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THE DOMAINS OF NORMALITY OF HOLOMORPHIC SELF-MAPS OF C^*

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1. Introduction and results

The iteration theory of Fatou and Julia applies to analytic maps $f\colon D \to D$ where the domain D is contained in $\bar{\mathbf{C}}$, and introduces the sets $N(f) = \{z : z \in D, (f^n)$ is a normal family in some neighbourhood of $z\}$ and a Julia set $J(f) = D - N(f)$. In 1953, Rådström [10] showed that, to obtain interesting results, it is necessary to assume f not to be a Moebius transformation and the complement of D to consist of at most two points. One may assume that the complement of D is $\emptyset$, $\{\infty\}$, or $\{0,\infty\}$, and with this normalisation there are essentially the following cases:

I. $D = \bar{\mathbf{C}}$, f rational,
II. $D = \mathbf{C}$, f entire,
III. $D = \mathbf{C}^* = \{z : 0 < |z| < \infty\}$.

In the third case there are four types of function f, depending on the behaviour at the points 0, ∞ (see [1]):

a) $f(z) = kz^n$, $k \neq 0$, $n \in \mathbf{Z} - \{0, \pm 1\}$,
b) $f(z) = \exp\bigl(G(z)\bigr)$, G non-constant entire,
c) $f(z) = z^{-n} \exp\bigl(G(z)\bigr)$, G non-constant entire,
d) $f(z) = z^m \exp\bigl(F(z^{-1}) + G(z)\bigr)$, F, G non-constant entire, $m \in \mathbf{Z}$.

Note that (a) and (b) belong to cases I and II, respectively, and for $k \geq 2$ and f of type (c) we have f^k of type (d).

In this paper we consider the dynamics of functions of class III(d), denoted by $\mathcal{R}$. This class of functions was first discussed by Rådström [10] and next by Baker [1], Bhattacharyya [3], Keen [6, 7], Kotus [8] and Makienko [9].

For $f \in \mathcal{R}$, the Julia set $J(f)$ is a non-empty perfect subset of $\mathbf{C}^*$ and also completely invariant, i.e. $f\bigl(J(f)\bigr) = f^{-1}\bigl(J(f)\bigr) = J(f)$ and $J(f^p) = J(f)$ for $p \in \mathbf{N}$ (see [10]). The dynamics on $N(f)$ is better understood for rational than for entire functions or for functions of class $\mathcal{R}$. It is a consequence of Sullivan's theorem which states that every component is eventually periodic. This theorem is true only for a certain subclass of $\mathcal{R}$:

Theorem A. *Let $f \in \mathcal{R}$ have finitely many singular values. Then every component of $N(f)$ is eventually periodic.*

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Theorem A was proved independently by Keen [6], Kotus [8] and Makienko [9]. A component D of $N(f)$ is called parabolic at 0 (or at ∞) if $f(D) \subset D$ and $f^n \to 0$ in D (or $f^n \to \infty$ in D).

In [8] the following theorem was proved:

Theorem B. *Let $f \in \mathcal{R}$ have finitely many singular values. Then f does not have a parabolic component at 0 and ∞.*

A consequence of both theorems is a classification of periodic components. The aim of further investigation is a description of wandering components. The next theorem, proved by Baker [1], implies that any such component is simply-connected for $n \geq n_0$.

Theorem C. *If $f \in \mathcal{R}$, components of $N(f)$ are simply or doubly-connected. There is at most one doubly-connected component.*

The known examples of wandering components were constructed by Baker [1] and belong to class III(b). Now we show the following examples.

Theorem 1. *There exist functions f_1, $f_2 \in \mathcal{R}$ such that:*

a) *f_1 has a wandering component D of $N(f_1)$ such that the limit set of $f_1^n(D)$ equals one of the two essential singularities,*

b) *f_2 has a wandering component D of $N(f_2)$ such that the limit set of $f_2^n(D)$ equals $\{0, \infty\}$.*

Theorem 2. *There is a function $f \in \mathcal{R}$ which has a wandering component D of $N(f)$ with an infinite limit set.*

The construction of these examples depends on the results obtained on complex approximation. This method of construction of components for entire functions was introduced by A. Eremenko and M.Yu. Lyubich [4] and improved by I.N. Baker [2]. We modify their constructions to obtain analogous examples of wandering components for functions of class $\mathcal{R}$. Also applying the results on complex approximation, we give examples of parabolic domains.

Theorem 3. *There are functions of class $\mathcal{R}$ admitting parabolic domains at 0 or at ∞.*

2. Preliminary lemmas

We shall make use of the following results.

Lemma 1 (Runge, see e.g. [5]). *Suppose that K is compact in $\mathbf{C}$ and f is holomorphic on K; let also $\varepsilon > 0$. Let E be a set such that E meets every component of $\bar{\mathbf{C}} - K$. Then there exists a rational function r with poles in E such that*

$$\left|f(z) - r(z)\right| < \varepsilon, \qquad z \in K.$$

Lemma 2. ([5], p. 131) *Suppose that E is a closed set in $\mathbf{C}$ and f is a function defined on E. Then f can be uniformly approximated on E by meromorphic functions without poles in E if and only if f can be uniformly approximated by rational functions on each compact subset of E.*

Lemma 3 ([5], p. 137). *Suppose that E is a closed set in $\mathbf{C}$ and that z_1, z_2 lie in the same component of $\mathbf{C} - E$. Then for each function m meromorphic in $\mathbf{C}$ with a pole at z_1 and for each $\varepsilon > 0$ there exists a function m^* meromorphic in $\mathbf{C}$ which is analytic at z_1, has a pole at z_2, has no other poles except those of m, and for which*

$$\left|m(z) - m^*(z)\right| < \varepsilon, \qquad z \in E.$$

Lemma 4 ([5], p. 140). *Suppose that E is a closed set in $\mathbf{C}$ such that*
(i) $\bar{\mathbf{C}} \setminus E$ *is locally connected at* ∞.
If the meromorphic function m has no poles on E, then for each $\varepsilon > 0$ there exist a rational function r with poles outside E and an entire function g such that $\left|m(z) - (r+g)(z)\right| < \varepsilon$, $z \in E$.

Remark 1. Let m be a meromorphic function such that $m(z) = w(z^{-1}) + g(z)$, where w is a polynomial, while g is entire. Then the function $f(z) = \exp\left(w(z^{-1}) + g(z)\right)$ belongs to $\mathcal{R}$.

The next lemma is based on the main lemma proved in [4], p. 460.

Lemma 5. *Let K_n, L_n, $n = 1, 2, \ldots$, be compact subsets of $\mathbf{C}^*$ with the following properties:*
(i) K_n, L_n *are simply-connected for every* n;
(ii) $K_n \cap K_m = \emptyset$ *and* $L_n \cap L_m = \emptyset$ *for* $n \neq m$;
(iii) $\left(\bigcup_{n=1}^{\infty} K_n\right) \cap \left(\bigcup_{n=1}^{\infty} L_n\right) = \emptyset$;
(iv) $\max\{|z| : z \in K_n\} \to 0$, $\min\{|z| : z \in L_n\} \to \infty$ *as* $n \to \infty$ *and* $2\max\{|z| : z \in K_n\} < \frac{1}{2}\min\{|z| : z \in L_n\}$.
Let $w_n \in K_n$, $z_n \in L_n$, $\varepsilon_n > 0$ and the function h be analytic on $\bigcup_{n=1}^{\infty} K_n \cup \bigcup_{n=1}^{\infty} L_n$. Then there exist non-constant entire functions F, G such that $g(z) = \exp\left(F(z^{-1}) + G(z)\right)$ satisfies

$$\left|g(z) - h(z)\right| < \varepsilon, \qquad z \in K_n \cup L_n, \tag{1}$$

$$g(w_n) = h(w_n), \qquad g(z_n) = h(z_n), \tag{2}$$

$$g'(w_n) = h'(w_n), \qquad g'(z_n) = h'(z_n), \qquad n = 1, 2, \ldots. \tag{3}$$

In the proof we apply the following lemma.

Lemma 6 ([4], p. 460). *Let A be a locally convex topological space, V a domain in A, W a convex subset in V and S an affine subspace of A of finite codimension, such that $S \cap V \neq \emptyset$. Then $S \cap W$ is dense in $S \cap V$.*

Proof of Lemma 2. Let U be a union of two simply-connected sets which are neighbourhoods of K_1 and L_1 such that h is analytic in U and $U \cap K_n = \emptyset = U \cap L_n = \emptyset$ for $n \geq 2$. Consider the space A of all functions analytic in U with the topology of uniform convergence on compact sets. Then

$$V = \{g : |g(z) - h(z)| < \tfrac{1}{2}\varepsilon_1, z \in K_1 \cup L_1\}$$

is a convex domain in A. Let W be the subset of rational functions with poles at 0 and at ∞ only. By Lemma 1, W is dense in V. Clearly W is also convex. We also consider the affine subspace

$$S = \{g \in A \ : \ g(w_1) = h(w_1), \qquad g(z_1) = h(z_1), \\ g'(w_1) = h'(w_1), \qquad g'(z_1) = h'(z_1)\}.$$

By Lemma 6 there exists a rational function $g_1 \in W \cap S$ such that $g_1(z) = F_1(z^{-1}) + G_1(z)$, where F_1, G_1 are polynomials and

$$|g_1(z) - h(z)| < \tfrac{1}{2}\varepsilon_1, \qquad z \in K_1 \cup L_1, \\ g_1(w_1) = h(w_1), \qquad g_1(z_1) = h(w_1), \\ g_1'(w_1) = h(w_1), \qquad g_1'(z_1) = h'(z_1).$$

For $n > 1$ there is a rational function g_n such that $g_n(z) = F_n(z^{-1}) + G_n(z)$, where F_n, G_n are polynomials and

$$\left|\sum_{k=1}^{n} g_k(z) - h(z)\right| < \tfrac{1}{2}\varepsilon_n, \qquad z \in K_n \cup L_n, \tag{4}$$

$$|g_k(z)| < 2^{-n+k}\varepsilon_k, \qquad z \in K_k \cup L_k \quad \text{and} \quad k < n, \tag{5}$$

$$|g_n(z)| < 2^{-n}\ 2\max\{|\zeta| : \zeta \in K_n\} < |z| < \tfrac{1}{2}\min\{|\zeta| : \zeta \in L_n\} \tag{6}$$

$$\sum_{k=1}^{n} g_k(w_i) = h(w_i), \qquad \sum_{k=1}^{n} g_k(z_i) = h(z_i), \quad 1 \leq i \leq n, \tag{7}$$

$$\sum_{k=1}^{n} g_k'(w_i) = h'(w_i), \qquad \sum_{k=1}^{n} g_k'(z_i) = h(z_i), \quad 1 \leq i \leq n. \tag{8}$$

It follows from (6) that the series $g = \sum_{n=1}^{\infty} g_n$ converges uniformly on the compact subsets of $\mathbf{C}^*$ and defines a function $g \colon \mathbf{C}^* \to \mathbf{C}$ of the form $g(z) = F(z^{-1}) + G(z)$. Next, (4) and (5) imply (1), while (7) and (8) imply (2) and (3).

Remark 2. Let g be the function of the form $g(z) = F(z^{-1}) + G(z)$, with F, G non-constant entire. Then $f = \exp g$ belongs to $\mathcal{R}$.

Lemma 7 ([4], p. 461). *Let $f(z) = z + g(z)$ be an analytic function in the disk $\{z : |z| < R\}$ such that $g(0) = g'(0) = 0$ and $|g(z)| < \varepsilon R$ and some $\varepsilon < \frac{1}{2}$. Then*

$$|z|\Big(1 - \frac{\varepsilon}{R}|z|\Big) \leq |f(z)| \leq |z|\Big(1 + \frac{\varepsilon}{R}|z|\Big), \tag{9}$$

$$|\arg f(z) - \arg z| < 2\frac{\varepsilon}{R}|z|, \qquad |z| < R. \tag{10}$$

Lemma 8 ([4], p. 461). *Let $q > 1$. Then there exists a number $s(q)$ such that the estimates*

$$s_0 \sum_{k=0}^{n-1} \varepsilon_k < s(q), \qquad s_0 > 0, \qquad \varepsilon_k > 0,$$

$$s_k(1 - \varepsilon_k s_k) \leq s_{k+1} \leq s_k(1 + \varepsilon_k s_k), \qquad 0 \leq k \leq n-1,$$

imply that

$$\frac{1}{q}s_0 \leq s_k \leq q s_0, \qquad 0 \leq k \leq n.$$

Lemma 9. *If $f \in \mathcal{R}$ and D_i, $i = 1, 2$, are components of $N(f)$ such that $f^p(D_i) \subset D_i$ for some $p \in \mathbf{N}$ and $f^{np} \to 0$ in D_1, $f^{np} \to \infty$ in D_2. Then there are f^p invariant curves γ_i and positive constants a_i, b_i such that γ_1 tends to 0 in D_1, γ_2 to ∞ in D_2 and $a_i|z| < |f^p(z)| < b_i|z|$, $z \in \gamma_i$.*

The proof of this lemma is analogous to the proof of Theorem 2 in [2], p. 503.

3. Proof of Theorem 3

Denote

$$A = \{z : |z+2| < \tfrac{1}{2}\},$$

$$A' = \{z : |z+2| < 1\},$$

$$B = \{z : 10 < |z| \text{ and } 0 < \arg z < \tfrac{1}{2}\pi\},$$

$$B' = \{z : 9 < |z| \text{ and } -0.1 < \arg z < \tfrac{1}{2}\pi + 0.1\},$$

$$D = \{z : 0 < |z| < 0.1 \text{ and } -\tfrac{1}{2}\pi < \arg z < 0\},$$

$$D' = \{z : 0 < |z| < 0.2 \text{ and } -(\tfrac{1}{2}\pi + 0.1) < \arg z < 0.1\}.$$

Define functions g and h on the set $A' \cup B \cup \partial B'$ by

$$g(z) = \begin{cases} \log 2 + i\pi & \text{if } z \in A' \cup \partial B' \\ \log(10|z|^{1/2}) + i(\frac{1}{2}\arg z + \frac{1}{6}\pi) & \text{if } z \in B, \end{cases}$$

$$h(z) = \begin{cases} \log 2 + i\pi & \text{if } z \in A' \cup \partial B' \\ -\log(100|z|^{1/2}) - i(\frac{1}{2}\arg z + \frac{1}{6}\pi) & \text{if } z \in B. \end{cases}$$

Then the functions $\exp g$ and $\exp h$ have the form

$$\exp g(z) = \begin{cases} -2 & \text{if } z \in A \cup \partial B' \\ 10 z^{1/2} e^{i\pi/6} & \text{if } z \in B, \end{cases} \tag{11}$$

$$\exp h(z) = \begin{cases} -2 & \text{if } z \in A \cup \partial B' \\ 100^{-1} z^{-1/2} e^{-i\pi/6} & \text{if } z \in B, \end{cases} \tag{12}$$

where $z^{1/2}$ means one branch of the root function.

Clearly $\exp g(B) \subset B$, $\exp g(A') \subset A$, $\exp h(A') \subset A$, $\exp h(B) \subset D$. Let $C = \bar{A}' \cup \bar{B} \cup \partial B'$. Then C and g, h satisfy the assumptions of Lemma 2, so there are meromorphic functions m_1, m_2 with no poles on C such that

$$\left|m_1(z) - g(z)\right| < \tfrac{1}{2}\delta, \qquad z \in C, \tag{13}$$

$$\left|m_2(z) - h(z)\right| < \tfrac{1}{2}\delta, \qquad z \in C, \tag{14}$$

where δ is a small positive number. Since C also satisfies the assumptions of Lemma 4, there exist rational functions r_1, r_2 with poles outside C, and entire functions e_1, e_2 such that

$$\left|m_1(z) - \big(r_1(z) + e_1(z)\big)\right| < \tfrac{1}{2}\delta, \qquad z \in C, \tag{15}$$

$$\left|m_2(z) - \big(r_2(z) + e_2(z)\big)\right| < \tfrac{1}{2}\delta, \qquad z \in C. \tag{16}$$

Applying Lemma 3 we can choose r_1 such that it has exactly one pole in $\mathbf{C} - C$ at say $a = 0$. We can suppose that r_1 really has a pole at a since the addition of λ_1/z, where λ_1 is a sufficiently small constant, will bring this about without spoiling the approximation properties listed above. Thus we may assume that $m_1(z) = \lambda_1 z^{-1} + e_1(z)$ and $m_2(z) = \lambda_2 z^{-1} + e_2(z)$. By Remark 1, $f_1 = \exp m_1$ and $f_2 = \exp m_2$ belong to the class $\mathcal{R}$. Now, we show

$$\delta = \log(1+\mu) + i\varepsilon \tag{17}$$

with $0 < \mu < 0.1$, $0 < \varepsilon < 0.1$, $f_1(B) \subset B$, $f_2(B) \subset D$. By (13–16) and (17) we have

$$\operatorname{Re} m_1(z) > \operatorname{Re} g(z) - \log(1+\mu) = \log\big(10|z|^{1/2}(1+\mu)^{-1}\big) > \log 10$$

and

$$\operatorname{Re} m_2(z) < \operatorname{Re} h(z) + \log(1+\mu) = \log\bigl(100^{-1}|z|^{-1/2}(1+\mu)^{-1}\bigr) > \log 10^{-1}.$$

As $w(z) = z^{-1}$ maps D onto B for $f_3 = f_2 \circ w$, we have $f_3(D) \subset D$ and $f_3 \in \mathcal{R}$.
Next, (11), (12) and (17) imply

$$\bigl|f_1(z)\bigr| > \bigl|\exp m_1(z)\bigr|(1+\mu) > 2|z|, \qquad z \in B,$$

$$\bigl|f_3(z)\bigr| < \bigl|\exp m_2(z^{-1})\bigr|(1+\mu)^{-1} < \tfrac{1}{2}|z|, \qquad z \in D.$$

Hence all orbits of f_1 in B tend to ∞ and all orbits of f_3 tend to 0 in D. Thus $B \subset N(f_1)$, $D \subset N(f_3)$. The functions f_i, $i = 1$ or 3 map A' into A, so A' contains attractive fixed points ζ_i, $f_i^n \to \zeta_i$ in A' and $A' \subset N(f_i)$. Further, $f_1(\partial B') \subset A'$, $f_3(\partial D') \subset A'$, B is contained in a parabolic component of $N(f_1)$ at ∞ and D in a parabolic component of $N(f_3)$ at 0.

4. Proof of Theorem 1

We first construct an example of a function $f \in \mathcal{R}$ which has wandering components D_k of $N(f)$ such that $f^n \to \infty$ in D_k, $k, n \in \mathbf{N}$.

Define $\varepsilon_n = 10^{-n-3}$, $n \in \mathbf{N}$, and $\eta_1 = 0$, $\eta_2 = \varepsilon_1$, $\eta_n = \sum_{k=1}^{n-1} 2^{-n+k+1}\varepsilon_k$ for $n > 2$, so $\eta_{n+1} = \frac{1}{2}\eta_n + \varepsilon_n$. Write $c_1 = 2$, $c_n = 2^{c-n-1}$, $n \in \mathbf{N}$ and, further, $a_1 = 1$, $b_1 = 2$, $a_n = \frac{2}{3}c_{n-1}$ and $b_n = c_{n-1}^2$ for $n > 1$. Set

$$L_n = \Bigl\{z : a_n < |z| < b_n \text{ and } \Bigl|\arg z - \frac{\pi}{2^n}\Bigr| < \frac{1}{2^{n+3}} + \eta_n\Bigr\},$$

$$H_n = \Bigl\{z : \ln a_n < \operatorname{Re} z < \ln b_n \text{ and } \Bigl|\operatorname{Im} z - \frac{\pi}{2^n}\Bigr| < \frac{1}{2^{n+3}} + \eta_n\Bigr\}, \qquad n \in \mathbf{N}.$$

Define the function h on the set $\cup_{n=1}^{\infty} L_n$ by

$$h(z) = \log\bigl(c_n|z|^{1/2}\bigr) + i\tfrac{1}{2}\arg z$$

for $z \in L_n$. Then $\exp h(z) = c_n z^{1/2}$, $z \in L_n$, where $z^{1/2}$ means one branch of the root function. It is easy to check that $h(L_n) \subset H_{n+1}$, so $\exp h(L_n) \subset L_{n+1}$. The sets L_n, $n \in \mathbf{N}$, and the function h satisfy the assumptions of Lemma 5. Thus there is a function $g\colon \mathbf{C}^* \to \mathbf{C}$, $g(z) = F(z^{-1}) + G(z)$ where F and G are non-constant entire functions such that

$$\bigl|g(z) - h(z)\bigr| < \delta_n, \qquad z \in L_n; \tag{18}$$

here δ_n is a small positive number. Analogously the proof of Theorem 3 one can check that if

$$\delta_n = \log(1+\mu_n) + i\varepsilon_n, \tag{19}$$

where $\mu_n > 0$ and $\mu_n \to 0$, then $\exp g(L_n) \subset L_{n+1}$ for $n \in \mathbf{N}$. By (18) and (19) for $z \in L_n$,

$$\left|\operatorname{Re} g(z) - \operatorname{Re} h(z)\right| < \log(1+\mu_n).$$

So

$$\begin{aligned}\operatorname{Re} g(z) &< \operatorname{Re} h(z) + \log(1+\mu_n) < \log\bigl(c_n|z|^{1/2}(1+\mu_n)\bigr) \\ &< \log c_n|c_{n-1}^2|^{1/2}(1+\mu_n) < \log c_n^2 = \log b_{n+1}\end{aligned}$$

and

$$\begin{aligned}\operatorname{Re} g(z) &> \operatorname{Re} h(z) - \log(1+\mu_n) > \log\bigl(c_n|z|^{1/2}(1-\mu_n')\bigr) \\ &> \log c_n\left|\tfrac{2}{3}c_{n-1}\right|^{1/2}(1-\mu_n') > \log \tfrac{2}{3}c_n = \log a_{n+1}.\end{aligned}$$

Thus for $z \in L_n$

$$a_{n+1} < \left|\exp g(z)\right| < b_{n+1}. \tag{20}$$

Also by (18) and (19) we have

$$\begin{aligned}\left|\operatorname{Im} g(z) - \frac{\pi}{2^{n+1}}\right| &< \left|\operatorname{Im} h(z) - \frac{\pi}{2^{n+1}}\right| + \varepsilon_n \\ &< \left|\tfrac{1}{2}\left(\frac{\pi}{2^n} + \frac{1}{2^{n+3}} + \eta_n\right) + \varepsilon_n - \frac{\pi}{2^{n+1}}\right| = \frac{1}{2^{n+4}} + \eta_{n+1}.\end{aligned} \tag{21}$$

Let $f = \exp g$. Then, by Remark 2, g belongs to $\mathcal{R}$. It follows from (20) and (21) that $f(L_n) \subset L_{n+1}$ and so $f^n \to \infty$ uniformly in L_n. Thus $L_n \subset N(f)$. Let N_n be a component of $N(f)$ containing L_n. Clearly $f\colon N_n \to N_{n+1}$. If some $N_n = N_{n+1}$, then N_n is in fact an unbounded domain which is mapped into itself and in which $f^n(z) \to \infty$. By Lemma 9 there is a path γ to ∞ in N_n and positive constants c, d such that $c|z| < \left|f(z)\right| < d|z|$ as $z \to \infty$ on γ. But this contradicts the growth implied by the construction on

$$\left\{z : |z| = c_{n-1} \text{ and } \left|\arg z - \frac{\pi}{2^n}\right| < \frac{1}{2^{n+3}} + \eta_n\right\} \subset N_n.$$

We have $\left|f(z)\right| > a_{n+1} = \frac{2}{3}2^{c_{n-1}}$. Thus all N_n are different and each is a wandering component of $N(f)$. Analogously one can construct an example of wandering components in which $f^n \to 0$.

Now, we give an example of $f \in \mathcal{R}$ which has wandering components with limit set equals to $\{0,\infty\}$.

Let L_n, H_n, ε_n, η_n be defined as above. Write $d_n = c_n^{-1}$, $r_n = b_n^{-1}$, $s_n = a_n^{-1}$ and set

$$K_n = \left\{z : r_n < |z| < s_n \text{ and } \left|\arg z + \frac{\pi}{2^n}\right| < \frac{1}{2^{n+3}} + \eta_n\right\},$$

$$D_n = \left\{ z : \ln r_n < \operatorname{Re} z < \ln s_n \text{ and } \left|\operatorname{Im} z + \frac{\pi}{2^n}\right| < \frac{1}{2^{n+3}} + \eta_n \right\}, \qquad n \in \mathbf{N}.$$

Define the function h on the set $\cup_{n=1}^{\infty}(K_n \cup L_n)$ by

$$h(z) = \begin{cases} -\left(\log d_{2n}|z|^{1/2} + i\frac{1}{2}\arg x\right) & \text{if } z \in K_{2n}, \\ -\left(\log c_{2n-1}|z|^{1/2} + i\frac{1}{2}\arg x\right) & \text{if } z \in L_{2n-1}, n \in \mathbf{N}. \end{cases}$$

Clearly $h(L_{2n-1}) \subset D_{2n}$, $h(K_{2n}) \subset H_{2n+1}$. By Lemma 5 there is a function $g(z) = F(z^{-1}) + G(z)$, F and G non-constant entire functions, satisfying

$$\left|g(z) - h(z)\right| < \delta_{2n-1}, \qquad z \in L_{2n-1},$$

$$\left|g(z) - h(z)\right| < \delta_{2n}, \qquad z \in K_{2n}, n \in \mathbf{N},$$

where δ_{2n-1} and δ_{2n} are so chosen that $f = \exp g$ maps L_{2n-1} into K_{2n} and K_{2n} into L_{2n+1}. Hence $f^{2n} \to \infty$ and $f^{2n-1} \to 0$ uniformly in each L_{2n-1}, and conversely $f^{2n-1} \to \infty$, $f^{2n} \to 0$ in each K_{2n}. Let N_{2n-1} and N_{2n} be components of $N(f)$ containing L_{2n-1} and K_{2n}, respectively. On

$$\left\{ z : |z| = d_{2n-1} \text{ and } \left|\arg z + \frac{\pi}{2^{2n}}\right| < \frac{1}{2^{2n+3}} + \eta_{2n} \right\} \subset N_{2n}$$

we have

$$\left|f^2(z)\right| < S_{2n+2} = \tfrac{3}{2}\left(2^{c_{2n}}\right)^{-1} = \tfrac{3}{2}2^{-d_{2n}}$$

while on

$$\left\{ z : |z| = c_{2n-2} \text{ and } \left|\arg z - \frac{\pi}{2^{2n-1}}\right| < \frac{1}{2^{2n+2}} + \eta_{2n-1} \right\} \subset N_{2n-1}$$

we have

$$\left|f^2(z)\right| > a_{2n+1} = \left(\tfrac{2}{3}2^{c_{2n}}\right).$$

This and Lemma 9 imply that all N_{2n-1} and N_{2n} are different components of $N(f)$.

5. Proof of Theorem 2

To construct this example, first fix ε and q such that $0 < \varepsilon < \frac{1}{2}$, $1 < q < 2^{1/3}$. Consider a sequence r_n, $0 \le n < \infty$, such that $0 < r_n < \frac{1}{2}r_{n-1}$, $n = 1, 2, \ldots$, and

$$\varepsilon \sum_{k=0}^{n-1} \frac{r_n}{r_k} < \min\left\{ s, \frac{\pi}{8} \right\}, \tag{22}$$

where $s = s(q)$ is chosen to satisfy Lemma 8. Define a sequence $a_0 = 2$, $a_{2k-1} = 4^k$, $a_{2k} = 4^{-k}$, $k \geq 1$. Let

$$B_n = \{z : |z - a_n| < \tfrac{1}{2} r_n\},$$
$$D_n = \{z : q^{-2} r_n < |z| < q^{-1} r_n, |\arg z| < \tfrac{1}{4}\pi\},$$
$$Q_n = \{z : q^{-3} r_n < |z - a_n| < r_n, |\arg z| < \tfrac{1}{4}\pi\}, \qquad n \geq 0,$$

and define the function h on the set $\cup_{n=0}^{\infty}(B_n \cup Q_n)$ by

$$h(z) = \begin{cases} \log(z + a_{n+1} - a_n) & \text{if } z \in B_n, n \geq 0, \\ \log(q^{-3/2} r_{n+1}) & \text{if } z \in Q_n, n \geq 0. \end{cases}$$

See Figure 1.

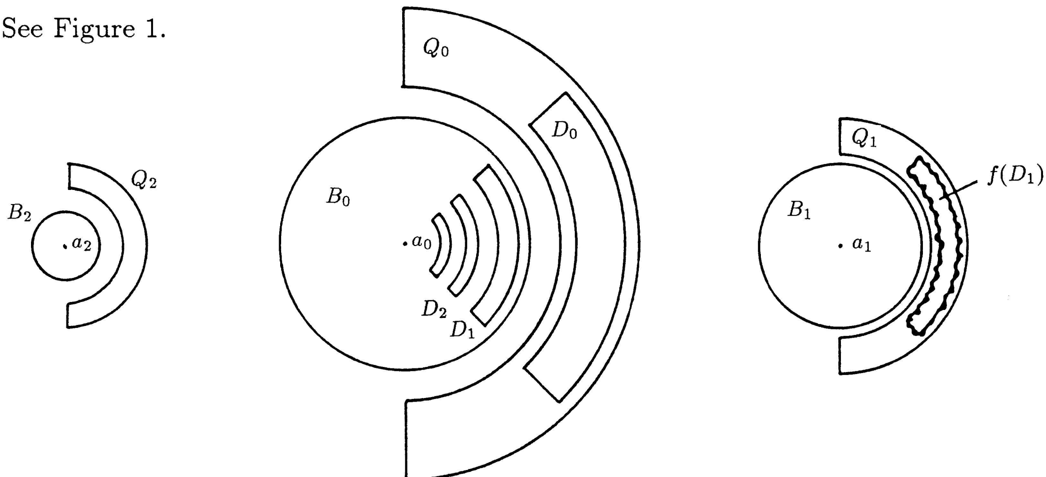

Figure 1.

By Lemma 5 there are non-constant entire functions F and G such that $f(z) = \exp\big(f(z^{-1}) + G(z)\big)$ satisfies

$$(23) \qquad f(a_n) = a_{n+1}, \qquad f'(a_n) = 1, \qquad n \geq 0,$$

$$(24) \qquad \big|f(z) - h(z)\big| < \tfrac{1}{4}\varepsilon r_n, \qquad z \in B_n,$$

and

$$(25) \qquad f(Q_n) \subset D_{n+1}.$$

Let $|z| < \frac{1}{2} r_n$, $s_k = \big|f^k(z) - a_k\big|$, $k \geq 0$. We shall prove that

$$q^{-1} s_0 \leq s_k \leq q s_0, \qquad 0 \leq i \leq k - 1 < n.$$

As $qs_0 < (\frac{1}{2}2^{1/3})r_n < r_n < \frac{1}{2}r_i$, it follows that $f^i(z) \in B_i$, $0 \le i \le k-1$. Then from (23) and (24) and Lemma 7 we have

$$s_i(1 - \varepsilon s_i/r_i) \le s_{i+1} \le s_i(1 + \varepsilon s_i/r_i), \qquad 0 \le i \le k-1.$$

Moreover,

$$s_0 \sum_{i=0}^{n-1} \frac{\varepsilon}{r_i} \le \sum_{i=0}^{n-1} \frac{r_n}{r_i} < s.$$

Using Lemma 8 we obtain $q^{-1}s_0 \le s_k \le qs_0$. Thus (26) is proved by induction, and we have $f^k(z) \in B_k$ for $0 \le k \le n-1$.

Using Lemma 7 once more we obtain

$$\left|\arg(f^{k+1}z - a_{k+1}) - \arg(f^k z - a_k)\right| \le 2\varepsilon \frac{s_k}{r_k} \le 2\varepsilon \frac{r_n}{r_k}, \quad 0 \le k \le n-1.$$

Therefore, by (22),

$$\left|\arg(f^n z - a_n) - \arg z\right| \le 2\varepsilon \sum_{k=0}^{n-1} \frac{r_n}{r_k} \le \tfrac{1}{4}\pi.$$

It follows that (taking $k = n$) $f^n(D_n) \subset Q_n$, $n \ge 0$. Since $f(Q_n) \subset D_{n+1}$ by (25), it follows that every orbit originating from D_0 has a limit point at $a_0 = 2$, and, consequently, further limit points at $a_1 = f(a_0)$, a_2, a_3, ... Moreover, it is clear from $f^{n+1}(D_n) \subset D_{n+1}$ that the limit functions are constants. As in the case of entire functions, the existence of an infinity of constant limit functions of (f^n) in a component of $N(f)$ implies that the component is wandering. Thus the component which contains D_0 is wandering and the limit set of $f^n|D_0$ contains an oscillating trajectory a_0, a_1, a_2, ...

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