficient systems, i.e. contravariant functors $\mathcal{O}_G \to Ab$.

Let X be a G-space and $A \subset X$ a G-subspace; we use the convention that the definition of a G-space includes the Hausdorff condition. In the following, a *G-covering* of X means an open covering $\mathcal{U} = \{U_x \mid x \in X\}$ of X satisfying

$$x \in U_x \quad \text{for each } x \in X, \qquad gU_x = U_{gx} \quad \text{for each } x \in X,\ g \in G$$

(here we follow an idea of Godement, cf. [G], p. 223). Given such a $\mathcal{U}$, we form the G-covering $\mathcal{U}' = \{U_x \cap A \mid x \in A\}$ of A.

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Let $\mathcal{U}$ be a G-covering of X, as above. For each $n \in \mathbf{N}$ we define a coefficient system $\underline{C}_n(\mathcal{U}) \in \mathcal{C}_G$ by

$$\underline{C}_n(\mathcal{U})\colon G/H \mapsto \text{ free abelian group with basis } \\ \{(x_0,\ldots,x_n) \in (X^H)^{n+1} \mid U_{x_0} \cap \cdots \cap U_{x_n} \cap X^H \neq \emptyset\}$$

for $H \leq G$. $\underline{C}_n(\mathcal{U})$ has obvious values on morphisms of $\mathcal{O}_G$. Namely, a morphism $\varphi\colon G/K \to G/H$ has the form $gK \mapsto gaH$, where $a^{-1}Ka \leq H$, and we set $\underline{C}_n(\mathcal{U})(\varphi)\colon (x_0,\ldots,x_n) \mapsto (ax_0,\ldots,ax_n)$; this is well-defined, because

$$\begin{aligned} U_{ax_0} \cap \cdots \cap U_{ax_n} \cap X^K &\supset U_{ax_0} \cap \cdots \cap U_{ax_n} \cap X^{aHa^{-1}} \\ &= a \cdot (U_{x_0} \cap \cdots \cap U_{x_n} \cap X^H) \end{aligned}$$

is non-empty, if $U_{x_0} \cap \cdots \cap U_{x_n} \cap X^H \neq \emptyset$. There are obvious boundary morphisms $\partial\colon \underline{C}_n(\mathcal{U}) \to \underline{C}_{n-1}(\mathcal{U})$ in $\mathcal{C}_G$ $(n \geq 1)$ determined by $(x_0,\ldots,x_n) \mapsto \sum_{i=0}^n (-1)^i (x_0,\ldots,\hat{x}_i,\ldots,x_n)$, and hence $\underline{C}_*(\mathcal{U})$ is a chain complex in $\mathcal{C}_G$.

The preceding construction applied to the G-covering $\mathcal{U}'$ of A yields the chain complex $\underline{C}_*(\mathcal{U}')$. Because $\underline{C}_*(\mathcal{U}')(G/H) \subset \underline{C}_*(\mathcal{U})(G/H)$ for each $H \leq G$, there is canonical monomorphism $\underline{C}_*(\mathcal{U}') \hookrightarrow \underline{C}_*(\mathcal{U})$. We define a chain complex $\underline{C}_*(\mathcal{U},\mathcal{U}')$ in $\mathcal{C}_G$ by

$$\underline{C}_*(\mathcal{U},\mathcal{U}') = \operatorname{coker}[\underline{C}_*(\mathcal{U}') \hookrightarrow \underline{C}_*(\mathcal{U})].$$

We call a G-covering $\mathcal{V}$ of X a *refinement* of $\mathcal{U}$ provided that $V_x \subset U_x$ for each $x \in X$. If $\mathcal{V}$ is a refinement of $\mathcal{U}$, there is a canonical monomorphism $\underline{C}_*(\mathcal{V},\mathcal{V}') \to \underline{C}_*(\mathcal{U},\mathcal{U}')$, which induces a cochain map $\operatorname{Hom}_{\mathcal{C}_G}(\underline{C}_*(\mathcal{U},\mathcal{U}'),m) \to \operatorname{Hom}_{\mathcal{C}_G}(\underline{C}_*(\mathcal{V},\mathcal{V}'),m)$ for any $m \in \mathcal{C}_G$ (here it is essential that we use only G-coverings of the particular kind specified above).

Definition 1.1. Let $m \in \mathcal{C}_G$ be a contravariant coefficient system. The *equivariant Čech cochain complex* of (X,A) with coefficients m is

$$\check{C}^*_G(X,A;m) = \varinjlim_{\mathcal{U}} \operatorname{Hom}_{\mathcal{C}_G}(\underline{C}_*(\mathcal{U},\mathcal{U}'),m).$$

The *equivariant Čech cohomology groups* of (X,A) with coefficients m are

$$\check{H}^n_G(X,A;m) = H^n(\check{C}^*_G(X,A;m)), \qquad n \in \mathbf{N}.$$

Next we give a description of the equivariant Alexander–Spanier cohomology of (X,A), which differs slightly from the definition given in [H]. If $\mathcal{U}$ is a G-covering of X, we can define chain complexes $\underline{D}_*(\mathcal{U})$, $\underline{D}_*(\mathcal{U}')$ and $\underline{D}_*(\mathcal{U},\mathcal{U}')$ in $\mathcal{C}_G$ such that

$$\underline{D}_n(\mathcal{U})\colon G/H \mapsto \text{ free abelian group with basis } \\ \{(y_0,\ldots,y_n) \in (X^H)^{n+1} \mid \{y_0,\ldots,y_n\} \subset U_x \text{ for some } x \in X^H\},$$

$$\underline{D}_*(\mathcal{U},\mathcal{U}') = \operatorname{coker}\left[\underline{D}_*(\mathcal{U}') \hookrightarrow \underline{D}_*(\mathcal{U})\right].$$

Proposition 1.2. *For any coeffcient system $m \in \mathcal{C}_G$,*

$$\varinjlim_{\mathcal{U}} \operatorname{Hom}_{\mathcal{C}_G}\left(\underline{D}_*(\mathcal{U},\mathcal{U}'), m\right)$$

is isomorphic to the equivariant Alexander–Spanier cochain complex $\overline{C}^*_G(X, A; m)$ as defined in [H].

Proof. Cf. the appendix. □

The main result of this paper is the following theorem which will be proved in Sections 2 and 3, as promised in the Introduction:

Theorem 1.3. *There is a natural isomorphism $\check{H}^*_G(X, A; m) \cong \overline{H}^*_G(X, A; m)$ between the equivariant Čech and Alexander–Spanier cohomology of (X, A) with arbitrary coefficients $m \in \mathcal{C}_G$.*

2. Subdivision

Let (X, A) be a G-pair and $\mathcal{U} = \{U_x \mid x \in X\}$ a G-covering of X, as above. If $H \leq G$, we denote $\mathcal{U}_H = \{U_x \cap X^H \mid x \in X^H\}$, an open covering of X^H. Let $K(\mathcal{U}_H)$ be the nerve of $\mathcal{U}_H$, i.e., $K(\mathcal{U}_H)$ is a simplicial complex, a simplex of $K(\mathcal{U}_H)$ being a finite subset $\{x_0, \dots, x_n\} \subset X^H$ satisfying $U_{x_0} \cap \dots \cap U_{x_n} \cap X^H \neq \emptyset$. Also, let $L(\mathcal{U}_H)$ be the simplicial complex, whose simplexes are those finite subsets $\{y_0, \dots, y_m\} \subset X^H$ which satisfy $\{y_0, \dots, y_m\} \subset U_x$ for some $x \in X^H$. We remark that a G-map $G/H_1 \to G/H_2$ induces obvious simplicial maps $K(\mathcal{U}_{H_2}) \to K(\mathcal{U}_{H_1})$ and $L(\mathcal{U}_{H_2}) \to L(\mathcal{U}_{H_1})$. In this notation we have

$$\begin{aligned}
\underline{C}_*(\mathcal{U})&: G/H \mapsto C_*\big(K(\mathcal{U}_H)\big)\\
\underline{C}_*(\mathcal{U}')&: G/H \mapsto C_*\big(K(\mathcal{U}'_H)\big)\\
\underline{C}_*(\mathcal{U},\mathcal{U}')&: G/H \mapsto C_*\big(K(\mathcal{U}_H), K(\mathcal{U}'_H)\big)\\
\underline{D}_*(\mathcal{U})&: G/H \mapsto C_*\big(L(\mathcal{U}_H)\big)\\
\underline{D}_*(\mathcal{U}')&: G/H \mapsto C_*\big(L(\mathcal{U}'_H)\big)\\
\underline{D}_*(\mathcal{U},\mathcal{U}')&: G/H \mapsto C_*\big(L(\mathcal{U}_H), L(\mathcal{U}'_H)\big)
\end{aligned} \tag{2.1}$$

for any $H \leq G$, where $C_*(K)$ means the ordered chain complex of the simplicial complex K.

We recall briefly the definition of the barycentric subdivision $\operatorname{Sd} K$ of an abstract simplicial complex K. The vertexes of $\operatorname{Sd} K$ are the simplexes of K, and a finite set of simplexes of K is a simplex of $\operatorname{Sd} K$ if it is linearly ordered by inclusion. Recall also that there is a natural subdivision chain map Sd: $C_*(K) \to C_*(\operatorname{Sd} K)$, which is a chain homotopy equivalence for any K (cf. [E-St], VI 8).

We write $K'(\mathcal{U}_H) = \operatorname{Sd} K(\mathcal{U}_H)$, $L'(\mathcal{U}_H) = \operatorname{Sd} L(\mathcal{U}_H)$ etc. $(H \leq G)$. We can now define the following new chain complexes of $\mathcal{C}_G$:

$$
\begin{aligned}
\underline{C}'_*(\mathcal{U})\colon G/H &\mapsto C_*\big(K'(\mathcal{U}_H)\big)\\
\underline{C}'_*(\mathcal{U}')\colon G/H &\mapsto C_*\big(K'(\mathcal{U}'_H)\big)\\
\underline{C}'_*(\mathcal{U},\mathcal{U}')\colon G/H &\mapsto C_*\big(K'(\mathcal{U}_H), K'(\mathcal{U}'_H)\big)\\
\underline{D}'_*(\mathcal{U})\colon G/H &\mapsto C_*\big(L'(\mathcal{U}_H)\big)\\
\underline{D}'_*(\mathcal{U}')\colon G/H &\mapsto C_*\big(L'(\mathcal{U}'_H)\big)\\
\underline{D}'_*(\mathcal{U},\mathcal{U}')\colon G/H &\mapsto C_*\big(L'(\mathcal{U}_H), L'(\mathcal{U}'_H)\big)
\end{aligned}
\tag{2.2}
$$

with obvious values on morphisms of $\mathcal{O}_G$. The above mentioned subdivision chain maps, being natural, define the following chain maps in $\mathcal{C}_G$: $\operatorname{sd} : \underline{C}_*(\mathcal{U}) \to \underline{C}'_*(\mathcal{U})$, $\operatorname{sd} : \underline{C}_*(\mathcal{U}') \to \underline{C}'_*(\mathcal{U}')$, $\operatorname{sd} : \underline{C}_*(\mathcal{U},\mathcal{U}') \to \underline{C}'_*(\mathcal{U},\mathcal{U}')$, and similarly with $\underline{C}_*$ replaced by $\underline{D}_*$.

If $\mathcal{U}$ is a refinement of a G-covering $\mathcal{V} = \{V_x \mid x \in X\}$, there are evident chain maps $\underline{C}'_*(\mathcal{U},\mathcal{U}') \to \underline{C}'_*(\mathcal{V},\mathcal{V}')$ and $\underline{D}'_*(\mathcal{U},\mathcal{U}') \to \underline{D}'_*(\mathcal{V},\mathcal{V}')$.

Lemma 2.3. *The subdivision chain maps are compatible with refinement, i.e., if $\mathcal{U}$ is a refinement of $\mathcal{V}$, then the squares*

$$
\begin{array}{ccc}
\underline{C}_*(\mathcal{U},\mathcal{U}') & \xrightarrow{\operatorname{sd}} & \underline{C}'_*(\mathcal{U},\mathcal{U}')\\
\downarrow & & \downarrow\\
\underline{C}_*(\mathcal{V},\mathcal{V}') & \xrightarrow{\operatorname{sd}} & \underline{C}'_*(\mathcal{V},\mathcal{V}')
\end{array}
\qquad
\begin{array}{ccc}
\underline{D}_*(\mathcal{U},\mathcal{U}') & \xrightarrow{\operatorname{sd}} & \underline{D}'_*(\mathcal{U},\mathcal{U}')\\
\downarrow & & \downarrow\\
\underline{D}_*(\mathcal{V},\mathcal{V}') & \xrightarrow{\operatorname{sd}} & \underline{D}'_*(\mathcal{V},\mathcal{V}')
\end{array}
$$

commute.

Proof. Taking values at G/H $(H \leq G)$ we obtain two squares, where the vertical arrows are induced by certain simplicial maps $\big(K(\mathcal{U}_H), K(\mathcal{U}'_H)\big) \to \big(K(\mathcal{V}_H), K(\mathcal{V}'_H)\big)$ and $\big(L(\mathcal{U}_H), L(\mathcal{U}'_H)\big) \to \big(L(\mathcal{V}_H), L(\mathcal{V}'_H)\big)$. The commutativity thus follows from the naturality of sd with respect to simplicial maps. □

The next two results, to be proved in Section 3, are main ingredients in the proof of Theorem 1.3:

Proposition 2.4. *The chain maps*

$$\operatorname{sd}\colon \underline{C}_*(\mathcal{U},\mathcal{U}') \to \underline{C}'_*(\mathcal{U},\mathcal{U}')$$

and

$$\operatorname{sd}\colon \underline{D}_*(\mathcal{U},\mathcal{U}') \to \underline{D}'_*(\mathcal{U},\mathcal{U}')$$

are chain homotopy equivalences for any G-covering $\mathcal{U}$ of X.

Proposition 2.5. *For each G-covering $\mathcal{U}$ of X there is a chain homotopy equivalence α: $\underline{C}'_*(\mathcal{U},\mathcal{U}') \to \underline{D}'_*(\mathcal{U},\mathcal{U}')$ in $\mathcal{C}_G$. The equivalences α for various $\mathcal{U}$ can be chosen in such a way that the square*

$$\begin{array}{ccc} \underline{C}'_*(\mathcal{U},\mathcal{U}') & \xrightarrow{\alpha} & \underline{D}'_*(\mathcal{U},\mathcal{U}') \\ \downarrow & & \downarrow \\ \underline{C}'_*(\mathcal{V},\mathcal{V}') & \xrightarrow{\alpha} & \underline{D}'_*(\mathcal{V},\mathcal{V}') \end{array}$$

commutes up to chain homotopy, whenever $\mathcal{U}$ is a refinement of $\mathcal{V}$ (i.e. α is compatible with refinement up to chain homotopy).

The proof of Theorem 1.3 is immediate, once we have proved 2.4 and 2.5. Namely, 2.4 implies that, for $n \in \mathbf{N}$,

$$\check{H}^n_G(X,A;m) \cong \varinjlim_{\mathcal{U}} H^n\big(\mathrm{Hom}_{\mathcal{C}_G}(\underline{C}'_*(\mathcal{U},\mathcal{U}'),m)\big),$$
$$\bar{H}^n_G(X,A;m) \cong \varinjlim_{\mathcal{U}} H^n\big(\mathrm{Hom}_{\mathcal{C}_G}(\underline{D}'_*(\mathcal{U},\mathcal{U}'),m)\big)$$

and the chain homotopy equivalences α of 2.5 induce the required isomorphism between these groups.

3. Construction of the chain maps and homotopies

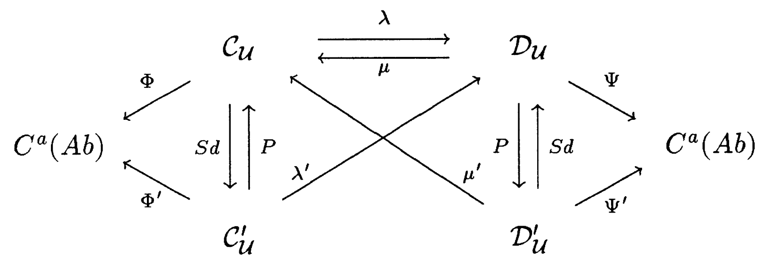
Let $\mathcal{U}$ be a covering of X as before. We define the following four categories:

	category	*objects*
	$\mathcal{C}_{\mathcal{U}}$	subcomplexes K of $K(\mathcal{U}_H)$ or $K(\mathcal{U}'_H)$, $H \le G$,
(3.1)	$\mathcal{C}'_{\mathcal{U}}$	subcomplexes K' of $K'(\mathcal{U}_H)$ or $K'(\mathcal{U}'_H)$, $H \le G$,
	$\mathcal{D}_{\mathcal{U}}$	subcomplexes L of $L(\mathcal{U}_H)$ or $L(\mathcal{U}'_H)$, $H \le G$,
	$\mathcal{C}'_{\mathcal{U}}$	subcomplexes L' of $L'(\mathcal{U}_H)$ or $L'(\mathcal{U}'_H)$, $H \le G$;

here subcomplexes of $K(\mathcal{U}_H)$ and $K(\mathcal{U}'_H)$, as well as those of $K(\mathcal{U}_{H_1})$ and $K(\mathcal{U}_{H_2})$ for $H_1 \neq H_2$, are considered distinct, and similarly in the other three cases. In each of the four categories, a morphism is a simplicial embedding induced by some $a \in G$ (for example, a morphism of $\mathcal{C}_{\mathcal{U}}$ is given by $x \mapsto ax$ on vertexes).

Let $C^a(Ab)$ be the category of augmented chain complexes of abelian groups. Next we define the functors appearing in the diagram below:

(3.2)



Each of the functors Φ, Φ', Ψ and Ψ' associates to a simplicial complex its ordered chain complex (for example, if $K \in \mathcal{C}_{\mathcal{U}}$, then $\Phi(K) = C_*(K)$). The functors Sd associate to a simplicial complex its barycentric subdivision.

(The alert reader may notice that for example the chain complex $\Phi(K)$ is augmented over $\mathbf{Z}$ only if $K \neq \emptyset$. It is, however, easy to see that this slight ambiguity does not affect the use of the acyclic model theorem in what follows.)

To define $P\colon \mathcal{C}'_{\mathcal{U}} \to \mathcal{C}_{\mathcal{U}}$, let s be a vertex of $K'(\mathcal{U}_H)$ (or $K'(\mathcal{U}'_H)$), $H \leq G$, i.e. s is a simplex of $K(\mathcal{U}_H)$ (or $K(\mathcal{U}'_H)$). We set $P(s) = \bar{s}$, the subcomplex of $K(\mathcal{U}_H)$ (or $K(\mathcal{U}'_H)$) consisting of the faces of s. If $K' \in \mathcal{C}'_{\mathcal{U}}$, i.e. K' is a subcomplex of $K'(\mathcal{U}_H)$ (or $K'(\mathcal{U}'_H)$), we let

$$P(K') = \bigcup \{ P(s) \mid s \in K' \text{ vertex} \} = \bigcup \{ \bar{s} \mid s \in K' \text{ vertex} \}.$$

The values of $P\colon \mathcal{D}'_{\mathcal{U}} \to \mathcal{D}_{\mathcal{U}}$ are defined similarly.

If $s = \{x_0, \ldots, x_n\}$ is again a simplex of $K(\mathcal{U}_H)$ (or $K(\mathcal{U}'_H)$), i.e. a vertex of $K'(\mathcal{U}_H)$ (or $K'(\mathcal{U}'_H)$), $H \leq G$, let $\lambda(s)$ be the subcomplex of $L(\mathcal{U}_H)$ (or $L(\mathcal{U}'_H)$) consisting of all simplexes $t \subset U_{x_0} \cap \cdots \cap U_{x_n}$. If $K \in \mathcal{C}_{\mathcal{U}}$ and $K' \in \mathcal{C}'_{\mathcal{U}}$, define

$$\lambda(K) = \bigcup \{ \lambda(x) \mid x \in K \text{ vertex} \},$$
$$\lambda'(K') = \bigcup \{ \lambda(s) \mid s \in K' \text{ vertex} \}.$$

This defines λ and λ' in 3.2.

If t is a simplex of $L(\mathcal{U}_H)$ (or $L(\mathcal{U}'_H)$), i.e. a vertex of $L'(\mathcal{U}_H)$ (or $L'(\mathcal{U}'_H)$), $H \leq G$, let $\mu(t)$ be the subcomplex of $K(\mathcal{U}_H)$ (or $K(\mathcal{U}'_H)$) consisting of all simplexes $\{x_0, \ldots, x_n\}$ such that $t \subset U_{x_0} \cap \cdots \cap U_{x_n}$. If $L \in \mathcal{D}_{\mathcal{U}}$ and $L' \in \mathcal{D}'_{\mathcal{U}}$, define

$$\mu(L) = \bigcup \{ \mu(y) \mid y \in L \text{ vertex} \},$$
$$\mu'(L') = \bigcup \{ \mu(t) \mid t \in L' \text{ vertex} \}.$$

This completes the construction of diagram 3.2.

Subdivision chain maps determine natural transformations

$$\text{sd}\colon \Phi \to \Phi' \circ \text{Sd}, \qquad \text{sd}\colon \Psi \to \Psi' \circ \text{Sd}. \tag{3.3}$$

Furthermore, because every vertex of a simplicial complex K is also a vertex of $\text{Sd}\, K$, we see that

$$\lambda' \circ \text{Sd} = \lambda, \qquad \mu' \circ \text{Sd} = \mu. \tag{3.4}$$

In the sequel we shall construct several natural transformations by aid of the acyclic model theorem, [S] 4.3.3. To this end, we fix some more notation. Let $\mathcal{M}_\mathcal{U} \subset \text{Ob}\,\mathcal{C}_\mathcal{U}$ be a set of representatives for the $\mathcal{C}_\mathcal{U}$-isomorphism classes of the complexes $\bar{s}$, s a simplex of $K(\mathcal{U}_H)$ or $K(\mathcal{U}'_H)$, $H \leq G$. Further, let $\mathcal{M}'_\mathcal{U} \subset \text{Ob}\,\mathcal{C}'_\mathcal{U}$ be a set of representatives for the $\mathcal{C}'_\mathcal{U}$-isomorphism classes of the complexes $\bar{\sigma}$, σ a simplex of $K'(\mathcal{U}_H)$ or $K'(\mathcal{U}'_H)$, $H \leq G$. Also, choose the sets $\mathcal{N}_\mathcal{U} \subset \text{Ob}\,\mathcal{D}_\mathcal{U}$ and $\nu'_\mathcal{U} \subset \text{Ob}\,\mathcal{D}'_\mathcal{U}$ analogously.

Proposition 3.5. *There are natural transformations* $p\colon \Phi' \to \Phi \circ P$ *and* $p\colon \Psi' \to \Psi \circ P$.

Proof. By the acyclic model theorem, it is enough to show that the functors Φ' and Ψ' are free with models $\mathcal{M}'_\mathcal{U}$ and $\mathcal{N}'_\mathcal{U}$, respectively, and the functors $\Phi \circ P$ and $\Psi \circ P$ are acyclic on the models $\mathcal{M}'_\mathcal{U}$ and $\mathcal{N}'_\mathcal{U}$, respectively.

The freeness of Φ' and Ψ' is clear by construction. To prove the acyclicity of $\Phi \circ P$, let $\bar{\sigma} \in \mathcal{M}'_\mathcal{U}$. If $s \in \sigma$ is the largest vertex of σ (with respect to inclusion), then $(\Phi \circ P)(\bar{\sigma}) = C_*(\bar{s}) \in C^a(Ab)$ which is, of course, acyclic. The acyclicity of $\Psi \circ P$ is proved similarly. □

We can form the following composite natural transformations:

$$\Phi \xrightarrow{\text{sd}} \Phi' \circ \text{Sd} \xrightarrow{(\text{Sd})^*(p)} \Phi \circ P \circ \text{Sd}$$

$$\Phi' \xrightarrow{p} \Phi \circ P \xrightarrow{P^*(\text{sd})} \Phi' \circ \text{Sd} \circ P$$

$$\Psi \xrightarrow{\text{sd}} \Psi' \circ \text{Sd} \xrightarrow{(\text{Sd})^*(p)} \Psi \circ P \circ \text{Sd}$$

$$\Psi' \xrightarrow{p} \Psi \circ P \xrightarrow{P^*(\text{sd})} \Psi' \circ \text{Sd} \circ P.$$

Further, if $K \in \mathcal{C}_\mathcal{U}$, then $K \subset P(\text{Sd}\, K)$, and this defines a natural transformation $I\colon \Phi \to \Phi \circ P \circ \text{Sd}$. In the same way we obtain natural transformations $I\colon \Phi' \to \Phi' \circ \text{Sd} \circ P$, $I\colon \Psi \to \Psi \circ P \circ \text{Sd}$ and $I\colon \Psi' \to \Psi' \circ \text{Sd} \circ P$.

Proposition 3.6. *There are natural chain homotopies*

$$(\text{Sd})^*(p) \circ \text{sd} \simeq I, \qquad P^*(\text{sd}) \circ p \simeq I.$$

Proof. For example, the assertion concerning the natural transformations between Φ and $\Phi \circ P \circ \text{Sd}$ follows from the trivial observation that Φ is free with models $\mathcal{M}_\mathcal{U}$ and $\Phi \circ P \circ \text{Sd}$ is acyclic on the same models. □

Proof of Proposition 2.4. We prove that the chain map sd: $\underline{C}_*(\mathcal{U},\mathcal{U}') \to \underline{C}'_*(\mathcal{U},\mathcal{U}')$ is a chain homotopy; the case of sd: $\underline{D}_*(\mathcal{U},\mathcal{U}') \to \underline{D}'_*(\mathcal{U},\mathcal{U}')$ is similar.

To construct a homotopy inverse p: $\underline{C}'_*(\mathcal{U},\mathcal{U}') \to \underline{C}_*(\mathcal{U},\mathcal{U}')$ of sd, we observe that the chain maps

$$\begin{array}{cccccc} p\colon & \Phi'(K'(\mathcal{U}_H)) & \longrightarrow & \Phi(P(K'(\mathcal{U}_H))) & = \Phi(K(\mathcal{U}_H)) \\ & \| & & & \| \\ & \underline{C}'_*(\mathcal{U})(G/H) & & & \underline{C}_*(\mathcal{U})(G/H) \end{array}$$

for $H \leq G$ are compatible with G-maps $G/H_1 \to G/H_2$ and hence determine a chain map p: $\underline{C}'_*(\mathcal{U}) \to \underline{C}_*(\mathcal{U})$. In the same way we get a chain map p: $\underline{C}'_*(\mathcal{U}') \to \underline{C}_*(\mathcal{U}')$, and the left hand square in the diagram

$$\begin{array}{ccccccccc} 0 \to & \underline{C}'_*(\mathcal{U}') & \longrightarrow & \underline{C}'_*(\mathcal{U}) & \longrightarrow & \underline{C}'_*(\mathcal{U},\mathcal{U}') & \to 0 \\ & \downarrow p & & \downarrow p & & & \\ 0 \to & \underline{C}_*(\mathcal{U}') & \longrightarrow & \underline{C}_*(\mathcal{U}) & \longrightarrow & \underline{C}_*(\mathcal{U},\mathcal{U}') & \to 0 \end{array}$$

commutes, because the horizontal arrows are induced by morphisms of $\mathcal{C}'_{\mathcal{U}}$. Thus the required p: $\underline{C}'_*(\mathcal{U},\mathcal{U}') \to \underline{C}_*(\mathcal{U},\mathcal{U}')$ exists. Proposition 3.6 then allows us to construct chain homotopies $p \circ \mathrm{sd} \simeq \mathrm{id}$ and $\mathrm{sd} \circ p \simeq \mathrm{id}$, proving that p is a homotopy inverse of sd. □

Proposition 3.7. *There are natural transformations* α: $\Phi' \to \Psi' \circ \mathrm{Sd} \circ \lambda'$ *and* β: $\Psi' \to \Phi' \circ \mathrm{Sd} \circ \mu'$.

Proof. Because Φ' and Ψ' are free with models $\mathcal{M}'_{\mathcal{U}}$ and $\mathcal{N}'_{\mathcal{U}}$, respectively, it is sufficient to show that $\Psi' \circ \mathrm{Sd} \circ \lambda'$ and $\Phi' \circ \mathrm{Sd} \circ \mu'$ are acyclic on the same models. We consider only $\Psi' \circ \mathrm{Sd} \circ \lambda'$.

Let σ be a simplex of $K'(\mathcal{U}_H)$, $H \leq G$ (the case $\sigma \in K'(\mathcal{U}'_H)$ is similar). Then $\lambda'(\bar{\sigma}) = \lambda(s)$, where s is the smallest vertex of σ. Write $s = \{x_0, \ldots, x_n\}$ and pick $z \in U_{x_0} \cap \cdots \cap U_{x_n} \cap X^H$. Because $\{y_0, \ldots, y_m\} \subset X^H$ is a simplex of $\lambda(s)$ if and only if $\{y_0, \ldots, y_m\} \subset U_{x_0} \cap \cdots \cap U_{x_n}$, the formula $(y_0, \ldots, y_m) \mapsto (z, y_0, \ldots, y_m)$ defines a chain contraction of $C_*(\lambda(s)) \in C^a(Ab)$. Because sd: $C_*(\lambda(s)) \to C_*(\mathrm{Sd}\,\lambda(s)) = (\Psi' \circ \mathrm{Sd} \circ \lambda')(\bar{\sigma})$ is a chain homotopy equivalence, $(\Psi' \circ \mathrm{Sd} \circ \lambda')(\bar{\sigma})$ is acyclic. □

Now we can form the following natural transformations:

$$\Phi' \xrightarrow{\alpha} \Psi' \circ \mathrm{Sd} \circ \lambda' \xrightarrow{(\mathrm{Sd} \circ \lambda')^*(\beta)} \Phi' \circ \mathrm{Sd} \circ \mu' \circ \mathrm{Sd} \circ \lambda' = \Phi' \circ \mathrm{Sd} \circ \mu \circ \lambda'$$

$$\Psi' \xrightarrow{\beta} \Phi' \circ \mathrm{Sd} \circ \mu' \xrightarrow{(\mathrm{Sd} \circ \mu')^*(\alpha)} \Psi' \circ \mathrm{Sd} \circ \lambda' \circ \mathrm{Sd} \circ \mu' = \Psi' \circ \mathrm{Sd} \circ \lambda \circ \mu'.$$

Further, let I: $\Phi' \to \Phi' \circ \mathrm{Sd} \circ \mu \circ \lambda'$ and I: $\Psi' \to \Psi' \circ \mathrm{Sd} \circ \lambda \circ \mu'$ be the natural transformations determined by the inclusions $K' \hookrightarrow \mathrm{Sd}(\mu(\lambda'(K')))$ and $L' \hookrightarrow \mathrm{Sd}(\lambda(\mu'(L')))$ for $K' \in \mathcal{C}'_{\mathcal{U}}$ and $L' \in \mathcal{D}'_{\mathcal{U}}$.

Proposition 3.8. *There are natural chain homotopies*

$$(\mathrm{Sd}\circ\lambda')^*(\beta)\circ\alpha\simeq I, \qquad (\mathrm{Sd}\circ\mu')^*(\alpha)\circ\beta\simeq I.$$

Proof. It is enough to show that the functors $\Phi'\circ\mathrm{Sd}\circ\mu\circ\lambda'$ and $\Psi'\circ\mathrm{Sd}\circ\lambda\circ\mu'$ are acyclic on the models $\mathcal{M}'_{\mathcal{U}}$ and $\mathcal{N}'_{\mathcal{U}}$, respectively. We consider only $\Phi'\circ\mathrm{Sd}\circ\mu\circ\lambda'$, and leave the (similar) other case to the reader.

Let σ be a simplex of $K'(\mathcal{U}_H)$, $H\leq G$, and $s=\{x_0,\dots,x_n\}$ the smallest vertex of σ. Then $\lambda'(\bar\sigma)=\lambda(s)$, as noted in the proof of 3.7. Now $t=\{z_0,\dots,z_q\}\subset X^H$ is a simplex of $\mu(\lambda(s))$, if and only if there is a vertex $y\in\lambda(s)$ (i.e. $y\in U_{x_0}\cap\dots\cap U_{x_n}\cap X^H$) such that $y\in U_{z_0}\cap\dots\cap U_{z_q}\cap X^H$; this is clearly equivalent to the condition $s\cup t\in K(\mathcal{U}_H)$ (i.e. $U_{x_0}\cap\dots\cap U_{x_n}\cap U_{z_0}\cap\dots\cap U_{z_q}\cap X^H\neq\emptyset$). Hence the formula $(z_0,\dots,z_q)\mapsto(x_0,z_0,\dots,z_q)$ defines a chain contraction of $C_*(\mu(\lambda(s)))\in C^a(Ab)$, and so $(\Phi'\circ\mathrm{Sd}\circ\mu\circ\lambda')(\bar\sigma)=C_*(\mathrm{Sd}(\mu(\lambda(s))))\simeq C_*(\mu(\lambda(s)))$ is acyclic. □

Assume now that $\mathcal{U}$ is a refinement of another G-covering $\mathcal{V}$ of X. If $H\leq G$, then every subcomplex K' of $K'(\mathcal{U}_H)$ (or $K'(\mathcal{U}'_H)$) can also be regarded as a subcomplex of $K'(\mathcal{V}_H)$ (or $K'(\mathcal{V}'_H)$). This defines a functor $j\colon\mathcal{C}'_{\mathcal{U}}\to\mathcal{C}'_{\mathcal{V}}$, and there is an analogous functor $j\colon\mathcal{D}'_{\mathcal{U}}\to\mathcal{D}'_{\mathcal{V}}$, too. There are also obvious natural transformations

$$\varrho\colon\Phi'_{\mathcal{U}}\to\Phi'_{\mathcal{V}}\circ j, \qquad \tau\colon\Psi'_{\mathcal{U}}\to\Psi'_{\mathcal{V}}\circ j;$$

for example, if $K'\in\mathcal{C}'_{\mathcal{U}}$, then $\varrho(K')$ is simply the identity of $C_*(K')$.

We consider the following composite natural transformations:

$$\Phi'\xrightarrow{\alpha}\Psi'_{\mathcal{U}}\circ\mathrm{Sd}\circ\lambda'\xrightarrow{(\mathrm{Sd}\circ\lambda')^*(\tau)}\Psi'_{\mathcal{V}}\circ j\circ\mathrm{Sd}\circ\lambda'$$

$$\Phi'_{\mathcal{U}}\xrightarrow{\varrho}\Phi'_{\mathcal{V}}\circ j\xrightarrow{j^*(\alpha)}\Psi'_{\mathcal{V}}\circ\mathrm{Sd}\circ\lambda'\circ j=\Psi'_{\mathcal{V}}\circ j\circ\mathrm{Sd}\circ\lambda'.$$

Proposition 3.9. *There is a natural chain homotopy*

$$(\mathrm{Sd}\circ\lambda')^*(\tau)\circ\alpha\simeq j^*(\alpha)\circ\varrho.$$

Proof. The functor $\Phi'_{\mathcal{U}}$ is free with models $\mathcal{M}'_{\mathcal{U}}$, and $\Psi'_{\mathcal{V}}\circ j\circ\mathrm{Sd}\circ\lambda'$ is acyclic on the same models. □

Proof of Proposition 2.5. The natural transformations α and β of 3.7 allow us to construct chain maps $\alpha\colon\underline{C}'_*(\mathcal{U},\mathcal{U}')\to\underline{D}'_*(\mathcal{U},\mathcal{U}')$ and $\beta\colon\underline{D}'_*(\mathcal{U},\mathcal{U}')\to\underline{C}'_*(\mathcal{U},\mathcal{U}')$ in the same way as the chain map $p\colon\underline{C}'_*(\mathcal{U},\mathcal{U}')\to\underline{C}_*(\mathcal{U},\mathcal{U}')$ was contructed in the proof of 2.4. From 3.9 it then follows that α is compatible with refinement up to chain homotopy, and 3.8. shows that β is a chain homotopy inverse of α. □

Appendix

In this appendix we prove Proposition 1.2. Let (X,A) be a G-pair and $m \in \mathcal{C}_G$ a coefficient system. There are exact sequences

$$\begin{aligned} 0 \to \varinjlim_{\mathcal{U}} \operatorname{Hom}_{\mathcal{C}_G}\big(\underline{D}_*(\mathcal{U},\mathcal{U}'),m\big) &\to \varinjlim_{\mathcal{U}} \operatorname{Hom}_{\mathcal{C}_G}\big(\underline{D}_*(\mathcal{U}),m\big) \\ &\to \varinjlim_{\mathcal{U}} \operatorname{Hom}_{\mathcal{C}_G}\big(\underline{D}_*(\mathcal{U}'),m\big), \end{aligned}$$

$$0 \to \bar{C}^*_G(X,A;m) \to \bar{C}^*_G(X;m) \to \bar{C}^*_G(A;m);$$

the former is a consequence of the left exactness of $\operatorname{Hom}_{\mathcal{C}_G}$ and exactness of $\varinjlim_{\mathcal{U}}$, while the latter follows from Definition 1.1 in [H]. Therefore it is enough to show that

$$\bar{C}^*_G(X;m) \cong \varinjlim_{\mathcal{U}} \operatorname{Hom}_{\mathcal{C}_G}\big(\underline{D}_*(\mathcal{U}),m\big)$$

and

$$\bar{C}^*_G(A;m) \cong \varinjlim_{\mathcal{U}} \operatorname{Hom}_{\mathcal{C}_G}\big(\underline{D}_*(\mathcal{U}'),m\big)$$

(provided that the isomorphisms are functorial enough). In fact it suffices to prove the first isomorphism, for the second is a special case of the first by the next lemma:

Lemma A.1. *Any G-covering $\mathcal{V}$ of A is of the form $\mathcal{U}'$ for some G-covering $\mathcal{U}$ of X.*

Proof. Given $\mathcal{V}$, a desired $\mathcal{U}$ can be constructed as follows: Pick a representative $x \in A$ for each G-orbit of A and choose an open $U_x \subset X$ for every x so that U_x is G_x-invariant and $U_x \cap A = V_x$. If $y = gx \in A$, $g \in G$, let $U_y = gU_x$. In this way U_y is defined for every $y \in A$. For $y \in X \setminus A$, choose U_y arbitrarily to satisfy $y \in U_y$ and $gU_y = U_{gy}$, $g \in G$. □

In [H] we defined $\bar{C}^n_G(X;m) = C^n_G(X;m)/C^n_{G,0}(X;m)$, where $C^n_{G,0}(X;m) \subset C^n_G(X;m)$ is the subgroup of locally zero cochains; a cochain $c \in C^n_G(X;m)$ is locally zero, if there is a G-covering $\mathcal{U}$ of X such that

$$(A.2) \qquad c(\varphi) = 0 \quad \text{for } \varphi \in V_n(X), \quad t(\varphi) = H, \\ \text{if } \big\{\varphi_0(eH), \dots, \varphi_n(eH)\big\} \subset U_x \quad \text{for some } x \in X$$

(here we have used the notation of [H]).

Remark A.3. When defining locally zero cochains in [H], we used a more general notion of a G-covering than in this paper. These different notions lead to the same concept of locally zero cochains, however, because any open G-covering $\mathcal{W}$ of X in the sense of [H] has a refining G-covering $\mathcal{U} = \big\{U_x \mid x \in X\big\}$ in the sense of this paper; such a $\mathcal{U}$ can be obtained by an orbitwise construction as in A.1.

Let us also consider the following condition for a cochain $c \in C^n_G(X;m)$ and a G-covering $\mathcal{U}$ of X:

$$(A.4) \qquad c(\varphi) = 0 \quad \text{for } \varphi \in V_n(X), \quad t(\varphi) = H, \quad \text{if } \{\varphi_0(eH), \dots, \varphi_n(eH)\} \subset U_x \quad \text{for some } x \in X^H.$$

Denote ${}'C^n_{G,0}(X;m) = \{c \in C^n_G(X;m) \,|\, c \text{ satisfies A.4 for some } \mathcal{U}\}$.

Lemma A.5. *We have*

$$\varinjlim_{\mathcal{U}} \operatorname{Hom}_{\mathcal{C}_G}\big(\underline{D}_n(\mathcal{U}), m\big) \cong C^n_G(X;m)/{}'C^n_{G,0}(X;m).$$

Proof. Any $\varphi = (\varphi_0, \dots, \varphi_n) \in V_n(X)$ determines an $(n+1)$-tuple $(x_0, \dots, x_n) \in (X^H)^{n+1}$, $H = t(\varphi)$, by $x_i = \varphi_i(eH)$, and conversely. From this it follows that if $\mathcal{U}$ is a G-covering of X and we denote

$$V'_n(X,\mathcal{U}) = \Big\{\varphi \in V_n(X) \,\big|\, \{\varphi_0(eH), \dots, \varphi_n(eH)\} \subset U_x \text{ for some } x \in X^H,\ H = t(\varphi)\Big\}$$

and $C'_n(\mathcal{U})$ = free abelian group with basis $V'_n(X,\mathcal{U})$, then

$$\begin{aligned}\operatorname{Hom}_{\mathcal{C}_G}\big(\underline{D}_n(\mathcal{U}), m\big) \cong\ &\Big\{u\colon C'_n(\mathcal{U}) \to \bigoplus_{H \le G} m(G/H) \,\big|\, u \text{ is a homomorphism;}\\ &u(\varphi) \in m\big(G/t(\varphi)\big) \text{ for each } \varphi \in V'_n(X,\mathcal{U});\\ &\text{if } \alpha\colon G/K \to G/t(\varphi) \text{ is a } G\text{-map, then } u(\varphi \circ \alpha) = m(\alpha)\big(u(\varphi)\big)\Big\}.\end{aligned}$$

The assertion of the lemma is then verified in the same way as Lemma 4.1 in [H]. □

The proof of the identity $\bar{C}^n_G(X;m) \cong \varinjlim \operatorname{Hom}_{\mathcal{C}_G}\big(\underline{D}_n(\mathcal{U}), m\big)$ is now completed by the lemma below:

Lemma A.6. *Let $c \in C^n_G(X;m)$ and let $\mathcal{U}$ be a G-covering of X.*

i) *If c satisfies* A.2 *for $\mathcal{U}$, then c satisfies* A.4 *for $\mathcal{U}$;*

ii) *if c satisfies* A.4 *for $\mathcal{U}$, then there is a refinement $\mathcal{V}$ of $\mathcal{U}$ such that c satisfies* A.2 *for $\mathcal{V}$.*

Proof. Part i) is trivial. As for ii), a required $\mathcal{V}$ can be constructed as follows: Pick a representative $x \in X$ from every G-orbit of X; for each such an x choose an open G_x-invariant neighborhood V_x such that $V_x \subset U_x$ and $V_x \cap X^H = \emptyset$, if $H \le G$ and $x \notin X^H$ (recall that X^H is closed, because X is Hausdorff); if $y = gx$, $g \in G$, let $U_y = gU_x$. □

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