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REMOVABILITY THEOREMS FOR QUASIREGULAR MAPPINGS

P. Koskela and O. Martio

1. Introduction

A continuous mapping f of an open set $G \subset \mathbf{R}^n$ into $\mathbf{R}^n$ is called K-quasiregular, $K \geq 1$, if f is ACL^n, i.e. the coordinate functions of f belong to the local Sobolev space $\operatorname{loc} W_n^1(G)$, and if

$$|f'(x)|^n \leq K\, J(x,f) \tag{1.1}$$

a.e. in G. Here $f'(x)$ is the (formal) derivative of f at x, $J(x,f)$ is the jacobian determinant of the matrix $f'(x)$ and $|f'(x)|$ stands for the supremum norm of the linear mapping $f'(x)\colon \mathbf{R}^n \to \mathbf{R}^n$. For $n = 2$ and $K = 1$ these mappings reduce to the class of analytic functions in G. Quasiregular mappings seem to form a proper generalization of analytic functions to higher dimensional euclidean spaces. For the theory of quasiregular mappings we refer to [MRV1–2] and [R].

Suppose that C is a relatively closed subset of G. The removability theorem [CL, p. 5] of Painlevé and Besicovitch says that if the one dimensional Hausdorff measure $\mathcal{H}^1(C)$ of C vanishes and if $f\colon G\backslash C \to \mathbf{R}^2$ is a bounded analytic function, then f extends to an analytic function of G. For general quasiregular mappings $f\colon G \setminus C \to \mathbf{R}^n$ the following, much weaker result was proved in [MRV2]:

1.2. Theorem. *Suppose that C is of zero n-capacity. Then every bounded K-quasiregular mapping $f\colon G \setminus C \to \mathbf{R}^n$ extends to a K-quasiregular mapping of G.*

The proof for this result has potential theoretic character: For bounded harmonic functions in the plane a set of zero 2-capacity is removable and there is a similar result in $\mathbf{R}^n$, $n \geq 2$, for coordinate functions of a quasiregular mapping f, see [HKM].

Rather precise removability theorems can be obtained in the plane.

1.3. Theorem. *Suppose that $\mathcal{H}^\lambda(C) = 0$ for all $\lambda > 0$, i.e. the Hausdorff dimension $\dim_{\mathcal{H}}$ of C is zero. Then every bounded plane K-quasiregular mapping $f\colon G \setminus C \to \mathbf{R}^2$ has a K-quasiregular extension f^* to G.*

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The proof uses the representation theorem [LV, p. 247] for plane quasiregular mappings: $f = g \circ h$ where h is a quasiconformal mapping, i.e. a quasiregular homeomorphism, and g is analytic. Since $\mathcal{H}^1(C) = 0$, h extends to a quasiconformal mapping h^* of G, see [LV, p. 206]. A K-quasiconformal mapping is locally Hölder continuous with exponent $1/K$. Thus $\dim_{\mathcal{H}}(C) = 0$ implies $\dim_{\mathcal{H}}\big(h^*(C)\big) = 0$. Hence the aforementioned theorem of Painlevé and Besicovitch shows that g has an analytic extension g^*. Now $f^* = g^* \circ h^*$ is the required extension of f.

A look at the above proof gives the following result.

1.4. Theorem. *Suppose that $f\colon G \setminus C \to \mathbf{R}^2$ is a bounded K-quasiregular mapping. If $\mathcal{H}^{1/K}(C) = 0$, then f extends to a K-quasiregular mapping of G.*

For $n \geq 3$ no results like Theorem 1.3 or 1.4 are known, except possibly for K near 1. The method of the proof certainly fails.

Theorem 1.2 has a remarkable extension, see [MRV2]: It holds if the mapping f omits a set of positive n-capacity—of course, the extended mapping may now take the value ∞. The proof for this result employs the geometric theory of quasiregular mappings—modulus and capacity estimates.

The purpose of this paper is twofold. We prove a removability theorem, Theorem 4.1, for general quasiregular mappings $f\colon G \setminus C \to \mathbf{R}^n$ which omit a set of positive n-capacity. Our assumptions allow the set C to be of positive n-capacity although C is quite thin, for example $\dim_{\mathcal{H}} C = 0$. The proof employs the geometric theory as in [MRV2]. We first show that f can be extended to a continuous mapping f^* of G. Thus we are naturally led to study removability questions for continuous mappings $f\colon G \to \mathbf{R}^n$ which are quasiregular in $G \setminus C$—this is done in Chapter 3. In Chapter 2 we introduce the conditions for the set C used in the main theorem.

For locally Hölder continuous functions $f\colon G \to \mathbf{R}^2$ analytic in $G \setminus C$ the removability of C is determined in terms of the Hölder exponent and the Hausdorff dimension of C. The following, very precise, result is due to L. Carleson, see e.g. [G, p.78]: If $\mathcal{H}^{\lambda}(C) = 0$ and if f is locally Hölder continuous in G with exponent $\alpha \geq \lambda - 1$ and analytic in $G \setminus C$, then f extends to an analytic function of G. Reasoning as for Theorem 1.3 we obtain

1.5. Theorem. *Suppose that $f\colon G \to \mathbf{R}^2$ is locally Hölder continuous with exponent $0 < \alpha \leq 1$ and K-quasiregular in $G \setminus C$. If $\mathcal{H}^{\lambda}(C) = 0$, $\lambda = \min(1, 1/K + \alpha/K^2)$, then f extends to a K-quasiregular mapping of G.*

For $n \geq 2$ a different technique produces results which, in general, are better than Theorem 1.5. It suffices to assume that $\lambda \leq \min\{1, \alpha/n\}$ and hence λ can be chosen independently of K, see Theorem 3.9. A careful analysis of a special semilocal Hölder class leads to results which allow removable sets C with $\dim_{\mathcal{H}}(C) > 1$. Such a result is Theorem 3.17 where the Minkowski dimension of C and the Hölder exponent of f are related.

1.6. Remark. For each $\varepsilon > 0$ there is a Cantor set $C \subset \mathbf{R}^2$ with $\dim_{\mathcal{H}}(C) < \varepsilon$ and a K-quasiregular mapping f of $\mathbf{R}^2 \setminus C$ that is locally Hölder continuous in $\mathbf{R}^2$ with some exponent $\alpha > 0$ but f fails to extend to a quasiregular mapping of $\mathbf{R}^2$. The mapping f can be constructed by composing a quasiconformal mapping as in [GV, Theorem 5] with an appropriate analytic function. Of course, K and α depend on ε.

Since this work was completed, we have become aware of three other manuscripts dealing with removability questions for quasiregular mappings. T. Iwaniec and G.J. Martin [IM] have proved that for each K and each n there is a $\lambda = \lambda(K,n) > 0$ such that closed sets F of the even dimensional space $\mathbf{R}^{2n}$ with $\mathcal{H}^\lambda(F) = 0$ are removable for bounded K-quasiregular mappings of $\mathbf{R}^{2n}$. Furthermore, P. Järvi and M. Vuorinen [JV] have established that certain self-similar Cantor sets are removable for quasiregular mappings omitting a finite but sufficiently large number of points. Finally, S. Rickman [Ri] has constructed examples of non-removable Cantor sets for bounded quasiregular mappings in $\mathbf{R}^3$.

2. Modulus conditions

We consider two modulus conditions, the M-condition and the UM-condition. The first was introduced in [M1] and further studied in [MS].

Let G be an open set in $\mathbf{R}^n$ and C a relatively closed subset of G. We say that a point $x_0 \in G$ satisfies the *M-condition* with respect to C if there exists a non-degenerate continuum $K \subset G$ such that

$$(K \setminus \{x_0\}) \cap C = \emptyset, \qquad x_0 \in K, \quad \text{and} \tag{2.1}$$

$$M\big(\Delta(K, C \cup \partial G; G \setminus \{x_0\})\big) < \infty. \tag{2.2}$$

Here $\Delta(E,F;A)$ stands for the family of all paths which join E to F in A and $M(\Gamma)$ is the n-modulus of the path family Γ, see [V]. Note that in (2.2) we can write $\mathbf{R}^n \setminus \{x_0\}$ instead of $G \setminus \{x_0\}$ as well—$G \setminus \{x_0\}$ instead of G is just used to avoid constant paths.

Clearly every point $x_0 \in G \setminus C$ satisfies the M-condition with respect to C. Hence only points $x_0 \in C$ are of interest in the M-condition. The M-condition seems also to depend on the domain G. However, writing $A = \mathbf{R}^n \setminus B^n(x_0, r_0)$, $x_0 \in G$, we see that

$$M_0 = M\big(\Delta(K, A; \mathbf{R}^n)\big) \le \omega_{n-1}\Big(\log \frac{r_0}{\delta}\Big)^{1-n} < \infty$$

whenever $K \subset G$ is a continuum with

$$\delta = \operatorname{dia}(K) < r_0$$

and with $x_0 \in K$. Since

$$M\Big(\Delta(K, C \cup \partial G; G \setminus \{x_0\})\Big) \le M_0 + M\Big(\Delta(K, (C \cup \partial G) \setminus A; G \setminus \{x_0\})\Big),$$

the points in $\partial G \cup C$ of distance $\varepsilon > 0$ from $x_0 \in G$ have no effect on the M-condition. Especially for points $x_0 \in C$ the M-condition is independent of G.

We say that C satisfies the *M-condition* (*with respect to* G) if each $x_0 \in C$ satisfies the M-condition.

Next we say that C satisfies the *UM-condition* (*with respect to* G) if for each compact set $F \subset G$ and for each $\varepsilon > 0$ there is $\delta > 0$ such that for every $x_0 \in F$ there exists a continuum $K \subset G$ with the property (2.1) and

$$\text{dia}(K) \ge \delta, \tag{2.3}$$

$$M\Big(\Delta(K, C \cup \partial G; G \setminus \{x_0\})\Big) \le \varepsilon. \tag{2.4}$$

If C satisfies the UM-condition, then C clearly satisfies the M-condition. The UM-condition (UM = uniform modulus) is a locally uniform version of the M-condition.

We shall frequently employ the following lemma which is a modification of a similar result in [M1] and [MS]. Note that the lemma will mostly be used for $C \cup \partial G$ instead of C.

2.5. Lemma. *Let C be a closed set in $\mathbf{R}^n$, $x_0 \in \mathbf{R}^n$ and K a continuum such that $x_0 \in K$ and $K \setminus \{x_0\} \subset \mathbf{R}^n \setminus C$. There are constants $\beta > 0$ and $b < \infty$ depending only on n such that if*

$$m = M\Big(\Delta(K, C; \mathbf{R}^n \setminus \{x_0\})\Big) \le \beta$$

then there are radii $r_i \in \big(\text{dia}(K)/2^{i+1}, \text{dia}(K)/2^i\big)$, $i = 1, 2, \ldots$, with

$$S^{n-1}(x_0, r_i) \subset \mathbf{R}^n \setminus C, \tag{2.6}$$

and

$$M\Big(\Delta(K', C; \mathbf{R}^n \setminus \{x_0\})\Big) \le bm, \tag{2.7}$$

where $K' = K \cup \cup_i S^{n-1}(x_0, r_i)$.

Proof. Let $t_i = \operatorname{dia}(K)/2^i$, $i = 1,2,\ldots$, and $A_0 = \mathbf{R}^n \setminus \overline{B}^n(x_0,t_2)$, $A_i = B^n(x_0,t_i) \setminus \overline{B}^n(x_0,t_{i+3})$, $i = 1,2,\ldots$. Write

$$\Gamma_i = \Delta(K,C;A_i).$$

First we prove that

$$\sum_{i=0}^{\infty} M(\Gamma_i) \leq 3\,M(\Gamma); \tag{2.8}$$

here $\Gamma = \Delta\bigl(K,C;\mathbf{R}^n \setminus \{x_0\}\bigr)$.

To this end, let ϱ be an admissible function for $M(\Gamma)$. Now ϱ is admissible for each $M(\Gamma_i)$ and no point $x \in \mathbf{R}^n$ belongs to more than three of the sets A_i; hence

$$\sum_{i=0}^{\infty} M(\Gamma_i) \leq \sum_{i=0}^{\infty} \int_{A_i} \varrho^n \, dm \leq 3 \int_{\mathbf{R}^n} \varrho^n \, dm.$$

The inequality (2.8) follows.

Observe that by [GM2, 2.18] and [HK, 2.6]

$$M(\Gamma_i) = \operatorname{cap}(E_i),$$

where $E_i = (C \cap \overline{A}_i, K \cap \overline{A}_i; A_i)$ is a condenser whose capacity is defined as

$$\operatorname{cap} E_i = \inf \int_A |\nabla u|^n \, dm; \tag{2.9}$$

here the infimum is taken over all functions $u \in C^1(A_i)$, continuous in $A_i \cup \bigl(\partial A_i \cap (C \cup K)\bigr)$ with $u|K \cap \overline{A}_i \geq 1$, $u|C \cap \overline{A}_i \leq 0$.

Next for each $i = 0,1,\ldots$ choose an admissible function for $\operatorname{cap} E_i$ such that

$$\int_{A_i} |\nabla u_i|^n \, dm \leq \operatorname{cap} E_i + M(\Gamma)/2^{i+1}; \tag{2.10}$$

note that we may assume $M(\Gamma) > 0$ since otherwise C is of zero capacity and the existence of the required radii r_i, $i = 1,2,\ldots$, follows easily. Consider the open sets

$$\mathcal{U}_i = \bigl\{x \in A_i \colon u_i(x) > 1/2\bigr\}, \qquad i = 1,2,\ldots.$$

If $\mathcal{U}_i$ does not contain any $S^{n-1}(x_0,r)$, $r \in (t_{i+1},t_i)$, then each such $S^{n-1}(x_0,r)$ meets both $A_i \setminus \mathcal{U}_i$ and K yielding by [V, 10.12]

$$\begin{aligned} &\operatorname{cap}(\overline{A_i \setminus \mathcal{U}_i}, K \cap \overline{A}_i; A_i) \\ &\qquad = M\bigl(\Delta(\overline{A_i \setminus \mathcal{U}_i}, K \cap \overline{A}_i; A_i)\bigr) \geq b_1 \log 2, \end{aligned} \tag{2.11}$$

where b_1 depends only on n. Next, observe that

$$\begin{aligned} \operatorname{cap}(\overline{A_i \setminus \mathcal{U}_i}, K \cap \overline{A}_i; A_i) &\le 2^n \int_{A_i} |\nabla u_i|^n \, dm \\ &\le 2^n\big(\operatorname{cap} E_i + M(\Gamma)/2^{i+1}\big) = 2^n\big(M(\Gamma_i) + M(\Gamma)/2^{i+1}\big) \\ &\le 2^n M(\Gamma), \end{aligned} \tag{2.12}$$

where we have used the definition of Γ_i and (2.9)–(2.10). Now (2.11) and (2.12) yield a contradiction provided that

$$M(\Gamma) \le \beta = b_1 2^{-n-2} \log 2.$$

We have shown the existence of $S^{n-1}(x_0, r_i) \subset \mathcal{U} \subset \mathbf{R}^n \setminus C$, $t_{i+1} < r_i < t_i$.

It remains to prove that

$$M(\Gamma') \le bm, \tag{2.13}$$

where $\Gamma' = \Delta\big(K', C; \mathbf{R}^n \setminus \{x_0\}\big)$ and $K' = K \cup \bigcup_{i=1}^{\infty} S^{n-1}(x_0, r_i)$.

To this end, write $B_0 = \mathbf{R}^n \setminus \overline{B}^n(x_0, r_1)$, $B_i = B^n(x_0, r_i) \setminus \overline{B}^n(x_0, r_{i+1})$, $i = 1, 2, \dots$, and let $\Gamma_i' = \Delta(K' \cap \overline{B}_i, C \cap \overline{B}_i; B_i)$, $i = 0, 1, 2, \dots$. Since each $\gamma \in \Gamma'$ has a subpath lying in some Γ_i' we conclude that

$$M(\Gamma') \le \sum_{i=0}^{\infty} M(\Gamma_i'). \tag{2.14}$$

Now we estimate $M(\Gamma_i')$.

For $i = 0, 1, \dots$ the function $2|\nabla u_i|$ is admissible for $M(\Gamma_i')$. Thus by (2.9) and (2.10)

$$\begin{aligned} M(\Gamma_i') &\le 2^n \int_{B_i} |\nabla u_i|^n \, dm \le 2^n \int_{A_i} |\nabla u_i|^n \, dm \\ &\le 2^n\big(\operatorname{cap} E_i + M(\Gamma)/2^{i+1}\big). \end{aligned} \tag{2.15}$$

Since $M(\Gamma_i) = \operatorname{cap} E_i$, (2.14) and (2.15) together with (2.8) yield

$$\begin{aligned} M(\Gamma') &\le \sum_{i=0}^{\infty} M(\Gamma_i') \le 2^n \Big(\sum_{i=0}^{\infty} M(\Gamma_i) + M(\Gamma)/2^{i+1}\Big) \\ &\le 2^{n+3} M(\Gamma). \end{aligned}$$

The claim follows with $\beta = b_1 2^{-n-2} \log 2$ and $b = 2^{n+3}$.

Next we produce a useful characterization for the UM-condition.

2.16. Lemma. *Suppose that C is a relatively closed subset of an open set G. Then C satisfies the UM-condition if and only if for every $x_0 \in G$ and every $\varepsilon > 0$ there is a neighborhood $\mathcal{U}$ of x_0 such that for each pair of points x_1, $x_2 \in \mathcal{U}$ there is a continuum $K = K_{x_1 x_2}$ with the properties:*

$$x_1, x_2 \in K, \quad K \setminus \{x_1, x_2\} \subset G \setminus C, \tag{2.17}$$

and

$$M\Big(\Delta(K, C \cup \partial G; G \setminus \{x_1, x_2\})\Big) \le \varepsilon. \tag{2.18}$$

Proof. It is immediate that the condition of the lemma implies the UM-condition. To prove the converse assume that C satisfies the UM-condition. Let $x_0 \in G$ and pick an $r > 0$ such that $F = \overline{B}^n(x_0, r)$ is a compact subset of G. By the UM-condition for each $\varepsilon > 0$ there is $\delta > 0$ such that for each $x \in F$ there is a continuum $K = K_x \subset G$ with properties (2.1), (2.3), and

$$M\Big(\Delta(K, C \cup \partial G; G \setminus \{x\})\Big) \le \frac{\varepsilon}{2\,b}; \tag{2.19}$$

here b is the constant of Lemma 2.5.

Next, fix $\varepsilon > 0$. We may assume that $\varepsilon/(2\,b) \le \beta$, see Lemma 2.5. Write

$$r_0 = \min(r, \delta/8)$$

and $\mathcal{U} = B(x_0, r_0)$. Then $\mathcal{U}$ is a neighborhood of x_0. Let x_1, $x_2 \in \mathcal{U}$ and pick continua $K_1 = K_{x_1}$, $K_2 = K_{x_2}$ as in (2.19). By Lemma 2.5 we may replace K_j with continua K'_j containing the spheres $S^{n-1}(x_j, r^j_i)$, $i = 1, 2, \ldots$, $j = 1, 2$; note that

$$M\Big(\Delta(K'_j, C \cup \partial G; \mathbf{R}^n \setminus \{x_0\})\Big) \le \frac{b\,\varepsilon}{2\,b} = \frac{\varepsilon}{2}, \tag{2.20}$$

cf. (2.7).

Since $|x_1 - x_2| < \delta/4$, the continua K'_1 and K'_2 meet each other. Hence $K = K'_1 \cup K'_2$ is a continuum with property (2.17). Furthermore, by (2.20)

$$\begin{aligned} &M\Big(\Delta(K, C \cup \partial G; G \setminus \{x_1, x_2\})\Big) \\ &\quad \le \sum_{j=1}^{2} M\Big(\Delta(K'_j, C \cup \partial G; G \setminus \{x_j\})\Big) \le \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon. \end{aligned}$$

This is (2.18) and thus the continuum K has the desired properties.

For $\alpha > 0$ and $C \subset \mathbf{R}^n$ we let $\mathcal{H}^\alpha(C)$ denote the usual α-dimensional (outer) Hausdorff measure of C. The Hausdorff dimension of C is written as $\dim_{\mathcal{H}}(C)$.

2.21. Lemma. *Suppose that C satisfies the M-condition. Then $\dim_{\mathcal{H}}(C) = 0$. In particular, C is totally disconnected.*

Proof. By [MS, 3.1] the n-capacity density of C is $= 0$ at each point $x \in C$. By [M2, 3.8] this implies that $\mathcal{H}^{\alpha}(C) = 0$ for every $\alpha > 0$. Consequently $\dim_{\mathcal{H}}(C) = 0$ as required.

2.22. Remarks. (a) If C satisfies the UM-condition, then C satisfies the M-condition as well. Hence Lemma 2.21 holds for sets C satisfying the UM-condition.

(b) In [M2] a set C satisfying the M-condition but of positive n-capacity was constructed. A closer look at the construction shows that C also satisfies the UM-condition. Thus there exist sets satisfying the UM-condition with positive n-capacity.

3. Continuous removability

Throughout this chapter G is a domain in $\mathbf{R}^n$ and C is a closed (relative to G) subset of G. We are mainly interested in the following problem: Suppose that $f\colon G \to \mathbf{R}^n$ is continuous and quasiregular in $G \setminus C$. Under which conditions is f quasiregular in G?

The most difficult part in proving removability theorems for quasiregular mappings $f\colon G \setminus C \to \mathbf{R}^n$ is to show that f is ACL^n in G. In most cases the ACL-property is trivial and hence it remains to show that $|f'|$ belongs to $\operatorname{loc} L^n(G)$. This fact is demonstrated in our first lemma.

3.1. Lemma. *Suppose that $\mathcal{H}^{n-1}(C) = 0$ and that $f\colon G \setminus C \to \mathbf{R}^n$ is a K-quasiregular mapping. If each $x_0 \in C$ has a neighborhood $\mathcal{U}$ such that*

$$\int_{\mathcal{U}\setminus C} |f'|^n \, dm < \infty, \tag{3.2}$$

then f extends to a K-quasiregular mapping $f^*\colon G \to \mathbf{R}^n$.

Proof. Let $x_0 \in C$ and pick a neighborhood $\mathcal{U}$ of x_0 as above. Since $\mathcal{H}^{n-1}(C) = 0$, f is ACL in $\mathcal{U}$ and by (3.2), f is ACL^n in $\mathcal{U}$. On the other hand, $|f'(x)|^n \le KJ(x,f)$ a.e. in $\mathcal{U}$, and these conditions imply that f has a continuous extension f^* to $\mathcal{U}$, see for example [BI, 2.1, 5.2]. The lemma follows.

For the next lemma we recall that a mapping $f\colon G \to \mathbf{R}^n$ is light if $f^{-1}(y)$ is a totally disconnected set for each $y \in \mathbf{R}^n$.

3.3. Lemma. *Let $f\colon G \to \mathbf{R}^n$ be continuous and light. If f is K-quasiregular in $G \setminus C$ and $m(f(C)) = 0$, then each $x \in C$ has a neighborhood $\mathcal{U}$ with*

$$\int_{\mathcal{U}\setminus C} |f'|^n \, dm < \infty.$$

Proof. Note first that f is discrete, open, and sense-preserving in each component of $G \setminus C$—this follows from the quasiregularity, see [MRV1, 2.26], and the lightness of f.

Fix $x_0 \in C$, and pick a domain D such that $x_0 \in D$, $\overline{D} \subset G$, and $f^{-1}(f(x_0)) \cap \partial D = \emptyset$; this is possible because f is light and hence $f^{-1}(f(x_0))$ is of topological dimension zero. Let V be the $f(x_0)$-component of $\mathbf{R}^n \setminus f(\partial D)$ and let $\mathcal{U}$ be the x_0-component of $f^{-1}(V)$. Then $\mathcal{U}$ is an open neighborhood of x_0. If $y \in V \setminus f(C)$, then

$$N(y, f, \mathcal{U}) \le \sum_{x \in f^{-1}(y) \cap \mathcal{U}} i(x, f) = \mu(y, f, \mathcal{U}) \tag{3.4}$$

by the properties of the topological index μ, see [MRV1, p. 6 and p. 11] for the definitions of N, i, and μ. Next observe that $f(\partial \mathcal{U}) \subset \partial V$, and hence y and $f(x_0)$ belong to the same component of $\mathbf{R}^n \setminus f(\partial \mathcal{U})$. But this means that

$$\mu(y, f, \mathcal{U}) = \mu\big(f(x_0), f, \mathcal{U}\big),$$

and hence by (3.4)

$$N(y, f, \mathcal{U}) \le \mu\big(f(x_0), f, \mathcal{U}\big) = m < \infty \tag{3.5}$$

for all $y \in V \setminus f(C)$. Since $m(f(C)) = 0$, (3.5) holds for a.e. y in V. On the other hand [MRV1, 2.14] yields

$$\begin{aligned}\int_{\mathcal{U} \setminus C} |f'|^n \, dm \le K \int_{\mathcal{U} \setminus C} J(x, f) \, dm &= K \int_{\mathbf{R}^n} N(y, f, \mathcal{U} \setminus C) \, dm \\ &\le \int_{\mathbf{R}^n} N(y, f, \mathcal{U}) \, dm \\ &= \int_{V \setminus f(C)} N(y, f, \mathcal{U}) \, dm < \infty,\end{aligned}$$

where (3.5) is used at the last step.

A mapping $f\colon G \to \mathbf{R}^n$ is said to be *locally Hölder continuous* if each $x_0 \in G$ has a neighborhood $\mathcal{U}$ such that for some constants $0 < \alpha \le 1$ and $M < \infty$

$$\big|f(x) - f(y)\big| \le M\,|x - y|^\alpha \qquad \text{for all } x, y \in \mathcal{U}.$$

Further, f is said to be *locally Hölder continuous with exponent* α if the above constant α is independent of x_0.

3.6. Theorem. *Suppose that $\mathcal{H}^\lambda(C) = 0$ for some λ, $0 < \lambda \le n - 1$, and let $f\colon G \to \mathbf{R}^n$ be light and locally Hölder continuous with exponent $\alpha \ge \lambda/n$. If f is K-quasiregular in $G \setminus C$, then f is K-quasiregular in G.*

Proof. By Lemmas 3.1 and 3.3 it suffices to show that $m(f(C)) = 0$.

To this end, let F be a compact subset of C. Since f is locally Hölder continuous with exponent α, there is a neighborhood $\mathcal{U}$ of F such that

$$|f(x) - f(y)| \le M\,|x-y|^\alpha \tag{3.7}$$

for all x, $y \in \mathcal{U}$, where M is independent of x and y.

Let $\varepsilon > 0$. Since $\mathcal{H}^\lambda(F) = 0$, there is a covering of F by balls $B^n(x_i, r_i)$, $r_i \le 1$, such that $B^n(x_i,r_i) \subset \mathcal{U}$ and

$$\sum_{i=1}^\infty r_i^\lambda < \varepsilon. \tag{3.8}$$

Now $f\bigl(B^n(x_i,r_i)\bigr)$, $i = 1,2,\ldots$, is a covering of $f(F)$ and hence

$$\begin{aligned} m(f(F)) &\le \Omega_n \sum_{i=1}^\infty \operatorname{dia}\Bigl(f\bigl(B^n(x_i,r_i)\bigr)\Bigr)^n \\ &\le \Omega_n\, M^n \sum_{i=1}^\infty \operatorname{dia}\bigl(B^n(x_i,r_i)\bigr)^{\alpha n} \le \Omega_n\, M^n\, 2^{\alpha n} \sum_{i=1}^\infty r_i^\lambda \\ &\le \Omega_n\, M^n\, 2^{\alpha n}\, \varepsilon; \end{aligned}$$

here (3.7) and (3.8) are also used. Letting $\varepsilon \to 0$ we obtain $m(f(F)) = 0$. Thus $m(f(C)) = 0$ as desired.

3.9. Theorem. *Suppose that $\mathcal{H}^\lambda(C) = 0$, $0 < \lambda \le 1$, and that $f\colon G \to \mathbf{R}^n$ is locally Hölder continuous with exponent $\alpha \ge \lambda/n$. If f is K-quasiregular in $G \setminus C$, then f is K-quasiregular in G.*

Proof. Since $\mathcal{H}^1(C) = 0$, $G \setminus C$ is a domain. If $f|G\setminus C$ is constant, then the claim is clear. Otherwise $f|G\setminus C$ is discrete and since $\mathcal{H}^1(C) = 0$, C is totally disconnected. Hence f is light. The proof now follows from Theorem 3.6.

3.10. Remarks (a) For large values of K Theorem 3.9 is better than Theorem 1.5. Note that the inequality $\alpha \ge \lambda/n$ does not include K.

(b) Theorem 3.9 gives the following result: If $\dim_{\mathcal{H}}(C) = 0$ and if $f\colon G \to \mathbf{R}^n$ is locally Hölder continuous in G and K-quasiregular in $G \setminus C$, then f is K-quasiregular in G.

The preceding results have their roots in [MRV2, 4.1]. Next we relax the Hölder continuity condition of Theorem 3.6 slightly. If D is an open, proper subset of $\mathbf{R}^n$, we let $W = \{Q\}$ denote the Whitney decomposition of D into cubes Q. This means that each $Q \in W$ is a closed cube whose edges are of length 2^{-i} for some integer i and parallel to the axes and the diameter of Q is approximately proportional to the distance $d(Q, \mathbf{R}^n \setminus D)$, more precisely

$$\operatorname{dia}(Q) \le d(Q, \mathbf{R}^n \setminus D) \le 4 \operatorname{dia}(Q).$$

Moreover, the interiors of Q are mutually disjoint and $\cup Q = D$. For the construction of a Whitney decomposition W see [S, p. 16]. The Whitney decomposition W of D is not unique but this fact has no importance in the following.

Suppose that $f\colon D \to \mathbf{R}^n$ and $0 < \alpha \le 1$. We say that f belongs to $\operatorname{loc Lip}_\alpha(D)$ if there is $M < \infty$ such that

$$|f(x) - f(y)| \le M\,|x-y|^\alpha$$

for each x, $y \in Q$ and for each $Q \in W$ where $W = \{Q\}$ is a Whitney decomposition of D. For the properties of the class $\operatorname{loc Lip}_\alpha(D)$ see [GM1]. Note that the class $\operatorname{loc Lip}_\alpha(D)$ is properly contained in the class of locally Hölder continuous mappings in D with exponent α.

Finally we recall the definition of the Minkowski dimension of a compact set $F \subset \mathbf{R}^n$. For $\lambda > 0$ and $r > 0$ write

$$M_r^\lambda(F) = \inf\Bigl\{k\,r^\lambda : F \subset \bigcup_{i=1}^{k} B^n(x_i, r)\Bigr\}$$

and let

$$M^\lambda(F) = \limsup_{r \to 0} M_r^\lambda(F).$$

The Minkowski dimension of F is then defined similarly to the Hausdorff dimension as

$$\dim_{\mathcal{M}}(F) = \inf\{\lambda > 0 : M^\lambda(F) < \infty\}.$$

Note that $\dim_{\mathcal{M}}(F) \ge \dim_{\mathcal{H}}(F)$—for the properties of $\dim_{\mathcal{M}}$ see e.g. [MV].

3.11. Lemma. *Let $f\colon G \setminus C \to \mathbf{R}^n$ be K-quasiregular. Suppose that $x \in C$ has a neighborhood $\mathcal{U}$ such that $\dim_{\mathcal{M}}(C \cap \overline{\mathcal{U}}) = \lambda < n$ and f lies in $\operatorname{loc Lip}_\alpha(\mathcal{U} \setminus C)$, $\alpha > \lambda/n$. Then there is a neighborhood V of x with*

$$\int_{V \setminus C} |f'|^n \, dm < \infty.$$

Proof. We may assume that $\mathcal{U} = B^n(x, r)$ and that $\overline{\mathcal{U}} \subset G$. Write $F = C \cap \overline{\mathcal{U}}$, and let W be the Whitney decomposition of $\mathcal{U} \setminus F$.

For each $Q \in W$ let Q' denote the cube with the same center as Q, sides parallel to those of Q and edge length $\ell(Q') = (3/2)\,\ell(Q)$. Note that Q' is covered with cubes $\widetilde{Q} \in W$ satisfying $\widetilde{Q} \cap Q \ne \emptyset$ and that

$$\tfrac14 \ell(Q) \le \ell(\widetilde{Q}) \le 4\,\ell(Q)$$

for each such cube $\widetilde{Q}$.

Next pick a constant $b_2 = b_2(n)$ so that $Q' \subset B^n(x,r)$ whenever $Q \in W$ satisfies $Q \cap B^n(x, r/b_2) \neq \emptyset$; this is possible by the properties of W since $x \in F$. We complete the proof by showing that

$$\int_{V \setminus C} |f'|^n \, dm < \infty,$$

where $V = B^n(x, r/b_2)$. Clearly we may assume that $2r \leq 1$.

Notice first that

$$\int_{V \setminus C} |f'|^n \, dm \leq \sum_{i=1}^{\infty} \sum_{j=1}^{N_i} \int_{Q_{ij}} |f'|^n \, dm \tag{3.12}$$

where each $Q_{ij} \in W$ satisfies $\ell(Q_{ij}) = 2^{-i}$, $Q'_{ij} \subset B^n(x,r) \setminus F \subset G \setminus C$, and N_i is the number of the cubes $Q_{ij} \in W$ that intersect V. Since $Q'_{ij} \subset G \setminus C$ and $f\colon G \setminus C \to \mathbf{R}^n$ is K-quasiregular, [GLM, Lemma 4.2], see also [BI, 6.1], yields

$$\int_{Q_{ij}} |f'|^n \, dm \leq b_3 \max_{y \in Q'_{ij}} \left| f(y) - f(y_{ij}) \right|^n, \tag{3.13}$$

where b_3 depends only on K and n and y_{ij} is the center of Q_{ij}. Note that (3.13) follows from the standard estimate of [GLM, p. 54] since each coordinate function of $f - f(y_{ij})$ is an F-extremal, cf. [GLM, p. 71], with an appropriate F and $\operatorname{cap}_n(Q_{ij}, \operatorname{int} Q'_{ij}) = c_n$ where c_n depends only on n—here cap_n refers to the n-capacity.

Next, since $f \in \operatorname{loc} \operatorname{Lip}_\alpha(\mathcal{U} \setminus C)$, we obtain

$$\left| f(y) - f(y_{ij}) \right| \leq 5 \sqrt{n}\, M \, 2^{-i\alpha} = b_4 \, 2^{-i\alpha} \tag{3.14}$$

for each $y \in Q'_{ij}$; here we have used the fact that every $y \in Q'_{ij} \setminus Q_{ij}$ is contained in a cube $Q \in W$ meeting Q_{ij} and hence $\ell(Q) \leq 4\ell(Q_{ij})$. On the other hand, by [MV, 3.9]

$$N_i \leq b_5 \, 2^{i\lambda_1}, \qquad i = 1, 2, \ldots \tag{3.15}$$

for any $\lambda_1 > \lambda = \dim_{\mathcal{M}}(F)$ for some b_5 independent of i. Combining (3.12)–(3.15) we obtain

$$\int_{V \setminus C} |f'|^n \, dm \leq b_6 \sum_{i=1}^{\infty} 2^{i(\lambda_1 - \alpha n)}$$

where $b_6 = b_3 \, b_4^n \, b_5$. Since $\alpha > \lambda/n$, the claim follows.

3.16. Theorem. *Let $f\colon G\setminus C \to \mathbf{R}^n$ be K-quasiregular. Suppose that each $x \in C$ has a neighborhood $\mathcal{U}$ such that $\dim_{\mathcal{M}}(C \cap \overline{\mathcal{U}}) = \lambda < n-1$ and f lies in $\operatorname{loc}\operatorname{Lip}_\alpha(\mathcal{U}\setminus C)$ for some $\alpha > \lambda/n$. Then f extends to a K-quasiregular mapping $f^*\colon G \to \mathbf{R}^n$.*

Proof. The claim follows from Lemmas 3.1 and 3.11.

3.17. Theorem. *Suppose that $\dim_{\mathcal{M}}(F) \le \lambda < n-1$ for each compact $F \subset C$ and that $f\colon G \to \mathbf{R}^n$ is locally Hölder continuous with exponent $\alpha > \lambda/n$. If f is K-quasiregular in $G\setminus C$, then f is K-quasiregular in G.*

Proof. If f is locally Hölder continuous with exponent α in G, then each $x \in C$ has a neighborhood $\mathcal{U}$ such that $f \in \operatorname{loc}\operatorname{Lip}_\alpha(\mathcal{U}\setminus C)$. Since $\dim_{\mathcal{M}}(C\cap\overline{\mathcal{U}}) \le \lambda$, the claim follows from Theorem 3.17.

3.18. Remark. It may happen that $\dim_{\mathcal{M}}(C) > \dim_{\mathcal{H}}(C)$, hence Theorem 3.17 does not imply Theorem 3.9. Note that there are countable closed sets C with $\dim_{\mathcal{M}}(C) > 0$.

4. A removability theorem for quasiregular mappings

Suppose that G is a domain in $\mathbf{R}^n$ and C is a relatively closed subset of G.

4.1. Theorem. *Suppose that $f\colon G\setminus C \to \mathbf{R}^n$ is a K-quasiregular mapping omitting a set of positive n-capacity. If C satisfies the UM-condition, then f has a K-quasimeromorphic extension $f^*\colon G \to \mathbf{R}^n \cup \{\infty\}$.*

The formulation of the theorem needs an explanation. First, the mapping f^* may take the value ∞. Hence, as in the classical analytic plane case, we say that $f^*\colon G \to \mathbf{R}^n \cup \{\infty\}$ is K-quasimeromorphic if for each $x \in G$ either f^* is K-quasiregular or, in the case $f^*(x) = \infty$, $g \circ f$ is K-quasiregular at a neighborhood of x; here g is a sense-preserving Möbius transformation such that $g(\infty) \ne \infty$. Next let F be a closed proper subset of $\mathbf{R}^n$. If $F_1 \subset \mathbf{R}^n \setminus F$ is a nondegenerate continuum, then we write $\Gamma(F_1) = \Delta(F_1, F; \mathbf{R}^n)$. Now $M\bigl(\Gamma(F_1)\bigr) > 0$ or $M\bigl(\Gamma(F_1)\bigr) = 0$ for each such continuum F_1. In the first case we say that F is of *positive n-capacity* and write $\operatorname{cap}_n F > 0$. In the second case F is said to be of *zero n-capacity*; this we write $\operatorname{cap}_n F = 0$. Since

$$M\bigl(\Gamma(F_1)\bigr) = \operatorname{cap}_n(F_1, \mathbf{R}^n \setminus F),$$

this definition agrees with the usual definition of a set of zero n-capacity, see [MRV2, p. 6] or [HKM].

Finally, note that if f is bounded, then f omits a set of positive n-capacity and the mapping f^* in Theorem 4.1 is K-quasiregular.

To prove Theorem 4.1 we need three lemmas; we assume that $f\colon G\setminus C \to \mathbf{R}^n$ and C satisfy the conditions of the theorem.

4.2. Lemma. *The mapping f has a continuous extension $f^*\colon G \to \mathbf{R}^n \cup \{\infty\}$.*

Proof. We may assume that f is non-constant. It suffices to show that f can be extended continuously to each point $x_0 \in C$. Fix $x_0 \in C$ and let $\varepsilon > 0$. Pick a neighborhood $\mathcal{U} \subset G$ of x_0 such that the conditions (2.17) and (2.18) of Lemma 2.16 hold. Let x_1, $x_2 \in \mathcal{U} \setminus C$ and let $K_{x_1 x_2}$ be a continuum with the properties in Lemma 2.16. Write Γ for the family of paths joining $f(K_{x_1x_2})$ to $\mathbf{R}^n \setminus f(G \setminus C)$ in $\mathbf{R}^n \cup \{\infty\}$; note that $f(G \setminus C)$ is an open subset of $\mathbf{R}^n$ because f is open. Let Γ^* be the family of maximal lifts (under f) of the paths in Γ starting at $K_{x_1x_2}$, see [MRV3, 3.11]. Then each $\gamma^* \in \Gamma^*$ ends either in C or in $\partial G \cup \{\infty\}$. By the fundamental modulus inequality for quasiregular mappings, see [P],

$$M(\Gamma) \le K^{n-1}\, M(\Gamma^*); \tag{4.3}$$

note that the inner dilatation $K_I(f)$ of f satisfies $K_I(f) \le K^{n-1}$, see [MRV1, pp. 14–15]. On the other hand, condition (2.18) of Lemma 2.16 yields

$$M(\Gamma^*) \le \varepsilon. \tag{4.4}$$

Thus (4.3) and (4.4) imply

$$M(\Gamma) \le K^{n-1}\, \varepsilon. \tag{4.5}$$

Next write $t = q\big(f(K_{x_1x_2})\big)$—the spherical diameter of $f(K_{x_1x_2})$, see [MRV2, 3.10]. Since $\operatorname{cap}_n\big(\mathbf{R}^n \setminus f(G \setminus C)\big) > 0$, [MRV2, Lemma 3.1] together with (4.5) shows that $t \le \delta$ where $\delta \to 0$ as $\varepsilon \to 0$. Since the spherical distance $q\big(f(x_1), f(x_2)\big)$ of $f(x_1)$ and $f(x_2)$ satisfies

$$q\big(f(x_1), f(x_2)\big) \le q\big(f(K_{x_1x_2})\big) \le \delta,$$

the Cauchy criterion shows that f has a continuous extension to x_0.

4.6. Remark. It was proved in [Vu] that if $f\colon G \setminus C \to \mathbf{R}^n$ is quasiregular and omits a set of positive n-capacity, then f has a unique asymptotic limit at $x_0 \in C$ provided that x_0 satisfies the M-condition with respect to C. Lemma 4.2 shows that a slightly stronger assumption yields a continuous extension.

4.7. Lemma. *The mapping $f^*\colon G \to \mathbf{R}^n \cup \{\infty\}$ is either a constant or light and open.*

Proof. Suppose that f^* is not a constant. Fix $y \in \mathbf{R}^n \cup \{\infty\}$. Then

$$f^{*-1}(y) \subset f^{-1}(y) \cup C$$

and since $f^{-1}(y)$ is a discrete set of points in $G \setminus C$ and since C is totally disconnected, see Lemma 2.21 and Remark 2.22 (a), $f^{*-1}(y)$ is a subset of a totally disconnected set. Thus f^* is light.

Next we show that f^* is open. Note that f is open at any $x_0 \in G \setminus C$. Suppose that $x_0 \in C$. Since f^* is light and C is totally disconnected, there are arbitrarily small connected neighborhoods $D \subset G$ of x_0 such that

$$\partial D \subset G \setminus C \tag{4.8}$$

and

$$f^{*-1}(x_0) \cap \partial D = \emptyset. \tag{4.9}$$

Fix such a domain D. It suffices to show that $f^*(x_0) \in \operatorname{int} f^*(D)$. Since f^* is continuous, we may assume that $f^*(x_0) \neq \infty$ and that $f^*(\overline{D})$ is a compact subset of $\mathbf{R}^n$.

Denote the $f^*(x_0)$-component of $\mathbf{R}^n \setminus f(\partial D)$ by D', and let $V = D' \setminus f^*(\overline{D})$. Since $f^*(\overline{D})$ is compact, V is open. We shall show that $V = \emptyset$.

Suppose not. Pick a connected component V' of V. If $\partial V' \cap D' = \emptyset$, then $\partial V' \subset \partial D'$ and hence $V' = D'$ which is impossible because $f^*(x_0) \in D' \setminus V'$. Thus there is $y \in \partial V' \cap D'$. Now $y \in f^*(\overline{D}) \setminus f(\partial D)$, hence there is a point x in D with $f^*(x) = y$. Pick a continuum K_x as in the M-condition for x with $K_x \setminus \{x\} \subset D \setminus C$ and

$$M\Big(\Delta(K_x, C \cup \partial G; G \setminus \{x\})\Big) \leq 1.$$

On the other hand, $y = f^*(x)$ is a boundary point of a domain V', hence for each $T > 0$ there is a non-degenerate continuum $K' \subset V'$ such that

$$M\Big(\Delta(f^*(K_x), K'; \mathbf{R}^n)\Big) \geq T; \tag{4.10}$$

note that $f^*(K_x)$ is a non-degenerate continuum containing the point y.

Next, write $\Gamma = \Delta(f^*(K_x), K'; \mathbf{R}^n)$, and let Γ^* be the family of all maximal lifts (under $f|D \setminus C$) of Γ starting at $K_x \setminus \{x\}$. Since $K' \cap f^*(\overline{D}) = \emptyset$, each $\gamma^* \in \Gamma^*$ ends either at $C \cap D$ or at ∂D. Thus

$$\begin{aligned} M(\Gamma^*) &\leq M(\Delta(K_x, C; \mathbf{R}^n \setminus \{x\})) + M(\Delta(K_x, \partial D; \mathbf{R}^n)) \\ &\leq 1 + M < \infty \end{aligned} \tag{4.11}$$

where M is independent of K'; note that $M < \infty$ because K_x is a compact subset of D. Since f is K-quasiregular, we conclude that

$$M(\Gamma) \leq K^{n-1} M(\Gamma^*) \leq K^{n-1}(1 + M).$$

Choosing T in (4.10) large enough we obtain a contradiction. Hence $V = \emptyset$.

Now $D' \setminus f^*(\overline{D}) = V = \emptyset$ and thus $D' \subset f^*(\overline{D})$. Since D' does not meet $f^*(\partial D)$, $D' \subset f^*(D)$ and since D' is an open neighborhood of $f^*(x_0)$ in $f^*(D)$, we have the desired conclusion $f^*(x_0) \in \operatorname{int} f^*(D)$.

4.12. Lemma. *The mapping f^* is locally Hölder continuous in $G\setminus f^{*-1}(\infty)$.*

Proof. Since $f\colon G\setminus C\to\mathbf{R}^n$ is locally Hölder continuous as a quasiregular mapping, see [MRV2, 3.2], it suffices to show that any $x_0\in C$ with $f^*(x_0)\neq\infty$ has a neighborhood $\mathcal{U}$ with $|f^*(x)-f^*(y)|\le M|x-y|^\alpha$ for all x, $y\in\mathcal{U}$, where $\alpha>0$ and $M<\infty$ are independent of the points x and y.

To this end, fix such a point $x_0\in C$ and pick a ball $B^n(x_0,8r)\subset G$ such that $\infty\notin f^*(\overline{B}^n(x_0,6r))$ and for any $x\in B^n(x_0,r)$ there is a continuum K_x with $x\in K_x$, $K_x\setminus\{x\}\subset G\setminus C$, $8r\le \operatorname{dia}(K_x)\le 9r$, and $M(K,C\cup\partial G;\mathbf{R}^n\setminus\{x\})\le\beta$, where β is the constant of Lemma 2.5. This is possible because C satisfies the UM-condition and f^* is continuous. Let $x\in B^n(x_0,r)$ and pick a continuum K_x as above. Let $r_1>r_2>\dots$ be the sequence of radii given by Lemma 2.5. From this sequence we select every second and still denote this new sequence by (r_i). Write

$$L_i=\max_{y\in\overline{B}(x,r_i)}|f^*(y)-f^*(x)|,\qquad i=1,2,\dots. \tag{4.13}$$

Since f^* is open by Lemma 4.7,

$$L_i=\max_{y\in S^{n-1}(x,r_i)}|f(y)-f^*(x)|; \tag{4.14}$$

note that $S^{n-1}(x,r_i)\subset G\setminus C$.

For each $i=1,2,\dots$ let Γ_i be the family of paths which connect $f(S^{n-1}(x,r_{i+1}))$ to $f(S^{n-1}(x,r_i))$ in $\mathbf{R}^n$. Let Γ_i^* be the family of maximal lifts under $f|B^n(x,r_i)\setminus C$ of Γ_i starting at $S^{n-1}(x,r_{i+1})$. Each path γ^* in Γ_i^* ends either in C or in $S^{n-1}(x,r_i)$. Thus

$$\begin{aligned}M(\Gamma_i^*)&\le M\Big(\Delta\big(S^{n-1}(x,r_{i+1}),C;\mathbf{R}^n\big)\Big)\\&\quad+M\Big(\Delta\big(S^{n-1}(x,r_{i+1}),S^{n-1}(x,r_i);\mathbf{R}^n\big)\Big)\\&\le b\beta+\omega_{n-1}(\log r_i/r_{i+1})^{1-n}\\&\le b\beta+\omega_{n-1}\Big(\log\frac{\operatorname{dia}(K_x)/2^{2i+2}}{\operatorname{dia}(K_x)/2^{2(i+1)+1}}\Big)^{1-n}\\&=b\beta+\omega_{n-1}(\log 2)^{1-n}=b_1;\end{aligned} \tag{4.15}$$

here we used the fact that we had chosen every second of the original radii of Lemma 2.5. On the other hand, (4.15), [V, 6.4], and the quasiregularity of f imply

$$\begin{aligned}M(\Gamma_i)&\le M\Big(\Delta\big(f(S^{n-1}(x,r_{i+1})),\\&\qquad f(S^{n-1}(x,r_i)\cup\partial f(G\setminus C);f(G\setminus C))\Big)\\&\le K^{n-1}M(\Gamma_i^*)\le K^{n-1}b_1.\end{aligned} \tag{4.16}$$

Let $y_0 \in S^{n-1}(x, r_{i+1})$ be such that $L_{i+1} = |f(y_0) - f(x)|$ and write $z_0 = L_i(f(y_0) - f(x)) + f(x)$. Since f^* is open, for each $s \in (L_i - L_{i+1}, L_i)$ the sphere $S^{n-1}(z_0, s)$ meets both $f(S^{n-1}(x, r_i))$ and $f(S^{n-1}(x, r_{i+1}))$. Hence [V, 10.12] yields

$$M(\Gamma_i) \geq b_2 \log \frac{L_i}{L_i - L_{i+1}}. \tag{4.17}$$

Here b_2 depends only on n. Now (4.16) and (4.17) give

$$L_{i+1} \leq b_3 L_i, \qquad i = 1, 2, \ldots, \tag{4.18}$$

where $b_3 = (e^t - 1)/e^t$, $t = K^{n-1} b_1 / b_2$, is independent of x and i.

From (4.18) we obtain by iteration

$$L_i \leq b_3^{i-1} L_1, \qquad i = 1, 2, \ldots. \tag{4.19}$$

Finally, let $y \in B^n(x_0, r)$. Note that $r_1 > \operatorname{dia}(K_x)/4 \geq 2r > |x - y|$; hence we may pick an integer i such that

$$r_{i+1} \leq |x - y| < r_i.$$

Now (4.13) and (4.19) imply

$$|f^*(x) - f^*(y)| \leq L_i \leq b_3^{i-1} L_1. \tag{4.20}$$

On the other hand,

$$|x - y| \geq r_{i+1} \geq \operatorname{dia}(K_x)/2^{2i+2} \geq 2r/2^i;$$

hence

$$i \geq \log\left(\frac{2r}{|x - y|}\right)^{1/(2\log 2)}.$$

By (4.20) this yields (observe that $b_3 < 1$)

$$|f^*(x) - f^*(y)| \leq M|x - y|^{\alpha},$$

where $\alpha = -\log b_3 / 2\log 2 > 0$ and

$$\begin{aligned} M &= b_3^{-1} L_1 (2r)^{(\log b_3)/(2\log 2)} \\ &\leq 2 b_3^{-1} (2r)^{(\log b_3)/(2\log 2)} \max_{z \in \overline{B}^n(x_0, 6r)} |f(z)| < \infty \end{aligned}$$

are independent of x and y. The lemma follows.

Proof for Theorem 4.1. Since the definition of quasimeromorphic mappings is local, it suffices to show that any $x_0 \in C$ has a neighborhood $\mathcal{U}$ with $f^*|\mathcal{U}$ K-quasimeromorphic; here f^* is the mapping given by Lemma 4.2. Fix $x_0 \in C$, and pick a sense-preserving Möbius transformation g with $g(\infty) \neq \infty$. Assume first that $f^*(\infty) \neq \infty$; then Lemmas 4.2 and 4.12 imply that f^* is locally Hölder continuous in a neighborhood $\mathcal{U}$ of x_0. Thus f^* is K-quasiregular in $\mathcal{U}$ by Lemma 2.21 and Remark 3.10 (b).

Suppose finally that $f^*(x_0) = \infty$. Now $g \circ f^*$ is bounded in a neighborhood $\mathcal{U}$ of x_0 and K-quasiregular in $\mathcal{U} \setminus C$. Hence Lemma 4.2 yields a continuous extension $(g \circ f)^*\colon \mathcal{U} \to \mathbf{R}^n \cup \{\infty\}$. Moreover, $(g \circ f)^* = g \circ f^*$ and $(g \circ f)^*(x_0) \neq \infty$; hence the proof follows by applying the above reasoning to $(g \circ f)^*$.

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P. Koskela
University of Jyväskylä
Department of Mathematics
Seminaarinkatu 15
SF-40100 Jyväskylä
Finland

O. Martio
University of Jyväskylä
Department of Mathematics
Seminaarinkatu 15
SF-40100 Jyväskylä
Finland

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