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MEAN VALUES AND THERMIC MAJORIZATION OF SUBTEMPERATURES

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1. Introduction

A classical result of F. Riesz states that the mean values of subharmonic functions over concentric spheres of radius r, form convex functions of $\log r$ or r^{2-n}, depending on the dimension of the space [2, p. 24]. The corresponding result for subtemperatures, in which the mean values are taken over level surfaces of the Green function, was presented in [6], along with some of its consequences. In the present paper, we give a different, more elementary proof of the theorem for subtemperatures, as well as two new results on thermic majorization, one of which gives a criterion in terms of the mean values and depends upon consequences of the convexity theorem.

The principal result on thermic majorization, Theorem 3, is analogous to a well-known, elementary result on the harmonic majorization of subharmonic functions: a subharmonic function on $\mathbf{R}^n$ has a harmonic majorant there if and only if its mean values over all spheres centred at the origin form a bounded function. Again the mean values of a subtemperature over level surfaces of the Green function are used, and because of their geometry the whole space is replaced by a half-space $\mathbf{R}^n \times]-\infty, a[$. The result requires the boundedness of means associated with a sequence of points rather than just one, and there are many technical difficulties which do not arise in the subharmonic case. For example, we have to prove that a subtemperature which has a thermic majorant on the sets $\mathbf{R}^n \times]-\infty, a_j[$ for all $j \in \mathbf{N}$, must also possess one on the union of those half-spaces. This result has an illuminating generalization to arbitrary open sets, which is given in Theorem 2.

We work in $\mathbf{R}^{n+1}$, and denote a typical point by p or (x,t), as convenient. A particular point p_0 is assumed without comment to be (x_0, t_0). A temperature is a solution of the heat equation

$$\sum_{i=1}^{n} D_i^2 u - D_t u = 0.$$

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We use θ to denote the heat operator, and θ^* is adjoint (obtained from θ by changing the sign of D_t).

For all $x \in \mathbf{R}^n$, we put $W(x,t) = (4\pi t)^{-n/2}\exp(-\|x\|^2/4t)$ if $t > 0$, and $W(x,t) = 0$ if $t \leq 0$. Then the Green function G for θ on $\mathbf{R}^{n+1}$ is given by $G(p,q) = W(x-y,t-s)$, where $p = (x,t)$ and $q = (y,s)$.

If $p_0 \in \mathbf{R}^{n+1}$ and $c > 0$, the fundamental domain $\Omega(p_0,c)$ is defined as $\{p \in \mathbf{R}^{n+1} : G(p_0,p) > (4\pi c)^{-n/2}\}$; it is convex and bounded. Its boundary is a smooth surface with equation $\|x_0 - x\| = \left[2n(t_0-t)\log\left(c/(t_0-t)\right)\right]^{1/2}$, together with $\{p_0\}$. If $(x,t) \in \mathbf{R}^n \times]0,\infty[$, we put

$$Q(x,t) = \|x\|^2\left[4\|x\|^2t^2 + \left(\|x\|^2 - 2nt\right)^2\right]^{-1/2};$$

we also put $Q(0,0) = 1$. For each fixed $c > 0$, the restriction to $\partial\Omega(p_0,c)$ of the function $(x,t) \mapsto Q(x_0 - x, t_0 - t)$ is continuous, and is positive except for a zero at $(x_0, t_0 - c)$. If w is a function on $\partial\Omega(p_0,c)$, we put

$$\mathcal{M}(w,p_0,c) = (4\pi c)^{-n/2}\int_{\partial\Omega(p_0,c)} Q(x_0-x,t_0-t)w(x,t)\,d\sigma(x,t),$$

where σ denotes surface measure, provided that the integral exists. If u is a temperature on an open set D, and $\bar{\Omega}(p_0,c) \subseteq D$, then $u(p_0) = \mathcal{M}(u,p_0,c)$. In particular, $\mathcal{M}(1,p_0,c) = 1$ for any p_0 and c.

If D is an open set and $p_0 \in D$, we denote by $\Lambda(p_0)$ the set of all points $q \in D \setminus \{p_0\}$ which can be joined to p_0 by a polygonal line in D along which t is strictly increasing as the line is described from q to p_0. A function w on D is called a subtemperature if it is upper semicontinuous, extended real valued but never $+\infty$, real valued on a sequence $\{p_i\}$ such that $D = \bigcup_{i=1}^{\infty}\Lambda(p_i)$, and satisfies $w(p_0) \leq \mathcal{M}(w,p_0,c)$ whenever $\bar{\Omega}(p_0,c) \subseteq D$. If w is a subtemperature on D, a thermic majorant of w on D is a temperature u such that $w \leq u$ on D. If w has a thermic majorant on D, then it has a least one. The basic properties of subtemperatures are given in [4] and [5], and the equivalent class of subparabolic functions is discussed in [2].

2. Convexity of mean values of subtemperatures

In this section we present a more elementary proof of [6, Theorem 2] than was given in [6]. We consider subtemperatures on a domain of the form $A(p_0,c_1,c_2) = \Omega(p_0,c_2) \setminus \bar{\Omega}(p_0,c_1)$, where $p_0 \in \mathbf{R}^{n+1}$ and $0 < c_1 < c_2$. Such a domain corresponds to an annulus in the subharmonic case. We show that, if w is a subtemperature on an open superset of $\bar{A}(p_0,c_1,c_2)$ then $\mathcal{M}(w,p_0,c)$ is a convex function of $c^{-n/2}$ for $c \in [c_1,c_2]$. Our method is based on an idea due to Dinghas [1] in the subharmonic case.

By a smooth function, we mean one for which the partial derivatives that occur in θ exist as continuous functions. For a smooth function u on a domain in $\mathbf{R}^{n+1}$, we put $\nabla_x u = (D_1 u, \dots D_n u)$ and $\|\nabla_x u\| = \left(\sum_{i=1}^n (D_i u)^2\right)^{1/2}$. We use $\langle \cdot, \cdot \rangle$ to denote the inner product in $\mathbf{R}^n$.

It is convenient to first establish some notation and list some elementary formulas. In addition to the functions W and Q given above, for all $(x,t) \in \mathbf{R}^n \times]0, \infty[$ we put

$$L(x,t) = \left[4\|x\|^2 t^2 + \left(\|x\|^2 - 2nt\right)^2\right]^{-1/2}$$

and

$$J(x,t) = 2nt \exp\left(-\|x\|^2/2nt\right) L(x,t);$$

note that $Q(x,t) = \|x\|^2 L(x,t)$. If $F \in \{W, Q, L, J\}$ and $(x_0, t_0) \in \mathbf{R}^{n+1}$, we put $F_0(x,t) = F(x_0 - x, t_0 - t)$ for all $(x,t) \in \mathbf{R}^n \times]-\infty, t_0[$. On $\partial\Omega(p_0, c)$, where

$$(t_0 - t)\exp\left(\|x_0 - x\|^2/2n(t_0 - t)\right) = c,$$

the outward unit normal (ν_x, ν_t) is given by

$$\nu_x = -2(t_0 - t)(x_0 - x)L_0(x,t), \qquad \nu_t = \left(\|x_0 - x\|^2 - 2n(t_0 - t)\right)L_0(x,t).$$

It is useful to have (ν_x, ν_t) in terms of J_0. Since

$$cJ_0(x,t) = 2n(t_0 - t)^2 L_0(x,t) \tag{1}$$

whenever $(x,t) \in \partial\Omega(p_0, c)$, we have

$$\nu_x = -\frac{c(x_0 - x)}{n(t_0 - t)} J_0(x,t), \tag{2}$$

$$\nu_t = \frac{c\left(\|x_0 - x\|^2 - 2n(t_0 - t)\right)}{2n(t_0 - t)^2} J_0(x,t) \tag{3}$$

for such points (x,t). Next,

$$\nabla_x W_0(x,t) = \frac{x_0 - x}{2(t_0 - t)} W_0(x,t), \tag{4}$$

$$D_t W_0(x,t) = \frac{2n(t_0 - t) - \|x_0 - x\|^2}{4(t_0 - t)^2} W_0(x,t), \tag{5}$$

and if $g = W_0^{-2/n}$ we have

$$\theta^* g = \frac{2}{n}\left(\frac{2}{n}+1\right)\frac{\|\nabla_x W_0\|^2}{W_0^2} g \tag{6}$$

since $\theta^* W_0 = 0$. Finally, on $\partial\Omega(p_0, c)$ we have

$$-\langle \nabla_x W_0, \nu_x \rangle = \|x_0 - x\|^2 L_0(x,t) W_0(x,t) = (4\pi c)^{-n/2} Q_0(x,t). \tag{7}$$

We need certain Green identities. If v and w are smooth functions, it is elementary that

$$v\theta w = \sum_{i=1}^{n} D_i(vD_i w) - \langle \nabla_x v, \nabla_x w \rangle - D_t(vw) + wD_t v$$

and

$$w\theta^* v = \sum_{i=1}^{n} D_i(wD_i v) - \langle \nabla_x w, \nabla_x v \rangle + wD_t v.$$

Therefore, if A is any domain for which the divergence theorem is applicable,

$$\iint_A \left(v\theta w + \langle \nabla_x v, \nabla_x w \rangle - wD_t v\right) dx\, dt = \int_{\partial A} \left(\langle v\nabla_x w, \nu_x \rangle - vw\nu_t\right) d\sigma \tag{8}$$

and

$$\iint_A \left(w\theta^* v + \langle \nabla_x w, \nabla_x v \rangle - wD_t v\right) dx\, dt = \int_{\partial A} \langle w\nabla_x v, \nu_x \rangle\, d\sigma. \tag{9}$$

Lemma 1. *Let $p_0 \in \mathbf{R}^{n+1}$, let $0 < c_1 < c_2$, let u be a smooth function on an open superset of $\bar{A}(p_0, c_1, c_2)$, and put $\Omega(c) = \Omega(p_0, c)$ and $\mathcal{M}(c) = \mathcal{M}(u, p_0, c)$ for all $c \in [c_1, c_2]$. Then, if $\kappa_n = 2^{n+1}\pi^{n/2}n^{-1}$ and $c \in]c_1, c_2]$, we have*

$$\kappa_n c^{(n/2)+1} \mathcal{M}_c(c) = \int_{\partial\Omega(c)} \left(\langle \nabla_x u, \nu_x \rangle - u\nu_t\right) d\sigma \tag{10}$$

and

$$\kappa_n \left(c^{(n/2)+1} \mathcal{M}_c(c)\right)_c = \int_{\partial\Omega(c)} J_0 \theta u\, d\sigma. \tag{11}$$

Proof. Let $c \in]c_1, c_2[$, and put $A = A(p_0, c_1, c)$. We want to use (9) with this choice of A and w smooth, but with $v = W_0^{-2/n}/4\pi$, so that the smoothness of

v breaks down at p_0. To prove that this is permissible, we use an approximation argument. Let $t \in]t_0 - c_1, t_0[$, and put

$$\begin{aligned} F_1(t) &= \partial\Omega(p_0, c_1) \cap \big(\mathbf{R}^n \times [t_0 - c_1, t]\big), \\ F_2(t) &= \partial\Omega(p_0, c) \cap \big(\mathbf{R}^n \times [t_0 - c, t]\big), \\ V(t) &= A(p_0, c_1, c) \cap \big(\mathbf{R}^n \times [t_0 - c, t]\big). \end{aligned}$$

Applying (9) on $V(t)$ with $v = W_0^{-2/n}/4\pi$, and using (6) and (7), we obtain

$$\begin{aligned} (12) \iint_{V(t)} &\Big(\frac{2}{n}\Big(\frac{2}{n}+1\Big)\frac{\|\nabla_x W_0\|^2}{W_0^2} vw - \frac{2v}{nW_0}\langle \nabla_x w, \nabla_x W_0\rangle + \frac{2vw}{nW_0} D_t W_0\Big)\, dx\, dt \\ &= \int_{\partial V(t)} \frac{-2wv}{nW_0}\langle \nabla_x W_0, \nu_x\rangle\, d\sigma = \frac{2}{n}\Big(c\int_{F_2(t)} - c_1 \int_{F_1(t)}\Big) wQ_0\, d\sigma. \end{aligned}$$

Since wQ_0 is bounded on $\partial A(p_0, c_1, c)$, as $t \to t_0-$ the last expression tends to

$$(13) \qquad \frac{2}{n}\Big(c\int_{\partial\Omega(c)} - c_1\int_{\partial\Omega(c_1)}\Big) wQ_0\, d\sigma.$$

For the integral over $V(t)$ in (12), the integrand is

$$\frac{2}{n}\Big(\frac{2}{n}+1\Big)\frac{\|x_0 - x\|^2}{4(t_0-t)^2} vw - \frac{v}{n(t_0-t)}\langle \nabla_x w, x_0 - x\rangle + \frac{vw\big(2n(t_0-t) - \|x_0 - x\|^2\big)}{2n(t_0-t)^2}$$

by (4) and (5). Since v, w and $\|\nabla_x w\|$ are bounded, this expression is dominated by a multiple of

$$(14) \qquad \frac{\|x_0 - x\|^2}{(t_0-t)^2} + \frac{2n}{(t_0-t)},$$

which is obviously integrable on $V(t_0 - c_1 e^{-1})$. Furthermore, in $A(p_0, c_1, c)$,

$$\|x_0 - x\|^2 \geq 2n(t_0 - t)\log\big(c_1/(t_0-t)\big),$$

so that on $A(p_0, c_1, c) \setminus V(t_0 - c_1 e^{-1})$ we have

$$\frac{\|x_0 - x\|^2}{(t_0-t)^2} \geq \frac{2n}{(t_0-t)},$$

and therefore the expression (14) is dominated by $\|x_0 - x\|^2 (t_0 - t)^{-2}$, which is integrable by [4, Lemma 4]. It follows that we can make $t \to t_0-$ in (12); in view of (13), we thus obtain (9) with $A = A(p_0, c_1, c)$ and $v = W_0^{-2/n}/4\pi$. Next,

$$\iint_A f\, dx\, dt = \int_{c_1}^{c} d\gamma \int_{\partial\Omega(\gamma)} f J_0\, d\sigma$$

for any function f such that either side exists [4, p. 388]. It therefore follows from (9) that

$$\Big(\int_{\partial\Omega(c)} - \int_{\partial\Omega(c_1)}\Big) w\langle\nabla_x v, \nu_x\rangle\, d\sigma = \int_{c_1}^{c} d\gamma \int_{\partial\Omega(\gamma)} \big(w\theta^* v + \langle\nabla_x w, \nabla_x v\rangle - wD_t v\big) J_0\, d\sigma,$$

so that

$$(15)\qquad \Big(\int_{\partial\Omega(c)} w\langle\nabla_x v, \nu_x\rangle\, d\sigma\Big)_c = \int_{\partial\Omega(c)} \big(w\theta^* v + \langle\nabla_x w, \nabla_x v\rangle - wD_t v\big) J_0\, d\sigma.$$

In (15), we take $w = u$ and $v = W_0^{-2/n}/4\pi$. Then, by (7),

$$\begin{aligned}\int_{\partial\Omega(c)} w\langle\nabla_x v, \nu_x\rangle\, d\sigma &= \frac{1}{4\pi}\int_{\partial\Omega(c)} u\Big(-\frac{2}{n}\Big) W_0^{-(2/n)-1}\langle\nabla_x W_0, \nu_x\rangle\, d\sigma \\ &= \kappa_n c^{(n/2)+1}\mathcal{M}(c),\end{aligned}$$

so that the left side of (15) is

$$\kappa_n\Big(c^{(n/2)+1}\mathcal{M}(c)\Big)_c.$$

Next, by (6), (4), and (1),

$$\begin{aligned}\int_{\partial\Omega(c)} w(\theta^* v) J_0\, d\sigma &= \frac{1}{4\pi}\int_{\partial\Omega(c)} u\frac{2}{n}\Big(\frac{2}{n}+1\Big)\frac{\|\nabla_x W_0\|^2}{W_0^2} W_0^{-2/n} J_0\, d\sigma \\ &= \Big(\frac{2}{n}+1\Big)\int_{\partial\Omega(c)} \Big(\frac{c\,\|x_0 - x\|^2 J_0}{2n(t_0-t)^2}\Big) u\, d\sigma \\ &= \Big(\frac{2}{n}+1\Big)\int_{\partial\Omega(c)} Q_0 u\, d\sigma = \kappa_n\Big(1+\frac{n}{2}\Big)c^{n/2}\mathcal{M}(c).\end{aligned}$$

By (2), (3), (4) and (5), the remainder of the right side of (15) is

$$\begin{aligned}&\frac{1}{4\pi}\int_{\partial\Omega(c)}\Big(-\frac{2}{n}W_0^{-(2/n)-1}\langle\nabla_x u, \nabla_x W_0\rangle + \frac{2}{n}W_0^{-(2/n)-1} u D_t W_0\Big) J_0\, d\sigma \\ &\qquad = \int_{\partial\Omega(c)} \big(\langle\nabla_x u, \nu_x\rangle - u\nu_t\big)\, d\sigma.\end{aligned}$$

Hence (15) yields

$$\kappa_n\Big(c^{(n/2)+1}\mathcal{M}(c)\Big)_c = \kappa_n\Big(1+\frac{n}{2}\Big)c^{n/2}\mathcal{M}(c) + \int_{\partial\Omega(c)} \big(\langle\nabla_x u, \nu_x\rangle - u\nu_t\big)\, d\sigma,$$

which implies the identity (10).

Taking $A = A(p_0, c_1, c)$ in (8), and using an argument similar to the one that gave us (15) from (9), we obtain

$$(16)\quad \left(\int_{\partial\Omega(c)} \left(\langle v\nabla_x w, \nu_x\rangle - vw\nu_t\right) d\sigma\right)_c = \int_{\partial\Omega(c)} \left(v\theta w + \langle \nabla_x v, \nabla_x w\rangle - wD_t v\right) J_0 \, d\sigma.$$

In (16), we take $v = 1$ and $w = u$. Then the left side becomes

$$\left(\int_{\partial\Omega(c)} \left(\langle \nabla_x u, \nu_x\rangle - u\nu_t\right) d\sigma\right)_c = \kappa_n \left(c^{(n/2)+1}\mathcal{M}_c(c)\right)_c$$

in view of (10), and the right side becomes

$$\int_{\partial\Omega(c)} J_0 \theta u \, d\sigma;$$

thus (11) is established.

Lemma 1 provides the following elementary proof of [4, Theorem 12].

Corollary. *Let w be a subtemperature on an open set D, and let $p_0 \in D$. Then $\mathcal{M}(w, p_0, \cdot)$ is increasing on the set of c such that $\bar{\Omega}(p_0, c) \subseteq D$.*

Proof. If w is smooth on D, we can take $A = \Omega(p_0, c)$ and $v = 1$ in (8), to obtain

$$\iint_{\Omega(c)} \theta w \, dx \, dt = \int_{\partial\Omega(c)} \left(\langle \nabla_x w, \nu_x\rangle - w\nu_t\right) d\sigma.$$

Therefore, taking $u = w$ in (10), we obtain

$$\mathcal{M}_c(c) = \kappa_n^{-1} c^{-(n/2)-1} \iint_{\Omega(c)} \theta w \, dx \, dt.$$

Since $\theta w \geq 0$, this formula immediately implies that $\mathcal{M}$ is increasing. (A similar argument was given by Pini [3] for the case $n = 1$.) If w is an arbitrary subtemperature, we can take a decreasing sequence $\{w_j\}$ of smooth subtemperatures, with limit w on a neighbourhood of $\bar{\Omega}(p_0, c)$ [2, p. 281]. Then $\mathcal{M}(w_j, p_0, \cdot)$ is increasing for every j, so that the same is true of its limit, which is $\mathcal{M}(w, p_0, \cdot)$ by the monotone convergence theorem.

We can now give a proof of [6, Theorem 2] that does not rely upon knowledge of the Dirichlet problem for $A(p_0, c_1, c_2)$.

Theorem 1. *Let w be a subtemperature on an open superset of $\bar{A}(p_0, c_1, c_2)$. Then there is a real-valued, convex function ϕ such that $\mathcal{M}(w, p_0, c) = \phi(c^{-n/2})$ for all $c \in [c_1, c_2]$.*

Proof. We require the fact that $\mathcal{M}(w, p_0, \cdot)$ is real-valued, which was proved in [6]. That proof requires the special case of [6, Theorem 1] in which u is a temperature on an open superset of $\bar{A}(p_0, c_1, c_2)$, which was proved by elementary techniques (and could alternatively be deduced from (11)). It also requires the result given as an example in [6], which depends only upon the aforementioned special case of [6, Theorem 1] and the fact that $\mathcal{M}(w, p_0, \cdot)$ is increasing (which we have just given an elementary proof of). Suppose that w is smooth. Then $\theta w \geq 0$, so that

$$\left(c^{(n/2)+1}\mathcal{M}_c(c)\right)_c \geq 0 \tag{17}$$

by (11). Suppose also that $w > 0$, so that $\mathcal{M} > 0$. Then we can rearrange (17) to obtain

$$\frac{\mathcal{M}_{cc}}{\mathcal{M}} + \left(\frac{n}{2c} + \frac{1}{c}\right)\frac{\mathcal{M}_c}{\mathcal{M}} \geq 0. \tag{18}$$

Put $\lambda(c) = c^{n/2}\mathcal{M}(c)$. Then

$$\frac{\mathcal{M}_c}{\mathcal{M}} = \left(\frac{\lambda_c}{\lambda} - \frac{n}{2c}\right)$$

and

$$\frac{\mathcal{M}_{cc}}{\mathcal{M}} = \frac{\lambda_{cc}}{\lambda} - \frac{n\lambda_c}{c\lambda} + \frac{n^2}{4c^2} + \frac{n}{2c^2},$$

so that (18) becomes

$$\frac{\lambda_{cc}}{\lambda} + \left(1 - \frac{n}{2}\right)\frac{\lambda_c}{c\lambda} \geq 0.$$

Putting $\xi = c^{n/2}$, we obtain

$$\lambda_{\xi\xi} = \frac{4c^{2-n}}{n^2}\left(\lambda_{cc} + \left(1 - \frac{n}{2}\right)\frac{\lambda_c}{c}\right) \geq 0,$$

so that λ is a convex function of ξ. Hence $c^{n/2}\mathcal{M}(c)$ is a convex function of $c^{n/2}$, which implies that $\mathcal{M}(c)$ is a convex function of $c^{-n/2}$.

If w is smooth but not positive, we can find an open superset S of $\bar{A}(p_0, c_1, c_2)$ and a constant K such that $w - K > 0$ on S, so that

$$\mathcal{M}(w, p_0, c) = \mathcal{M}(w - K, p_0, c) + K$$

is a convex function of $c^{-n/2}$. If w is not smooth, take a decreasing sequence $\{w_j\}$ of smooth subtemperatures that converges to w on an open superset of $\bar{A}(p_0, c_1, c_2)$ [2, p. 281]. Then $\{\mathcal{M}(w_j, p_0, c)\}$ is a decreasing sequence with limit $\mathcal{M}(w, p_0, c) \in \mathbf{R}$, so that $\mathcal{M}(w, p_0, c)$ is also a convex function of $c^{-n/2}$.

We now present a simple consequence of Theorem 1 that was not considered in [6]. The subharmonic analogue can be found in [2, p. 24].

Corollary. *Let w be a subtemperature on an open superset of $\bar{A}(p_0, c_1, c_2)$. If*

$$v(x,t) = \mathcal{M}\Big(w, p_0, (t_0 - t)\exp\left(\|x_0 - x\|^2 / 2n(t_0 - t)\right)\Big)$$

for all $(x,t) \in A(p_0, c_1, c_2)$, then v is a θ^*-subtemperature (that is, a subtemperature relative to the adjoint equation).

Proof. By Theorem 1, there is a finite, convex function ϕ on $]c_2^{-n/2}, c_1^{-n/2}[$ such that $v(x,t) = \phi\big((4\pi)^{n/2} W(x_0 - x, t_0 - t)\big)$. By Lemma 1, Corollary, ϕ is decreasing, so that if $\psi(s) = \phi(c_2^{-n/2} + c_1^{-n/2} - s)$ then ψ is increasing on $]c_2^{-n/2}, c_1^{-n/2}[$ and $v(x,t) = \psi\big(c_2^{-n/2} + c_1^{-n/2} - (4\pi)^{n/2} W(x_0 - x, t_0 - t)\big)$. Since the function of (x,t) with which ψ is composed to get v, is a solution of the adjoint equation, the dual of [4, Theorem 2] implies that v is a θ^*-subtemperature.

3. Thermic majorization

Let w be a subharmonic function on $\mathbf{R}^n$, and for each $r > 0$ let $\mathcal{L}(w, 0, r)$ denote its mean over the sphere of radius r centred at the origin. It is a well-known, elementary result that w has a harmonic majorant on $\mathbf{R}^n$ if and only if $\mathcal{L}(w, 0, \cdot)$ is bounded above on $]0, \infty[$; and that, if w has such a majorant and u is the least one, then

$$u(0) = \sup_{r>0} \mathcal{L}(w, 0, r) = \lim_{r\to\infty} \mathcal{L}(w, 0, r).$$

We seek an analogous result for subtemperatures. First, we must replace the whole space ($\mathbf{R}^{n+1}$ in this case) by a lower half-space $\mathbf{R}^n \times]-\infty, a[$, because

$$\bigcup_{c>0} \Omega(p_0, c) = \mathbf{R}^n \times]-\infty, t_0[.$$

Note that a subtemperature on $\mathbf{R}^{n+1}$ can have a thermic majorant on a half-space $\mathbf{R}^n \times]-\infty, 0[$ without having one on $\mathbf{R}^{n+1}$. For example, it is well-known that there is a temperature u on $\mathbf{R}^2$ that is identically zero on $\mathbf{R}^n \times]-\infty, 0]$ but not on any open superset thereof [7, p. 86]. The subtemperature u^+ cannot have a thermic majorant on $\mathbf{R} \times]0, \infty[$, since that would imply that

$$u(x,t) = \int_{R^n} W(x - y, t) u(y, 0)\, dy = 0$$

whenever $t > 0$ [7, pp. 100–102].

We therefore seek a necessary and sufficient condition, in terms of the surface means $\mathcal{M}$, for a subtemperature w on a half-space $H_a = \mathbf{R}^n \times]-\infty, a[$ to have a thermic majorant there. There are two essential elements for this. The first is that, if $q = (y,s) \in H_a$ and $\mathcal{M}(w,q,\cdot)$ is bounded above on $]0,\infty[$, then w has a thermic majorant on H_s. This is far less elementary than the subharmonic theorem, and depends on the continuity of $\mathcal{M}(w,q,\cdot)$ that is implied by Theorem 1. The second is that, if $t_j \to a-$ and w has a thermic majorant on H_{t_j} for every j, then w has a thermic majorant on H_a. This result can be generalized to arbitrary open sets, and is given in Theorem 2 below.

Lemma 2. *Let w be a subtemperature on an open superset D of $H_a \cup \{p_a\}$ for some $p_a \in \partial H_a$. If $\mathcal{M}(w,p_a,\cdot)$ is bounded above on $]0,\infty[$, then there is an increasing family $\{w_c : c > 0\}$ of subtemperatures on D such that $w_\infty = \lim_{c\to\infty} w_c$ is the least thermic majorant of w on H_a. Furthermore,*

$$w_\infty(p_a) = \lim_{c\to\infty} \mathcal{M}(w,p_a,c). \tag{19}$$

Proof. If $c > 0$, then there is a unique subtemperature w_c on D such that w_c is a temperature on $\Omega(p_a,c)$, $w_c = w$ on $D\setminus(\Omega(p_a,c)\cup\{p_a\})$, and $w_c \geq w$ on D [6, Theorem 5]. If $d > c$, then the same theorem yields not only w_d, but also a unique subtemperature u_d on D such that u_d is a temperature on $\Omega(p_a,d)$, $u_d = w_c$ on $D\setminus(\Omega(p_a,d)\cup\{p_a\})$, and $u_d \geq w_c$ on D. Since $w_c = w$ on a superset of $D\setminus(\Omega(p_a,d)\cup\{p_a\})$ and $w_c \geq w$ on D, we see that $w_d = u_d \geq w_c$ on D, so that $w_\infty = \lim_{c\to\infty} w_c$ exists on D. By hypothesis, there is $\alpha \in \mathbf{R}$ such that $\mathcal{M}(w,p_a,c) \leq \alpha$ for all $c > 0$. Therefore

$$w_c(p_a) \leq \mathcal{M}(w_c,p_a,c) = \mathcal{M}(w,p_a,c) \leq \alpha$$

for every $c > 0$, so that $w_\infty(p_a) \leq \alpha$. If $0 < c \leq d$ then, since w_d is a subtemperature on D and a temperature on $\Omega(p_a,d)$, we have $w_d(p_a) = \mathcal{M}(w_d,p_a,c)$ by [6, Theorem 4]. Since the mean values of subtemperatures are real-valued (by Theorem 1), we can use the monotone convergence theorem to deduce that

$$\mathcal{M}(w_\infty,p_a,c) = \lim_{d\to\infty} \mathcal{M}(w_d,p_a,c) = w_\infty(p_a) \leq \alpha$$

for every $c > 0$. Hence w_∞ is finite σ-a.e. on $\partial\Omega(p_a,c)$, and therefore the Harnack convergence theorem [2, p. 276] implies that w_∞ is a thermic majorant of w on H_a. Since any thermic majorant of w on H_a will also majorize w_c for every $c > 0$, the function w_∞ is the least such majorant. Finally, if $c > 0$ we have

$$\mathcal{M}(w,p_a,c) = \mathcal{M}(w_c,p_a,c) = w_c(p_a)$$

(by [6, Theorem 4]), from which (19) follows immediately.

Theorem 2. *Let w be a subtemperature on an open set E, and let $\{p_j\}$ be a sequence in E such that*

$$E = \bigcup_{j=1}^{\infty} \Lambda(p_j). \tag{20}$$

If w has a thermic majorant on $\Lambda(p_j)$ for every j, then w has a thermic majorant on E; and if u is the least thermic majorant of w on E, then for any j the restriction of u to $\Lambda(p_j)$ is the least thermic majorant of w on $\Lambda(p_j)$.

Proof. Let u_j denote the least thermic majorant of w on $\Lambda(p_j)$. Then $u_j - w$ is a potential on $\Lambda(p_j)$, and since the Green function for $\Lambda(p_j)$ is the restriction to $\Lambda(p_j) \times \Lambda(p_j)$ of the Green function G_E for E [5; 2, p. 300], we have

$$u_j - w = \int_{\Lambda(p_j)} G_E(\cdot, q)\, d\mu_j(q)$$

on $\Lambda(p_j)$, for some positive Borel measure μ_j. Next, $G_E(p,q) > 0$ if and only if $q \in \Lambda(p)$ [5; 2,p. 300], so that

$$u_j(p) - w(p) = \int_{\Lambda(p)} G_E(p, q)\, d\mu_j(q)$$

for all $p \in \Lambda(p_j)$. Next, by the form of the Riesz decomposition theorem given in [5] and the uniqueness of representing measures, μ_j is the measure given by the distribution $-\theta(u_j - w) = \theta w$ on $\Lambda(p_j)$. Therefore, whenever $\Lambda(p_j) \cap \Lambda(p_k) \neq \emptyset$, the measures μ_j and μ_k coincide there. In view of (20), we can therefore define a measure μ on E by putting $\mu = \mu_j$ on every $\Lambda(p_j)$. This yields the representation

$$u_j(p) = w(p) + \int_{\Lambda(p)} G_E(p, q)\, d\mu(q)$$

for quasi every $p \in \Lambda(p_j)$. Since the right-hand side is independent of j, whenever $\Lambda(p_j) \cap \Lambda(p_k) \neq \emptyset$ we have $u_j = u_k$ q.e., and hence everywhere, on that intersection. We can therefore define a temperature u on E by putting $u = u_j$ on $\Lambda(p_j)$. Obviously $u \geq w$ on E, and if v is a temperature on E such that $v(q) < u(q)$ for some $q \in E$, then there is i such that $v(q) < u_i(q)$ and so v does not majorize w on $\Lambda(p_i)$.

Theorem 3. *Let w be a subtemperature on $H_a = \mathbf{R}^n \times]-\infty, a[$. Then w has a thermic majorant on H_a if and only if there is a sequence $\{p_j\}$ in H_a such that*

$$H_a = \bigcup_{j=1}^{\infty} \Lambda(p_j)$$

and $\mathcal{M}(w, p_j, \cdot)$ *is bounded above on* $]0, \infty[$ *for every* j. *If* w *has a thermic majorant on* H_a *and* u *is the least one, then*

$$u(p) = \sup_{c>0} \mathcal{M}(w,p,c) = \lim_{c\to\infty} \mathcal{M}(w,p,c) \tag{21}$$

for every $p \in H_a$.

Proof. If there is a sequence $\{p_j\}$ as described, then w has a thermic majorant on every $\Lambda(p_j)$, by Lemma 2, so that w has a thermic majorant on H_a, by Theorem 2.

Conversely, if w has a thermic majorant v on H_a, then for any $p \in H_a$ and $c > 0$ we have

$$\mathcal{M}(w,p,c) \leq \mathcal{M}(v,p,c) = v(p) < \infty.$$

Finally, if w has a least thermic majorant u on H_a, and $p = (x,t) \in H_a$, then Theorem 2 shows that the restriction of u to H_t is the least thermic majorant of w on H_t. Therefore, by Lemma 2,

$$u(p) = \lim_{c\to\infty} \mathcal{M}(w,p,c),$$

and (21) follows because $\mathcal{M}(w,p,\cdot)$ is increasing.

References

[1] DINGHAS, A.: Über einige Konvexitätsätze für die Mittelwerte von subharmonischen Funktionen. - J. Math. Pures Appl. 44, 1965, 223–247.

[2] DOOB, J.L.: Classical potential theory and its probabilistic counterpart. - Springer-Verlag, New York, 1984.

[3] PINI, B.: Maggioranti e minoranti delle soluzioni delle equazioni paraboliche. - Ann. Mat. Pura Appl. 37, 1954, 249–264.

[4] WATSON, N.A.: A theory of subtemperatures in several variables. - Proc. London Math. Soc. (3) 26, 1973, 385–417.

[5] WATSON, N.A.: Green functions, potentials, and the Dirichlet problem for the heat equation. - Proc. London Math. Soc. (3) 33, 1976, 251–298.

[6] WATSON, N.A.: A convexity theorem for local mean values of subtemperatures. - Bull. London Math. Soc. 22, 1990, 245–252.

[7] WATSON, N.A.: Parabolic equations on an infinite strip. - Marcel Dekker, New York, 1989.

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